High School Math Contest
University of South Carolina

February 1, 2014

1. A nickel is placed flat on a table. What is the maximum number of nickels that can be placed around it, flat on the table, with each one tangent to it?

   (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

   Answer: (c)

   Solution: Connect the midpoints of the given nickel and two others placed around it so that all three touch. This creates an equilateral triangle with 60 degree angles. Since $360/60 = 6$, six nickels can be placed around the given nickel.

2. A man saved \$2.50 in buying some lumber on sale. If he spent \$25 for the lumber, which of the following is closest to the percentage he saved?

   (a) 8% (b) 9% (c) 10% (d) 11% (e) 12%

   Answer: (b)

   Solution: $2.5/27.5 = 1/11 = 9/99$ which is close to 9%.

3. In a group of dogs and people, the number of legs was 28 more than twice the number of heads. How many dogs were there? [Assume none of the people or dogs is missing a leg.]

   (a) 4 (b) 7 (c) 12 (d) 14 (e) 28

   Answer: (d)

   Solution: If there were only people, there would be exactly twice the number of legs as heads. Each dog contributes two extra legs over and above this number. So 28 extra legs means 14 dogs.

4. Suppose a, b, and c are positive integers with $a < b < c$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. What is $a + b + c$?

   (a) 1 (b) 4 (c) 9 (d) 11 (e) no such integers exist

   Answer: (d)

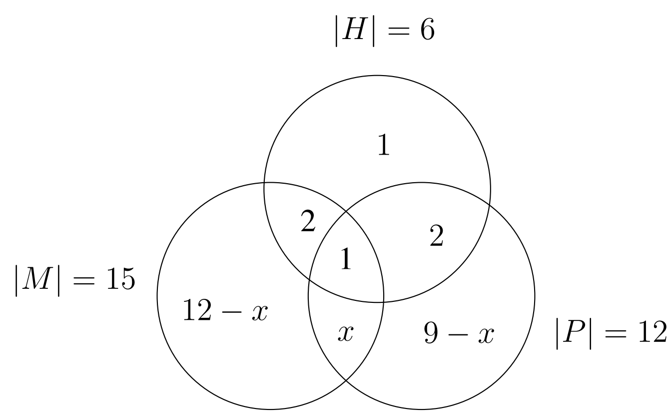
   Solution: First note that we must have $1/a < 1$, so $a > 1$. Since $1/a > 1/b > 1/c$, we must also have $1/a > 1/3$; so $a < 3$. Thus, $a = 2$. Now $1/b + 1/c = 1/2$ where $2 < b < c$. Similar to before, $1/b > 1/4$, so $b < 4$. Thus, $b = 3$. With $a = 2$ and $b = 3$ we have $1/2 + 1/3 + 1/c = 1$, which is satisfied when $c = 6$. To conclude, $a + b + c = 2 + 3 + 6 = 11$.

5. My cat keeps to himself most of the time. I only heard him meow, hiss, and purr on one day out of the last 23 days. But I did hear him make at least one of these sounds each day. I heard him hiss on 6 days, purr on 12 days, and meow on 15 days. On 2 days, I heard him meow and hiss but not purr, and on 2 days, I heard him purr and hiss but not meow. On how many days did I hear him meow and purr but not hiss?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

   Answer: (d)

   Solution: Let M denote the days on which the cat meowed, H the days on which he hissed, and P the days on which he purred. Given the data, the following Venn diagram represents the situation, where the expressions inside the circles are the cardinalities of disjoint sections:



   We are looking for x. The labeled sections of the Venn diagram are disjoint sets, so they should add to 23. That is,

$$23 = 1 + 1 + 2 + 2 + x + (9 - x) + (12 - x) = 27 - x.$$

   Therefore, $x = 4$.

   Solution 2: The inclusion-exclusion principle gives

$$\begin{aligned} 23 &= |M \cup H \cup P| = |M| + |H| + |P| - |M \cap H| - |H \cap P| - |M \cap P| + |M \cap H \cap P| \\ &= 15 + 6 + 12 - (2 + 1) - (2 + 1) - (x + 1) + 1. \end{aligned}$$

   So $x = 4$.

6. Alice the number theorist knows the rule for testing if a number n is divisible by 3:

   n is divisible by 3 if and only if the sum of the digits of n is divisible by 3.

   When Alice visits Mars, she finds that the Martians have six hands, and six fingers on each hand, so that they count in base 36. In base 36, Alice's divisibility test doesn't work for testing divisibility by $d = 3$. But it does work for one of the d listed below. Which one?

   (a) 4 (b) 7 (c) 10 (d) 11 (e) 15

   Answer: (b)

   Solution: First of all, why does the rule work in base 10 for 3? It works because $10-1 = 3\cdot 3$, $100-1 = 3\cdot 33$, $1000-1 = 3\cdot 333$, etc. As a result, the expressions

   $$a_0 + a_1\cdot 10 + a_2\cdot 10^2 + a_3\cdot 10^3 + \cdots + a_k\cdot 10^k$$

   and

   $$a_0 + a_1 + a_2 + a_3 + \cdots + a_k$$

   differ by a multiple of 3.

   The key point is that $3 \mid (10-1)$. It is quite easy to see that this is a *necessary* condition. We can see that it is sufficient because $100-1 = (10-1)\cdot(10+1)$, $1000-1 = (10-1)(100+10+1)$, etc. So, in base 36, we must have $d \mid (36-1)$. Therefore, d must be a divisor of 35, and the only such divisor listed is 7.

7. You play the following game with a friend. You share a pile of chips, and you take turns removing between one and four chips from the pile. (In particular, at least one chip must be removed on each turn.) The game ends when the last chip is removed from the pile; the one who removes it is the loser.

   It is your turn, and there are 2014 chips in the pile. How many chips should you remove to guarantee that you win, assuming you then make the best moves until the game is over?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) there is no way to guarantee a win, even with the best play

   Answer: (c)

   Solution: The key to winning this game is that if you leave your opponent with $5k+1$ chips, for any integer $k \geq 0$, then you win. When your opponent removes r chips, you remove $5-r$, and so there are now $5(k-1)+1$ chips left, and you keep using this strategy until there is only one chip left, at which point your opponent takes it and loses.

   So, you should take 3 chips to leave $2011 = 5\cdot 402 + 1$.

8. Two cylindrical candles of the same height but different diameters are lit at the same time. The first is consumed in 4 hours and the second in 3 hours. Assuming that they burn at a constant rate, how long after being lit was the first candle twice the height of the second candle?

   (a) 1 hr (b) 1 hr 12 min (c) 2 hr (d) 2 hr 12 min (e) 2 hr 24 min

   Answer: (e)

   Solution: Assume each candle is 1 unit high. Since the burn rates are $1/3$ and $1/4$ units per hour, after t hours they have been consumed by an amount of $t/3$ and $t/4$. We want $1 - t/4 = 2(1 - t/3)$. This occurs when $t = 2.4 = 2\frac{24}{60}$ hrs.

9. Let the number a be defined as follows.

$$\log_a(10) + \log_a(10^2) + \cdots + \log_a(10^{10}) = 110.$$

   What is a?

   (a) $\sqrt{10}$ (b) $e + 1$ (c) 10 (d) 20 (e) $10^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10}}$

   Answer: (a)

   Solution: By the rules for logs, the expression given is

$$\begin{aligned}\log_a(10 \cdot 10^2 \cdot 10^3 \cdots 10^{10}) &= \log_a(10^{1+2+3+\cdots 10}) \\ &= \log_a(10^{\frac{10\cdot 11}{2}}) = \log_a(10^{55}) \\ &= 55\log_a(10) = \frac{55}{\log_{10}(a)}.\end{aligned}$$

   Therefore, $\log_{10}(a) = \frac{1}{2}$, so that $a = \sqrt{10}$.

10. Let $x = \sqrt{16 + \sqrt{16 + \sqrt{16 + \cdots}}}$. What is the value of x?

   (a) $2\sqrt{2}$ (b) 4 (c) 4.52 (d) 8 (e) $\dfrac{1}{2} + \dfrac{\sqrt{65}}{2}$

   Answer: (e)

   Solution: The value of x can be found by solving the algebraic equation $x = \sqrt{16 + x}$. This gives $x^2 - x - 16 = 0$, and the positive root given by the quadratic formula is $x = \frac{1}{2} + \frac{\sqrt{65}}{2}$. The other root is extraneous.

   Remark: For this problem to have a real solution, it is essential for the associated sequence, $a_{n+1} = \sqrt{16 + a_n}$, to converge. To show this sequence does converge we use the fact that every nonempty set of real numbers that is bounded above has a least upper bound. That the sequence $\{a_n\}$ is increasing and bounded above (by 5) follows by an easy induction proof.

11. What is the smallest prime number with two sevens in it?

(a) 77 (b) 177 (c) 277 (d) 377 (e) 108794769

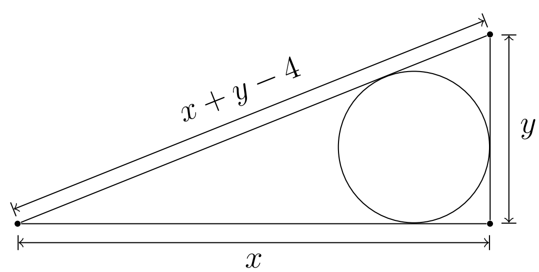
Answer: (c)

Solution: The smallest positive integers with two sevens are 77, 177, 277, $\ldots$. The first integer with two sevens that does not end with two sevens is 707.

$77 = 7 \cdot 11$, and 177 is divisible by 3 because the sum of its digits is. So now you just need to check that 277 is prime. There is no particular shortcut to see this – test divisibility by 2, 3, 5, 7, 11, and 13.

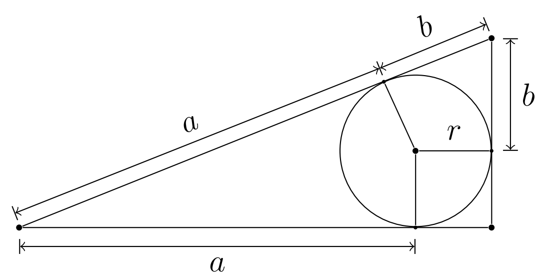
12. The lengths of the legs of a right triangle are x and y, while the length of the hypotenuse is $x + y - 4$. What is the maximum radius of a circle inscribed in this triangle?

(a) 1 (b) 2 (c) 4 (d) 22 (e) cannot be determined from the information given



Answer: (b)

Solution:



Length of hypotenuse: $x + y - 4 = a + b$. Hence $x + y = a + b + 4$.
Sum of lengths of legs: $x + y = a + b + 2r$.
Therefore $2r = 4$, so $r = 2$.

13. How many real solutions does the equation $x^5 + 2x^3 + 8x^2 + 16 = 0$ have?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 5

Answer: (b)

Solution: Note that $x^5+2x^3+8x^2+16 = (x^3+8)(x^2+2) = (x+2)(x^2-2x+4)(x^2+2)$. Since the quadratic equations $x^2 - 2x + 4 = 0$ and $x^2 + 2 = 0$ have no real solutions, the original equation has just one real solution, $x = -2$.

14. What is the value of the sum $\cos(\frac{\pi}{6}) + \cos(\frac{2\pi}{6}) + \cos(\frac{3\pi}{6}) + \cdots + \cos(\frac{2014\pi}{6})$?

(a) 0 (b) 1007 (c) $-1007\sqrt{3}$ (d) $-\dfrac{2+\sqrt{3}}{2}$ (e) $-\dfrac{1+\sqrt{3}}{2}$

Answer: (d)

Solution: The cosine function has a period 2π and $\cos(\frac{\pi}{6}) + \cos(\frac{2\pi}{6}) + \cos(\frac{3\pi}{6}) + \cdots + \cos(\frac{12\pi}{6}) = 0$. Now $2014 = 12 \times 167 + 10$, and, therefore, $\cos(\frac{\pi}{6}) + \cos(\frac{2\pi}{6}) + \cos(\frac{3\pi}{6}) + \cdots + \cos(\frac{2014\pi}{6}) = -(\cos(\frac{2015\pi}{6}) + \cos(\frac{2016\pi}{6})) = -(\cos(\frac{11\pi}{6}) + \cos(2\pi)) = -\left(\frac{\sqrt{3}}{2} + 1\right)$.

15. What is the smallest integer n that satisfies both of the following equations, in which p and q are positive integers?

$$\begin{aligned} n &= p^2 + p \\ n &= q^2 + q + 2014 \end{aligned}$$

(a) 2024 (b) 2034 (c) 2056 (d) 2070 (e) 2196

Answer: (d)

Solution: The term 2014 in the second equation suggests trying small numbers for q, and a little experimentation with p close to the square root of 2014 and $n = p(p+1)$ leads fairly quickly to $q = 7$, giving $n = 2070$, which is $(45)(46)$.

Solution 2: From $p^2+p = q^2+q+2014$ it follows that $(p-q)(p+q+1) = 2014 = 2\cdot 19\cdot 53$, where $p, q \in \mathbb{N}$. Then $p > q$, $p+q+1 > p-q$, and the two factors above can be:

(a) $p - q = 2$ and $p + q + 1 = 19 \cdot 53 = 1007,$ or

(b) $p - q = 2 \cdot 19 = 38,$ and $p + q + 1 = 53,$ or

(c) $p - q = 19,$ and $p + q + 1 = 2 \cdot 53 = 106.$

The second system produces the smallest p and the smallest q, namely $p = 45$ and $q = 7$. The corresponding n is the smallest: $n = p^2 + p = q^2 + q + 2014 = 2070$.

16. The expression $(a+b+c)^{10}$ is expanded and simplified. How many terms are in the resulting expression?

(a) 30 (b) 44 (c) 55 (d) 66 (e) 133

Answer: (d)

Solution: $(a+b+c)^{10} = [(a+b)+c]^{10}$. Let $x = a+b$. We see that $(x+c)^{10} = x^{10} + 10x^9c + \cdots + c^{10}$ has 11 distinct terms. Also, the ith term of this expansion has $11 - i$ distinct terms (since $x^{10-i} = (a+b)^{10-i}$ has $11 - i$ distinct terms). Hence there are $11 + 10 + \cdots + 1 = 66$ terms in this expansion.

Solution 2: We can represent each term as a line of 10 stars and bars, where the numbers of stars before the first bar, between the two bars, and after the second bar represent the exponents of $a, b,$ and c respectively. For example, $*****|**|***$ represents the $a^5b^2c^3$ term. Note that it is okay to have the bars adjacent to each other or appearing at either end, because exponents of zero are allowable.

So, the number of terms is equal to the number of such diagrams, where we can put 2 bars in any of the 12 positions — in other words, 12 choose 2 which equals 66.

17. If $\sin\alpha = -\dfrac{\sqrt{2}}{2}$ and $\cos(\alpha - \beta) = \dfrac{1}{2}$ with $\beta > 0$, what is the minimum value of β?

(a) $\dfrac{\pi}{24}$ (b) $\dfrac{\pi}{18}$ (c) $\dfrac{\pi}{12}$ (d) $\dfrac{\pi}{6}$ (e) $\dfrac{\pi}{4}$

Answer: (c)

Solution: We see that α must be equal to $5\pi/4$ or $7\pi/4$, plus or minus any multiple of 2π. Similarly, $\alpha - \beta$ must be $\pi/3$ or $5\pi/3$, plus or minus any multiple of 2π.

We can obtain a value of $\beta = \pi/12$ by taking $\alpha = 7\pi/4$ and $\alpha - \beta = 5\pi/3$. Moreover, any value of β must be a multiple of $\pi/12$ because the denominators are 4 and 3 respectively. Therefore, $\pi/12$ is the minimum.

18. If a and b are positive numbers, and (x, y) is a point on the curve $ax^2 + by^2 = ab$, what is the largest possible value of xy?

(a) $\dfrac{\sqrt{ab}}{2}$ (b) $\sqrt{ab}$ (c) $\dfrac{ab}{a+b}$ (d) $\dfrac{2ab}{a+b}$ (e) $\dfrac{\sqrt{2}ab}{a+b}$

Answer: (a)

Solution: To put the left-hand side into the form of a perfect square, subtract $2\sqrt{ab}xy$ from both sides of the equation:

$$\begin{aligned} ax^2 - 2\sqrt{ab}xy + by^2 &= ab - 2\sqrt{ab}xy \\ \left(\sqrt{a}x - \sqrt{b}y\right)^2 &= ab - 2\sqrt{ab}xy. \end{aligned}$$

Now, this left-hand side is zero at two points on the curve: $(\sqrt{b/2}, \sqrt{a/2})$ and $(-\sqrt{b/2}, -\sqrt{a/2})$. Thus:

$$\begin{aligned} 0 &\leq ab - 2\sqrt{ab}xy \\ 2\sqrt{ab}xy &\leq ab \\ xy &\leq ab/2/\sqrt{ab} = \sqrt{ab}/2 \end{aligned}$$

The answer is (a).

Solution 2: Divide $ax^2 + by^2 = ab$ by ab on each side, obtaining $\frac{x^2}{b} + \frac{y^2}{a} = 1$. The left side is then twice the arithmetic mean of $(x/\sqrt{b})^2$ and $(y/\sqrt{a})^2$ while twice their geometric mean is $2(x/\sqrt{a})(y/\sqrt{b})$. So $2(x/\sqrt{a})(y/\sqrt{b}) \leq 1$, which gives $xy \leq \frac{\sqrt{ab}}{2}$.

Solution 3: The curve is an ellipse centered on the origin with one end of one axis at $(\sqrt{b}, 0)$ and one end of the other at $(0, \sqrt{a})$. Its parametric equations are $x = \sqrt{b}\cos\theta$ and $y = \sqrt{a}\sin\theta$. So

$$xy = \sqrt{ab}(\cos\theta)(\sin\theta) = \frac{\sqrt{ab}}{2}\sin 2\theta$$

and $\sin(2\theta) \leq 1$ and so $xy = \sqrt{ab}/2$ maximizes xy.

19. What is the smallest integer $a > 0$ such that the inequality

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n+1} < a - 2010 - \frac{1}{3}$$

is satisfied for all positive integers n?

(a) 2011 (b) 2012 (c) 2013 (d) 2014 (e) 2015

Answer: (b)

Solution: Plugging in $n = 1$, we see that $a > 2011$. On the other hand, since there are $n + 1$ terms, and all of them are at most $1/(n + 1)$, the expression is at most 1, which means that $a = 2012$ works.

Solution 2: Let $f(n) = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n+1}$. Since $f(n)$ is a monotonically decreasing function, its maximum value is $f(1)$. Therefore, from $f(1) < a - 2010 - \frac{1}{3}$, we get $a = 2012$.

20. Which of the five numbers below is closest to the following product?

$$\left(1 - \frac{1}{10}\right)\left(1 - \frac{1}{10^2}\right)\left(1 - \frac{1}{10^3}\right)\cdots\left(1 - \frac{1}{10^{2014}}\right)$$

(a) 0.8 (b) 0.89 (c) 0.899 (d) 0.9 (e) 1

Answer: (b)

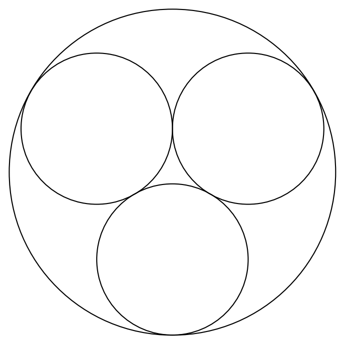
Solution: Denote the product by P. Clearly $P < .9(.99) = .891$. Also, for any positive real number a and any real number b between 0 and 1 we have, $b(1 - a) \geq b - a$. Applying this inequality repeatedly, we get

$$\begin{aligned}
P &= \underbrace{\left(1 - \frac{1}{10}\right)\left(1 - \frac{1}{10^2}\right)\left(1 - \frac{1}{10^3}\right)\cdots\left(1 - \frac{1}{10^{2013}}\right)}_{b}\left(1 - \underbrace{\frac{1}{10^{2014}}}_{a}\right) \\
&> \underbrace{\left(1 - \frac{1}{10}\right)\left(1 - \frac{1}{10^2}\right)\left(1 - \frac{1}{10^3}\right)\cdots\left(1 - \frac{1}{10^{2013}}\right)}_{b} - \underbrace{\frac{1}{10^{2014}}}_{a} \\
&> \left(1 - \frac{1}{10}\right)\left(1 - \frac{1}{10^2}\right)\left(1 - \frac{1}{10^3}\right)\cdots\left(1 - \frac{1}{10^{2012}}\right) - \frac{1}{10^{2013}} - \frac{1}{10^{2014}} \\
\cdots \\
&> 1 - .1 - .01 - .001 - \cdots - \frac{1}{10^{2014}} \\
&> 1 - \frac{0.1}{1 - 0.1} = \frac{8}{9} = 0.\overline{8}.
\end{aligned}$$

The number in the list closest to $0.\overline{8}$ is 0.89; the correct answer is (b).

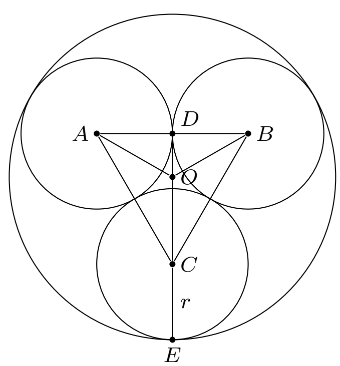
21. Three circles of equal size are inscribed inside a bigger circle of radius 1, so that every circle is tangent to every other circle. What is the radius of each of the smaller circles?

(a) $2-\sqrt{3}$ (b) $1/3$ (c) $\sqrt{3}/4$ (d) $2\sqrt{3}-3$ (e) $1/2$



Answer: (d)

Solution: Let r be the radius of each of the smaller circles. Draw an equilateral triangle $\triangle ABC$ connecting the centers of the smaller circles, as shown below. Each side of $\triangle ABC$ is of length $2r$, and the altitude of $\triangle ABC$ is $\sqrt{3}r$.



Let O be the center of the big circle. This is also the centroid of $\triangle ABC$, the point at which the medians meet. Since $\triangle ABC$ is equilateral, the altitudes and angle bisectors also meet at O. In particular, the line $\overline{OA}$ bisects $\angle A$.

Let D be the midpoint of the line segment from A to B. $\triangle ADO$ is a 30-60-90 triangle, so $DO = \frac{1}{2}OC$ and $DO + OC = \sqrt{3}r$, so $OC = 2r/\sqrt{3}$. The radial line of the big circle that passes through C to E as in the diagram thus has length

$$\frac{2r}{\sqrt{3}} + r = 1 \qquad \text{giving} \qquad r = \frac{\sqrt{3}}{\sqrt{3}+2}$$

Multiplying numerator and denominator by $\sqrt{3}-2$ and simplifying gives $r = 2\sqrt{3}-3$.

22. How many real numbers are solutions of the following equation?

$$\sqrt{x+1-4\sqrt{x-3}}+\sqrt{x+6-6\sqrt{x-3}}=1$$

(a) 0 (b) 1 (c) 2 (d) 4 (e) infinitely many

Answer: (e)

Solution: Note that $x+1-4\sqrt{x-3} = x-3-4\sqrt{x-3}+4 = (\sqrt{x-3}-2)^2$. Similarly, $x+6-6\sqrt{x-3} = (\sqrt{x-3}-3)^2$. Thus, the equation simplifies to $\left|\sqrt{x-3}-2\right| + \left|3-\sqrt{x-3}\right| = 1$. Clearly, every x such that $2 \le \sqrt{x-3} \le 3$, or $7 \le x \le 12$, is a solution. Thus, there are infinitely many solutions.

23. Let $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ and let $f_n(x) = \underbrace{f\big(f(f(\cdots(f}_{n}(x))\cdots))\big)$. In other words, $f_1(x) = f(x)$ and then we recursively define $f_{n+1}(x)$ as $f(f_n(x))$. What is $f_{99}(1)$?

(a) $\dfrac{1}{101}$ (b) $\dfrac{1}{100}$ (c) $\dfrac{1}{99}$ (d) $\dfrac{1}{10}$ (e) $\dfrac{1}{9}$

Answer: (d)

Solution: By the recursion formula,

$$f_1(x) = f(x) = \frac{x}{\sqrt{1+x^2}}, f_2(x) = f\big(f(x)\big) = \frac{x}{\sqrt{1+2x^2}}, \cdots, f_{99}(x) = \frac{x}{\sqrt{1+99x^2}}.$$

Thus $f_{99}(1) = 1/10$.

24. Suppose you answer the last three questions on this test at random. What is the most likely number of these three questions that you will answer correctly?

(a) 0 (b) 1 (c) 2 (d) 3 (e) it is impossible to determine

Answer: (a)

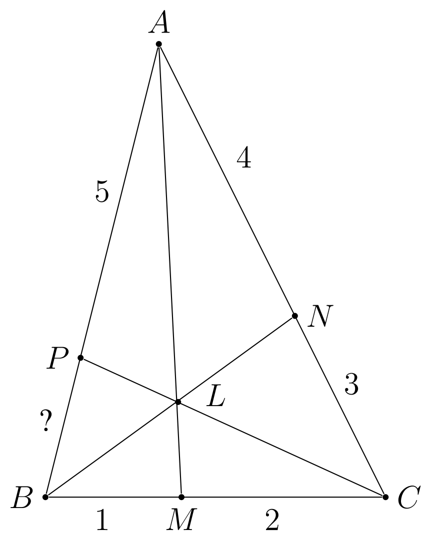
Solution: There are five answers to each question The probabilities you will correctly answer respectively 0, 1, 2, and 3 are

$$\begin{aligned} \frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5} &= \frac{64}{125}, \\ 3\cdot\frac{1}{5}\cdot\frac{4}{5}\cdot\frac{4}{5} &= \frac{48}{125}, \\ 3\cdot\frac{1}{5}\cdot\frac{1}{5}\cdot\frac{4}{5} &= \frac{12}{125}, \\ \frac{1}{5}\cdot\frac{1}{5}\cdot\frac{1}{5} &= \frac{1}{125}. \end{aligned}$$

The first is the most likely possibility. Indeed, it is more probable than the other three combined.

25. In $\triangle ABC$ below, $\overline{AM}$, $\overline{BN}$, and $\overline{CP}$ are concurrent at L. If $BM = 1$ in, $MC = 2$ in, $CN = 3$ in, $NA = 4$ in, and $AP = 5$ in, what is the length of $\overline{PB}$?

(a) 0.8 in (b) 1.875 in (c) 2.3125 in (d) 3 in (e) 5 in



Answer: (b)

Solution: Let L be the center of mass of a system of three particles located at A, B, and C. If the particle B has weight 1 and the particle C has weight $1/2$, then the system $B - C$ has the center of mass at M, because $BM/MC = 1/2$. The center of mass of the whole system, L, belongs then to the segment AM.

Furthermore, if the particle A has weight $3/8$, then the system $C - A$ has the center of mass at N, because $1/2 \cdot CN = 3/8 \cdot NA$. Then L belongs to the segment BN.

Since AM, BN, and CP are concurrent at L,

$$\frac{AP}{PB} = \frac{1}{3/8} = \frac{8}{3}.$$

Using $AP = 5$ this gives

$$PB = 5 \cdot \frac{3}{8} = \frac{15}{8} = 1.875.$$

Solution 2: By the Theorem of Ceva,

$$\frac{AP}{PB} \cdot \frac{BM}{MC} \cdot \frac{CN}{NA} = 1.$$

Therefore

$$PB = \frac{5 \cdot 1 \cdot 3}{2 \cdot 4} = \frac{15}{8} = 1.875.$$

26. A normal six-sided die bearing the numbers 1, 2, 3, 4, 5, and 6 is thrown until the running total surpasses 6. What is the most likely total that will be obtained?

(a) 7 (b) 8 (c) 9 (d) 10 (e) 11

Answer: (a)

Solution: The total before the last throw could be any number between one and six. The last throw will also be a number between one and six but with equal probability. Seven is the only final total that can come from each of these possibilities. Therefore seven is the most likely final total.

27. Suppose a game is played between two players A and B. On each turn of the game, exactly one of A or B gets a point. Suppose A is better than B and has a probability of 2/3 of getting a point on each turn of the game. The first person to get two points ahead in the game is the winner. What is the probability that A wins the game?

(a) $5/9$ (b) $4/7$ (c) $2/3$ (d) $4/5$ (e) $8/9$

Answer: (d)

Solution: Let W_0 be the event that Player A wins the game. The goal is to compute $P(W_0)$. Let W_1 be the event that Player A wins the game given that she is ahead by one point. Let W_{-1} be the event Player A wins the game given that she is behind by one point. Then, the probability that Player A wins the game is

$$\begin{aligned} P(W_0) &= P(\text{A is ahead by 1 point})P(W_1) + P(\text{A is behind by 1 point})P(W_{-1}) \\ &= (2/3)P(W_1) + (1/3)P(W_{-1}). \qquad (1) \end{aligned}$$

Now,

$$\begin{aligned} P(W_1) &= P(\text{Player A gets a point on the next turn}) + P(\text{Player B gets a point on the next turn})P(W_0) \\ &= (2/3) + (1/3)P(W_0), \end{aligned}$$

and

$$P(W_{-1}) = P(\text{Player A gets a point on the next turn})P(W_0) = (2/3)P(W_0).$$

Substituting these equations into (1) we have

$$P(W_0) = (2/3)[(2/3) + (1/3)P(W_0)] + (1/3)(2/3)P(W_0) = 4/9 + (4/9)P(W_0).$$

Therefore, $P(W_0) = (4/9)(9/5) = 4/5$.

28. The number sequence $\{a_n\}$ is given by $a_n = \dfrac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}$ where n is a positive integer. The sum of the first n terms in $\{a_n\}$ is defined as $S_n = \displaystyle\sum_{i=1}^{n} a_i$. How many terms in the sequence S_1, S_2, ..., S_{2014} are rational numbers? [Note: $S_1 = a_1 = 1/(2+\sqrt{2})$ is irrational.]

(a) 43 (b) 44 (c) 45 (d) 46 (e) 47

Answer: (a)

Solution: From

$$a_k = \frac{1}{\sqrt{k(k+1)}(\sqrt{k+1}+\sqrt{k})} = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}},$$

we have

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}.$$

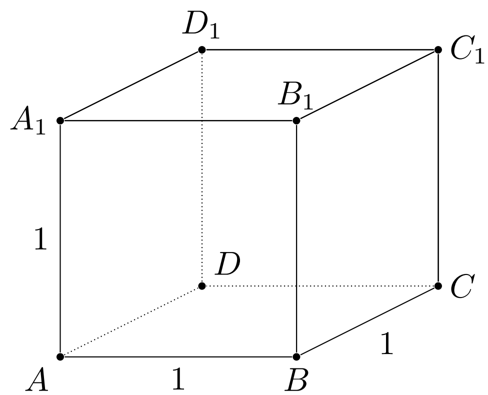
Thus, S_n is rational if and only if $n+1$ is a perfect square.

Since $44 < \sqrt{2015} < 45$, we also have $n+1 = 2^2, 3^2, 4^2, \cdots, 44^2$.

Therefore, among S_1, S_2, ..., S_{2014}, there are 43 rational terms.

29. The eight planes AB_1C, BC_1D, CD_1A, DA_1B, A_1BC_1, B_1CD_1, C_1DA_1, and D_1AB_1 cut the unit cube $ABCDA_1B_1C_1D_1$ into several pieces. What is the volume of the piece that contains the center of the cube? [Hint: As a help to visualizing this piece, consider the places where the cuts cross on the faces of the cube.]

(a) $\frac{1}{9}$ (b) $\frac{1}{8}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$ (e) $\frac{1}{3}$



Answer: (c)

Solution: The eight cuts leave an inscribed regular octahedron with its vertices in the centers of the faces of the cube. The hint helps to see this. Look at the front face AA_1B_1B. There are two cuts that include the diagonal A_1B and two more cuts that include the diagonal AB_1. All four cuts include the intersection of these two diagonals – the center point of this face.

By symmetry, similar things happen on all six faces of the cube. We conclude that the central piece touches each face only in its center. These six center points of the faces are the vertices of the central region; they form an octahedron. In fact, this octahedron is both regular and inscribed in the original cube.

To find the volume of this octahedron note that it is the union of two pyramids with square bases. The height of each pyramid is $1/2$; the square bases have side length $1/\sqrt{2}$, and area $1/2$. The pyramid has volume $1/3(1/2)(1/2) = 1/12$, and so the octahedron has volume $1/6$.

Remarks. (1) None of the cuts removes anything from the octahedron described above. The realization that the octahedron is the union of two pyramids helps to simplify the proof of this fact. By focusing on one pyramid, one verifies that none of the eight cuts removes any of its points, and symmetry does the rest.

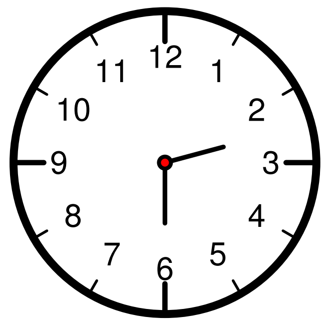
(2) Additional information about the octahedron can be found at numerous places on the web. The image at the right shows the cube and its inscribed regular octahedron (`http://commons.wikimedia.org/wiki/File:Dual_Cube-Octahedron.svg`). See also `http://mathworld.wolfram.com/Octahedron.html` and `http://www.mathsisfun.com/geometry/octahedron.html`.

(3) This octahedron is the *dual polyhedron* of the cube. For more on dual polyhedra, see `http://mathworld.wolfram.com/DualPolyhedron.html`.

30. A computer simulation of a 12 hour analog clock keeps perfect time while it is running, and has two hands—an hour hand and a minute hand—both of which move continuously around the 12 hour dial. (For example, at 2:30, the hour hand will be exactly halfway between 2 and 3.) Because of careless programming, the minute hand looks exactly like the hour hand, so that the two are indistinguishable.

    On one day, the clock stops at some time after 12am and before 12pm. How many times could it have stopped without it being possible after 12pm to tell what time it stopped?

    (a) 11 (b) 12 (c) 66 (d) 132 (e) 708



Answer: (d)

Solution:

A configuration of the hands is called "ambiguous" if you cannot use this broken clock to distinguish the minute hand from the hour hand.

In the above diagram it must be the minute hand that is pointing to 6 and the hour hand that is pointing midway between 2 and 3. (If the hour hand were to point to 6, the minute hand would have to point to 12.) 2:30 is not an ambiguous configuration.

Let $m \in (0, 1)$ represent the fraction of the way around the clock face for the minute hand and $h \in (0, 1)$ the corresponding fraction for the hour hand. For example, 2:30 has $m = 1/2 = 0.5$ and $h = 2.5/12 = 0.208\overline{3}$.

As the minute hand moves 12 times as fast as the hour hand, a configuration is ambiguous if m and $12h$ have the same fractional part. Because the hands are indistinguishable, a time is also ambiguous when h and $12m$ have the same fractional part. We express these conditions as: $12h \equiv m(\mod 1)$ and $12m \equiv h(\mod 1)$. Thus $144h \equiv h(\mod 1)$, and so $143h \equiv 0(\mod 1)$. There are 142 solutions to this equation in the open interval $(0, 1)$: $h \in H_1 := \{\frac{k}{143} : k = 1, 2, \ldots, 142\}$.

Times when the hour and minute hand perfectly overlap ($h = m$) are in H_1, but these are not ambiguous times. These configurations occur when $12h \equiv h(\mod 1)$. That is, $11h \equiv 0(\mod 1)$. There are 10 such times: $h \in H_2 := \left\{\frac{k}{11} : k = 1, 2, \ldots, 10\right\}$.

The number of ambiguous times is the cardinality of $H_1 \backslash H_2$. As $H_2 \subset H_1$ the cardinality of $H_1 \backslash H_2$ is $142 - 10 = 132$, which is answer (d).

This problem is also discussed in `http://plus.maths.org/content/puzzle-page-96`, but the problem there speaks of a real analog clock, where there is a third dimension with one of the hands above the other in the extra dimension.