High School Math Contest
University of South Carolina

February 1, 2014

1. A nickel is placed flat on a table. What is the maximum number of nickels that can be placed around it, flat on the table, with each one tangent to it?

   (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

2. A man saved \$2.50 in buying some lumber on sale. If he spent \$25 for the lumber, which of the following is closest to the percentage he saved?

   (a) 8% (b) 9% (c) 10% (d) 11% (e) 12%

3. In a group of dogs and people, the number of legs was 28 more than twice the number of heads. How many dogs were there? [Assume none of the people or dogs is missing a leg.]

   (a) 4 (b) 7 (c) 12 (d) 14 (e) 28

4. Suppose a, b, and c are positive integers with $a < b < c$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. What is $a + b + c$?

   (a) 1 (b) 4 (c) 9 (d) 11 (e) no such integers exist

5. My cat keeps to himself most of the time. I only heard him meow, hiss, and purr on one day out of the last 23 days. But I did hear him make at least one of these sounds each day. I heard him hiss on 6 days, purr on 12 days, and meow on 15 days. On 2 days, I heard him meow and hiss but not purr, and on 2 days, I heard him purr and hiss but not meow. On how many days did I hear him meow and purr but not hiss?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

6. Alice the number theorist knows the rule for testing if a number n is divisible by 3:

   n is divisible by 3 if and only if the sum of the digits of n is divisible by 3.

   When Alice visits Mars, she finds that the Martians have six hands, and six fingers on each hand, so that they count in base 36. In base 36, Alice's divisibility test doesn't work for testing divisibility by $d = 3$. But it does work for one of the d listed below. Which one?

   (a) 4 (b) 7 (c) 10 (d) 11 (e) 15

7. You play the following game with a friend. You share a pile of chips, and you take turns removing between one and four chips from the pile. (In particular, at least one chip must be removed on each turn.) The game ends when the last chip is removed from the pile; the one who removes it is the loser.

   It is your turn, and there are 2014 chips in the pile. How many chips should you remove to guarantee that you win, assuming you then make the best moves until the game is over?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) there is no way to guarantee a win, even with the best play

8. Two cylindrical candles of the same height but different diameters are lit at the same time. The first is consumed in 4 hours and the second in 3 hours. Assuming that they burn at a constant rate, how long after being lit was the first candle twice the height of the second candle?

   (a) 1 hr (b) 1 hr 12 min (c) 2 hr (d) 2 hr 12 min (e) 2 hr 24 min

9. Let the number a be defined as follows.

$$\log_a(10) + \log_a(10^2) + \cdots + \log_a(10^{10}) = 110.$$

   What is a?

   (a) $\sqrt{10}$ (b) $e+1$ (c) 10 (d) 20 (e) $10^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10}}$

10. Let $x = \sqrt{16+\sqrt{16+\sqrt{16+\cdots}}}$. What is the value of x?

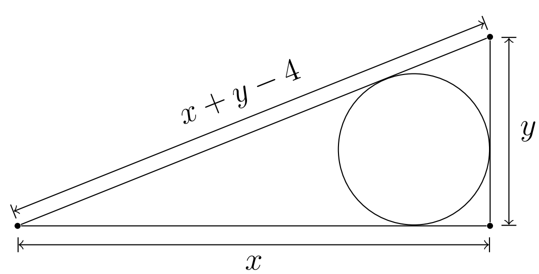
   (a) $2\sqrt{2}$ (b) 4 (c) 4.52 (d) 8 (e) $\frac{1}{2}+\frac{\sqrt{65}}{2}$

11. What is the smallest prime number with two sevens in it?

   (a) 77 (b) 177 (c) 277 (d) 377 (e) 108794769

12. The lengths of the legs of a right triangle are x and y, while the length of the hypotenuse is $x+y-4$. What is the maximum radius of a circle inscribed in this triangle?

   (a) 1 (b) 2 (c) 4 (d) 22 (e) cannot be determined from the information given



13. How many real solutions does the equation $x^5 + 2x^3 + 8x^2 + 16 = 0$ have?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 5

14. What is the value of the sum $\cos(\frac{\pi}{6}) + \cos(\frac{2\pi}{6}) + \cos(\frac{3\pi}{6}) + \cdots + \cos(\frac{2014\pi}{6})$?

(a) 0 (b) 1007 (c) $-1007\sqrt{3}$ (d) $-\dfrac{2+\sqrt{3}}{2}$ (e) $-\dfrac{1+\sqrt{3}}{2}$

15. What is the smallest integer n that satisfies both of the following equations, in which p and q are positive integers?

$$\begin{aligned} n &= p^2 + p \\ n &= q^2 + q + 2014 \end{aligned}$$

(a) 2024 (b) 2034 (c) 2056 (d) 2070 (e) 2196

16. The expression $(a+b+c)^{10}$ is expanded and simplified. How many terms are in the resulting expression?

(a) 30 (b) 44 (c) 55 (d) 66 (e) 133

17. If $\sin\alpha = -\dfrac{\sqrt{2}}{2}$ and $\cos(\alpha - \beta) = \dfrac{1}{2}$ with $\beta > 0$, what is the minimum value of β?

(a) $\dfrac{\pi}{24}$ (b) $\dfrac{\pi}{18}$ (c) $\dfrac{\pi}{12}$ (d) $\dfrac{\pi}{6}$ (e) $\dfrac{\pi}{4}$

18. If a and b are positive numbers, and (x, y) is a point on the curve $ax^2 + by^2 = ab$, what is the largest possible value of xy?

(a) $\dfrac{\sqrt{ab}}{2}$ (b) $\sqrt{ab}$ (c) $\dfrac{ab}{a+b}$ (d) $\dfrac{2ab}{a+b}$ (e) $\dfrac{\sqrt{2}ab}{a+b}$

19. What is the smallest integer $a > 0$ such that the inequality

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n+1} < a - 2010 - \frac{1}{3}$$

is satisfied for all positive integers n?

(a) 2011 (b) 2012 (c) 2013 (d) 2014 (e) 2015

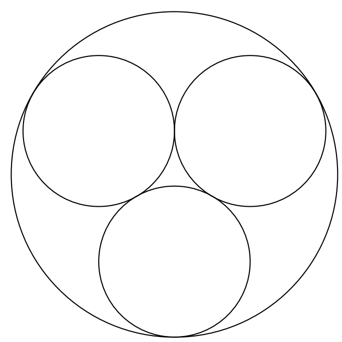
20. Which of the five numbers below is closest to the following product?

$$\left(1-\frac{1}{10}\right)\left(1-\frac{1}{10^2}\right)\left(1-\frac{1}{10^3}\right)\cdots\left(1-\frac{1}{10^{2014}}\right)$$

(a) 0.8 (b) 0.89 (c) 0.899 (d) 0.9 (e) 1

21. Three circles of equal size are inscribed inside a bigger circle of radius 1, so that every circle is tangent to every other circle. What is the radius of each of the smaller circles?

(a) $2-\sqrt{3}$ (b) $1/3$ (c) $\sqrt{3}/4$ (d) $2\sqrt{3}-3$ (e) $1/2$



22. How many real numbers are solutions of the following equation?

$$\sqrt{x+1-4\sqrt{x-3}}+\sqrt{x+6-6\sqrt{x-3}}=1$$

(a) 0 (b) 1 (c) 2 (d) 4 (e) infinitely many

23. Let $f(x)=\dfrac{x}{\sqrt{1+x^2}}$ and let $f_n(x)=\underbrace{f\big(f(f(\cdots(f}_{n}(x))\cdots))\big)$. In other words, $f_1(x)=f(x)$ and then we recursively define $f_{n+1}(x)$ as $f(f_n(x))$. What is $f_{99}(1)$?

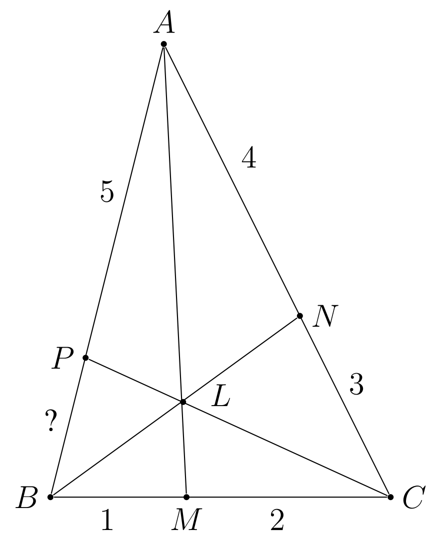
(a) $\dfrac{1}{101}$ (b) $\dfrac{1}{100}$ (c) $\dfrac{1}{99}$ (d) $\dfrac{1}{10}$ (e) $\dfrac{1}{9}$

24. Suppose you answer the last three questions on this test at random. What is the most likely number of these three questions that you will answer correctly?

(a) 0 (b) 1 (c) 2 (d) 3 (e) it is impossible to determine

25. In $\triangle ABC$ below, $\overline{AM}$, $\overline{BN}$, and $\overline{CP}$ are concurrent at L. If $BM = 1$ in, $MC = 2$ in, $CN = 3$ in, $NA = 4$ in, and $AP = 5$ in, what is the length of $\overline{PB}$?

(a) 0.8 in (b) 1.875 in (c) 2.3125 in (d) 3 in (e) 5 in



26. A normal six-sided die bearing the numbers 1, 2, 3, 4, 5, and 6 is thrown until the running total surpasses 6. What is the most likely total that will be obtained?

(a) 7 (b) 8 (c) 9 (d) 10 (e) 11

27. Suppose a game is played between two players A and B. On each turn of the game, exactly one of A or B gets a point. Suppose A is better than B and has a probability of 2/3 of getting a point on each turn of the game. The first person to get two points ahead in the game is the winner. What is the probability that A wins the game?

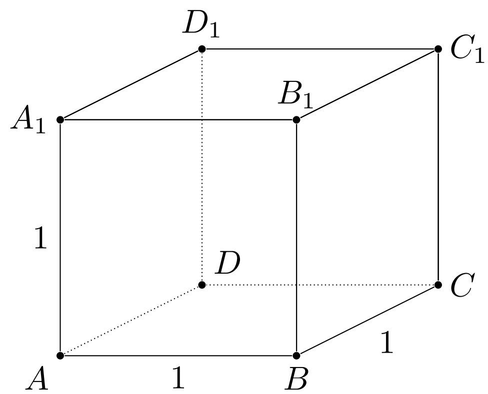
(a) $5/9$ (b) $4/7$ (c) $2/3$ (d) $4/5$ (e) $8/9$

28. The number sequence $\{a_n\}$ is given by $a_n = \dfrac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$ where n is a positive integer. The sum of the first n terms in $\{a_n\}$ is defined as $S_n = \displaystyle\sum_{i=1}^{n} a_i$. How many terms in the sequence S_1, S_2, …, S_{2014} are rational numbers? [Note: $S_1 = a_1 = 1/(2+\sqrt{2})$ is irrational.]

(a) 43 (b) 44 (c) 45 (d) 46 (e) 47

29. The eight planes AB_1C, BC_1D, CD_1A, DA_1B, A_1BC_1, B_1CD_1, C_1DA_1, and D_1AB_1 cut the unit cube $ABCDA_1B_1C_1D_1$ into several pieces. What is the volume of the piece that contains the center of the cube? [Hint: As a help to visualizing this piece, consider the places where the cuts cross on the faces of the cube.]

(a) $\frac{1}{9}$ (b) $\frac{1}{8}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$ (e) $\frac{1}{3}$



30. A computer simulation of a 12 hour analog clock keeps perfect time while it is running, and has two hands—an hour hand and a minute hand—both of which move continuously around the 12 hour dial. (For example, at 2:30, the hour hand will be exactly halfway between 2 and 3.) Because of careless programming, the minute hand looks exactly like the hour hand, so that the two are indistinguishable.

On one day, the clock stops at some time after 12am and before 12pm. How many times could it have stopped without it being possible after 12pm to tell what time it stopped?

(a) 11 (b) 12 (c) 66 (d) 132 (e) 708