Estimation of the Mean Function of Functional Data via Deep Neural Networks

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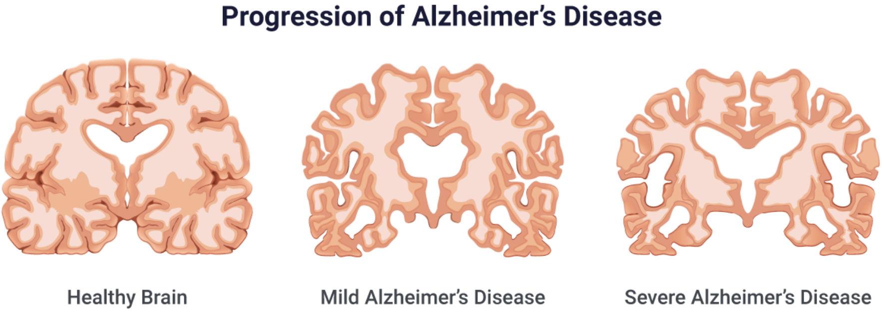
Outline

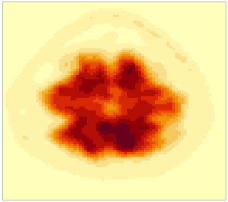
1. **Introduction**
   - Functional regression model
   - Deep neural networks
2. **Methods**
   - Estimation of mean function via deep neural networks
   - Non-asymptotic convergency rate
3. **Real data analysis**

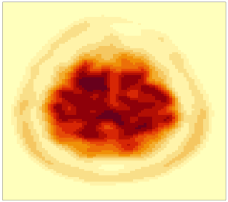
Brain Imaging

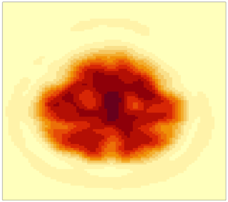


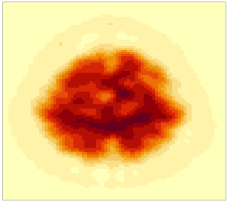
Source: https://www.caring.com/caregivers/alzheimers/

Positron Emission Tomography (PET) Images









Functional regression model

$$Y_{ij} = f_0(\mathbf{X}_j) + \eta(\mathbf{X}_j) + \epsilon_i(\mathbf{X}_j), \quad i = 1, 2, \ldots, n, j = 1, 2, \ldots, N,$$

- $f_0 : \mathbb{R}^d \to \mathbb{R}$, $\mathrm{E}(\mathrm{Y_{ij}}) = \mathrm{f_0}(\mathbf{X}_\mathrm{j})$; $\mathbf{X}_j \in \mathbb{R}^d$;
- $\eta(\cdot)$: individual curve variations; zero mean Gaussian process;
- $\epsilon_i(\cdot)$: zero mean measurement error;
- n: sample size;
- N: number of observations for each subject.

How to estimate mean function $f_0(\cdot)$?

Deep neural networks

Definition

$$f(\mathbf{x}) = \mathbf{W}_L\sigma(W_{L-1}\ldots\sigma(\mathbf{W}_2\sigma(\mathbf{W}_1\mathbf{x}+v_1)+v_2)\ldots+v_{L-1}),$$

- $d = d_0 \to d_1 \to \ldots, \ldots \to d_L \to d_{L+1} = 1$;
- $\sigma(x) = \max(x, 0)$: ReLU activation function;
- $\mathbf{W}_\ell : p_\ell \times p_{\ell+1}$ weight matrix;

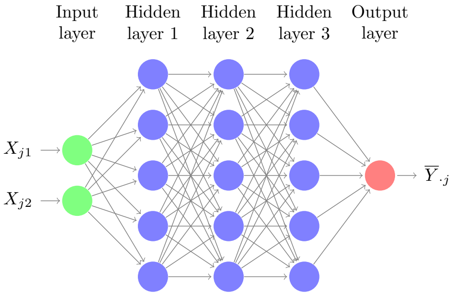
Sparse network space:

$$\mathcal{F}_{DNN}(L, \mathbf{p}, s) = \left\{ f : \max_{\ell=0,\ldots,L} \|\mathbf{W}_\ell\|_\infty + |\mathbf{v}_\ell|_\infty \leq 1, \sum_{\ell=0}^{L} \|\mathbf{W}_\ell\|_0 + |\mathbf{v}_\ell|_0 \leq s \right\}$$

Structured compositions of Hölder Functions

- $g_i : [a_i, b_i]^{d_i} \to [a_{i+1}, b_{i+1}]^{d_{i+1}}$, $g_i = \left(g_{ij}\right)^\top_{j=1,\ldots,d_{i+1}}$, ambient
- Each component g_{ij} is β_i-Hölder function with at most t_i-variate: $\left\{g_{ij} \in \mathcal{C}^{\beta_i}_{t_i}\left([a_i, b_i]^{t_i}, K_i\right), |a_i|, |b_i| \leq K_i\right\}$ intrinsic
- True underlying function space: $\mathcal{G}\left(q, \{d_i, t_i, \beta_i, K_i\}_{i\in[q]}\right)$ consists of $f = g_q \circ g_{q-1} \circ \ldots \circ g_1 \circ g_0$
- Smoothness of $f_i = g_q \circ g_{q-1} \circ \ldots \circ g_i$ $\beta^*_i := \beta_i \prod_{k=i+1}^{q}(\beta_k \wedge 1)$

Functional regression via Deep neural networks



Empirical risk minimization

$$\widehat{f} = \arg\min_{f \in \mathcal{F}_{DNN}} \frac{1}{N} \sum_{j=1}^{N} \left\{ \overline{Y}_{\cdot j} - f\left(\mathbf{X}_j\right) \right\}^2,$$

where $\overline{Y}_{\cdot j} = n^{-1} \sum_{i=1}^{n} Y_{ij}$, $\mathbf{X}_j = (X_{j1}, \ldots, X_{jd})$

Non-asymptotic convergency rate

Theorem 1

Under mild assumptions, with probability greater than $(1-\frac{2}{nN^{\varrho}})^{\log(nN^{\varrho})+1} \to 1$, we have

$$\|\widehat{f}-f_0\|_N^2 \le c(nN^{\varrho})^{-\frac{\theta}{\theta+1}}\log^6(nN^{\varrho}),$$

where $\varrho \ge 0$, $\theta = \min_{i=0,\dots,q} \frac{2\beta_i^*}{t_i}$ and c depend on true function class of f_0.

- $(nN^{\varrho})^{-\frac{\theta}{\theta+1}} = (nN^{\varrho})^{-\alpha}$ and $\alpha = \min_{i=0,\dots,q} \frac{2\beta_i^*}{2\beta_i^*+t_i}$
- If $\varrho = 0$, $\|\widehat{f}-f_0\|_N^2 \le cn^{-\frac{\theta}{\theta+1}}\log^6(n)$

ADNI

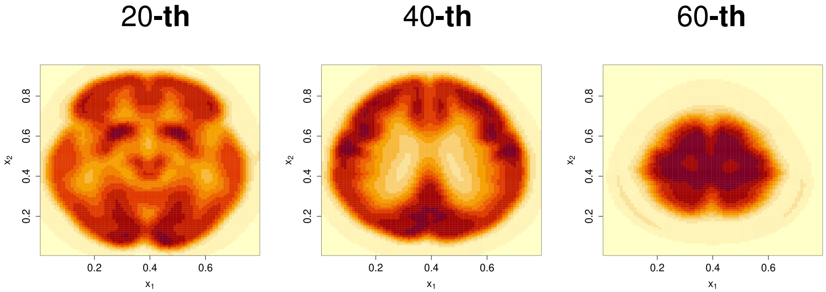
- 79 patients from the AD group.
  - 33 females
  - 46 males

ADNI

ADNI
ALZHEIMER'S DISEASE
NEUROIMAGING INITIATIVE

- 79 patients from the AD group.
  - 33 females
  - 46 males
- reoriented into $79 \times 95 \times 69$ voxels.
- each patient has 69 sliced 2D images with 79×95.

Recovery (79×95) from 3D scans



- Wang, S., Cao, G. and Shang, Z. (2020) Estimation of the Mean Function of Functional Data via Deep Neural Networks. *arxiv.org*

R Package

https://github.com/FDASTATAUBURN/FDADNN.

Assumptions

(A1) The true regression function f_0 has a composition structure.

Deep and wide neural networks

(A2) $\widehat{f} \in \mathcal{F}(L, \mathbf{p}, s)$, s.t.

- Depth: $L \asymp \log(nN^{\varrho})$, $\varrho \geq 0$;

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Assumptions

(A3) The maximal eigenvalue of the kernel matrix is $O(N^{-\varrho})$ for some constant $\varrho \geq 0$.