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Abstract

Tree approximation is a new form of nonlinear approximation which appears naturally in some applications such as image processing and adaptive numerical methods. It is somewhat more restrictive than the usual n-term approximation. We show that the restrictions of tree approximation cost little in terms of rates of approximation. We then use that result to design encoders for compression. These encoders are universal (they apply to general functions) and progressive (increasing accuracy is obtained by sending new bits). We show optimality of the encoders in the sense that they provide upper estimates for the Kolmogorov entropy of Besov balls.

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1 Introduction

Wavelets are utilized in many applications including image/signal processing and numerical methods for PDEs. Their usefulness stems in part from the fact that they provide efficient decompositions of functions into simple building blocks. For example, they provide unconditional bases, consisting of the shifted dilates of a finite number of functions, for many function spaces such as the L_p, H_p, Besov, and Triebel-Lizorkin spaces. The present paper is concerned with the following question: what is the most effective way to organize the terms in the wavelet decomposition of a function f? Of course the answer to this question depends on the potential application. We shall introduce a way of organizing the wavelet decomposition, by using tree-structures and certain ideas from nonlinear approximation, that is particularly well fitted to the application of data compression. This will result in an optimal encoding technique, which will be used to give simple proofs of upper estimates for the *Kolmogorov entropy* of Besov balls in L_p. The

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results from this paper have already been utilized in [5] for the design of encoders with optimal rate distortion performance with respect to deterministic and stochastic models for the signals.

1.1 Background and motivation

To describe the wavelet decompositions we have in mind, we introduce some notation which will be used throughout the paper. We let $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ denote the set of all dyadic cubes in $\mathbb{R}^d$, i.e. cubes of the type $2^{-j}(k + [0,1]^d-)$, with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, $\mathcal{D}_j := \mathcal{D}_j(\mathbb{R}^d)$ denote the set of dyadic cubes with sidelength 2^{-j}, and $\mathcal{D}_+ := \mathcal{D}_+(\mathbb{R}^d)$ the set of dyadic cubes with sidelength ≤ 1. We shall indicate the dependence of these sets on d only if there is a chance of confusion. If g is a function in $L_2(\mathbb{R}^d)$, and $I = 2^{-j}(k+[0,1]^d)$ is in $\mathcal{D}$, we define

$$g_I := g_{I,2} := 2^{jd/2} g(2^j \cdot -k).$$

Then, g_I is normalized in $L_2(\mathbb{R}^d)$: $\|g_I\|_{L_2(\mathbb{R}^d)} = \|g\|_{L_2(\mathbb{R}^d)}$, for each $I \in \mathcal{D}$. We shall also need normalizations in $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, given by

$$g_{I,p} := 2^{jd/p} g(2^j \cdot -k).$$

In order to explain our results in their simplest setting, we shall limit ourselves in this introduction to univariate decompositions using compactly supported orthogonal wavelets. Our results hold in much more generality and in fact will be developed in this paper in the multivariate setting for a general class of biorthogonal wavelets.

Let ψ be a univariate, compactly supported, orthogonal wavelet obtained from a compactly supported scaling function ϕ. Denoting by $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)dx$ the standard scalar product for $L_2(\mathbb{R})$, each function $f \in L_2(\mathbb{R})$ has the wavelet decomposition

$$f = \sum_{I \in \mathcal{D}_0} \langle f, \phi_I\rangle \phi_I + \sum_{I \in \mathcal{D}_+} a_I(f)\psi_I \tag{1.1}$$

with

$$a_I(f) := a_{I,2}(f) := \langle f, \psi_I\rangle.$$

The collection of functions Ψ appearing in (1.1) is an orthonormal basis for $L_2(\mathbb{R})$; it is also an unconditional basis for all of the $L_p(\mathbb{R})$ spaces, $1 < p < \infty$.

The most common way of organizing the decomposition (1.1) is

$$f = \sum_{I \in \mathcal{D}_0} \langle f, \phi_I\rangle \phi_I + \sum_{j=0}^{\infty} \sum_{I \in \mathcal{D}_j} a_I(f)\psi_I. \tag{1.2}$$

In other words, the terms are organized according to the *dyadic level* j (the frequency 2^j); low frequency terms appear first. This is analogous to the usual way of presenting Fourier decompositions. This organization of the wavelet series can be justified to a certain extent in that the membership of a function f in smoothness spaces (such as the Sobolev and Besov spaces) can be characterized by the decay of the wavelet coefficients of f with respect to frequency. Another justification comes from the viewpoint of approximation theory. Let V_n denote the linear space spanned by the functions $\phi(2^n \cdot -k)$, $k \in \mathbb{Z}$, or equivalently by the functions ϕ_I, $I \in \mathcal{D}_0$, and ψ_I, $I \in \mathcal{D}_j$, $j = 0, \ldots, n-1$. Then the partial sum

$$P_n(f) := \sum_{I \in \mathcal{D}_0} \langle f, \phi_I \rangle \phi_I + \sum_{j=0}^{n-1} \sum_{I \in \mathcal{D}_j} a_I(f) \psi_I$$

is the best $L_2(\mathbb{R})$ approximation to f from V_n. Also, it is a *near best approximation* to f in $L_p(\mathbb{R})$ whenever $1 < p < \infty$:

$$\|f - P_n(f)\|_{L_p(\mathbb{R})} \leq C_p \operatorname{dist}(f, V_n)_{L_p(\mathbb{R})}.$$

In most applications, we are dealing with functions defined on a domain Ω. In the univariate case, that we are now discussing, we shall take Ω to be a (possibly infinite) interval. In this case the wavelet decomposition (1.1) holds with $\mathcal{D}_0$ replaced by $\mathcal{D}_0(\Omega)$ and $\mathcal{D}_+$ replaced by $\mathcal{D}_+(\Omega)$ where the Ω indicates that we only take those I such that ϕ_I (respectively ψ_I) is not identically zero on Ω. It is also necessary to alter the definition of the coefficients $a_I(f)$ when the support of ψ_I intersects the boundary of Ω (see §2). If Ω is bounded, then the sets $\mathcal{D}_j(\Omega)$, $j \geq 0$, are finite.

A second way to organize the wavelet series comes forward when one considers the following problem of *nonlinear approximation*. For each positive integer $n \geq 1$, we let Σ_n denote the set of all functions which can be written as a linear combination of the scaling functions at level 0 together with at most n wavelets, i.e.,

$$S = \sum_{I \in \mathcal{D}_0(\Omega)} \langle f, \phi_I \rangle \phi_I + \sum_{I \in \Lambda} c_I \psi_I, \quad \#\Lambda \leq n,$$

where Λ is any subset of $\mathcal{D}_+(\Omega)$. (In order to simplify the presentation here in the introduction, we will not count the scaling functions appearing in the representation of S.) Note that Σ_n is *not* a linear space since, for example, the sum of two elements from Σ_n could require $2n$ wavelet terms in its representation. Approximation by the elements of Σ_n is called n-term approximation and is one of the simplest cases of nonlinear approximation. We define the error of n-term approximation in $L_p(\Omega)$ by

$$\sigma_n(f)_p := \inf_{S \in \Sigma_n} \|f - S\|_{L_p(\Omega)}. \tag{1.3}$$

In the case of approximation in $L_2(\mathbb{R})$, it is trivial to find best approximations to a function $f \in L_2(\mathbb{R})$ from Σ_n. Let $I_1 := I_1(f), I_2 := I_2(f), \ldots$, be a rearrangement of the intervals in $\mathcal{D}_+(\mathbb{R})$ so that

$$|a_{I_1}(f)| \geq |a_{I_2}(f)| \geq \cdots.$$

Then,

$$S_n := S_n(f) := \sum_{I \in \mathcal{D}_0(\mathbb{R})} \langle f, \phi_I \rangle \phi_I + \sum_{j=1}^{n} a_{I_j}(f) \psi_{I_j} \tag{1.4}$$

is a best n-term approximation to f and

$$\sigma_n(f)_2^2 = \sum_{j=n+1}^{\infty} |a_{I_j}(f)|^2. \tag{1.5}$$

It is remarkable that this same result is almost true when approximating in L_p. Now, we choose the intervals $I_j := I_j(f,p)$ such that

$$|a_{I_1,p}(f)| \geq |a_{I_2,p}(f)| \geq \dots$$

with $a_{I,p}(f) := |I|^{1/2-1/p} a_I(f)$ the L_p normalized coefficients. With this choice, Temlyakov [20] has shown that the corresponding approximant

$$S_{n,p} := \sum_{I \in \mathcal{D}_0(I\!\!R)} \langle f, \phi_I \rangle \phi_I + \sum_{j=1}^{n} a_{I_j,p}(f) \psi_{I_j,p}$$

is a *near best approximation* to f in L_p. Here, "near best" means that

$$\|f - S_{n,p}(f)\|_{L_p(I\!\!R)} \leq C_p \sigma_n(f)_p$$

with the constant C_p depending only on p.

We return now to the question of efficient decompositions of a function. If $f \in L_2(I\!\!R)$, we consider the following arrangement of the wavelet series:

$$f = \sum_{I \in \mathcal{D}_0(I\!\!R)} \langle f, \phi_I \rangle \phi_I + \sum_{j=1}^{\infty} a_{I_j}(f) \psi_{I_j}. \tag{1.6}$$

This arrangement of the terms is optimal in the sense that each partial sum is a best *n-term approximation*. Therefore, no other choice of n-terms could reduce the error more than this partial sum. Similarly, for functions in $L_p(I\!\!R)$, the arrangement

$$f = \sum_{I \in \mathcal{D}_0(I\!\!R)} \langle f, \phi_I \rangle \phi_I + \sum_{j=1}^{\infty} a_{I_j,p}(f) \psi_{I_j,p}. \tag{1.7}$$

is near optimal.

The decompositions (1.6), (1.7) have many impressive applications (see [11]). However, these decompositions do have the following deficiency. Consider the problem of encoding, where we want to transmit a finite number of bits which would allow the user to recover a good approximation to f with efficiency measured by the $L_p(I\!\!R)$ error between f and the approximant. A natural way to proceed would be to send a certain number of bits for each coefficient $a_{I_j,p}$ (how one might assign bits will be spelled out in § 4). However, for the receiver to reconstruct the approximant, he will also need to know the intervals I_j and their correspondence with the bitstream. To send this additional information may be very costly and in fact may dominate the total number of bits sent. The situation can be ameliorated by imposing more organization on the decomposition (1.7). We shall accomplish this by requiring that the intervals I appearing in the n-term approximation (1.4) be identified with nodes on a *tree*. This leads us to the concept of *tree approximation* which we now describe.

We consider the case when Ω is a finite interval. Any dyadic interval I has a parent and two children. We say that a collection of dyadic intervals $\mathcal{T} \subset \mathcal{D}_+(\Omega)$ is a tree if whenever $I \in \mathcal{T}$, with $|I| < 1$, then its parent is also in $\mathcal{T}$. The cubes $I \in \mathcal{T}$. with $|I| = 1$, are called the roots of the tree $\mathcal{T}$.

Given a positive integer n, we define Σ_n^t as the collection of all $S \in \Sigma_n$ such that

$$S = \sum_{I \in \mathcal{D}_0(\Omega)} c_I \phi_I + \sum_{I \in \mathcal{T}} c_I \psi_I, \quad \#\mathcal{T} \leq n, \tag{1.8}$$

with $\mathcal{T}$ a tree with cardinality $\leq n$. Given $f \in L_p(\Omega)$, the error in tree approximation is defined by

$$t_n(f)_p := \inf_{S \in \Sigma_n^t} \|f - S\|_{L_p(\Omega)}. \tag{1.9}$$

Then clearly, $\Sigma_n^t \subset \Sigma_n$ and $\sigma_n(f)_p \leq t_n(f)_p$. The question arises as to what is the cost in imposing the tree condition on the approximant. We shall show in § 4 that this cost is minimal in a certain sense which we now describe.

Let $\mathcal{A}^s := \mathcal{A}^s_\infty(L_p(\Omega))$ denote the class of functions $f \in L_p(\Omega)$ such that

$$\sigma_n(f)_p \leq Cn^{-s}, \quad n = 1, 2, \dots. \tag{1.10}$$

As we shall describe in more detail in § 3, it is possible to characterize $\mathcal{A}^s$ in terms of the wavelet coefficients of f. From this characterization, we can deduce the approximation properties of functions in the Besov spaces $B^s_q(L_\tau(\Omega))$. Recall that the functions in this Besov space have smoothness of order s in L_τ, in the sense that their modulus of smoothness $\omega_m(f,t)_{L_\tau} := \sup_{|h| \leq t} \|\Delta_h^m f\|_{L_\tau}$ behaves in $\mathcal{O}(t^s)$ for $m > s$ (where $\Delta_h^m f$ denotes the m-th order finite difference i.e. $\sum_{n=0}^m \binom{m}{n}(-1)^n f(\cdot - nh)$). The parameter q gives a fine tuning of smoothness: by definition, f is in $B^s_q(L_\tau(\Omega))$ if and only if $f \in L^p$ and the sequence $(2^{sj}\omega_m(f, 2^{-j})_{L_\tau})_{j \geq 0}$ is in ℓ^q. In particular for any non integer $s > 0$, $B^s_{\infty,\infty}$ is identical to the Hölder space C^s. Classical Sobolev spaces also fall in this class: $B^s_{p,p}$ is $W^{s,p}$ for all $s > 0$ if $p = 2$ and for all non integer $s > 0$ otherwise.

If $f \in B^s_q(L_\tau(\Omega))$, for some $0 < q \leq \infty$ and $\tau > (s + 1/p)^{-1}$, then $f \in \mathcal{A}^s$. This result is also true if $\tau = (s+1/p)^{-1}$ and $q \leq \tau$. In fact, in all these cases, n^{-s} is the best possible rate in the following sense. Let $\sigma_N(K)_p := \sup_{f \in K} \sigma_N(f)_p$ where K is any compact subset of $L_p(\Omega)$. Then, when U is the unit ball of $B^s_q(L_\tau(\Omega))$, one has $\sigma_N(U)_p \asymp n^{-s}$. Here $a \asymp b$ means that a can be bounded from above and below by some constant multiple of b uniformly with respect to any parameters on which a and b may depend.

This result has a simple geometrical interpretation given on Figure 1 below. We identify with each point (x, y) in the upper right quadrant of the plane the spaces $B^s_q(L_\tau(\Omega))$ with $x = 1/\tau$, $y = s$. Thus a given point has associated to it a family of spaces since we do not distinguish between different values of q. The line $1/\tau = s + 1/p$ is called the *critical line* for nonlinear approximation. Notice that it is also the critical line for the Sobolev embedding theorem. Each of the spaces corresponding to points to the left of the critical line is compactly embedded in $L_p(\Omega)$ (i.e. bounded sets are mapped into compact sets under the identity operator). Points on the critical line may or may not be embedded in $L_p(\Omega)$ depending on the value of q. To be precise, a space $B^s_q(L_\tau(\Omega))$ on the critical line is continuously embedded in $L_p(\Omega)$ if $q \leq p$ when $p < \infty$ and if $q \leq 1$ when $p = \infty$, see [13], p. 385. However, these embeddings are not compact.

We shall show in §5, that for each space $B^s_q(L_\tau(\Omega))$ strictly to the left of the critical line we have

$$t_n(f)_p \leq C(\tau, s)n^{-s}\|f\|_{B^s_q(L_\tau(\Omega))}. \tag{1.11}$$

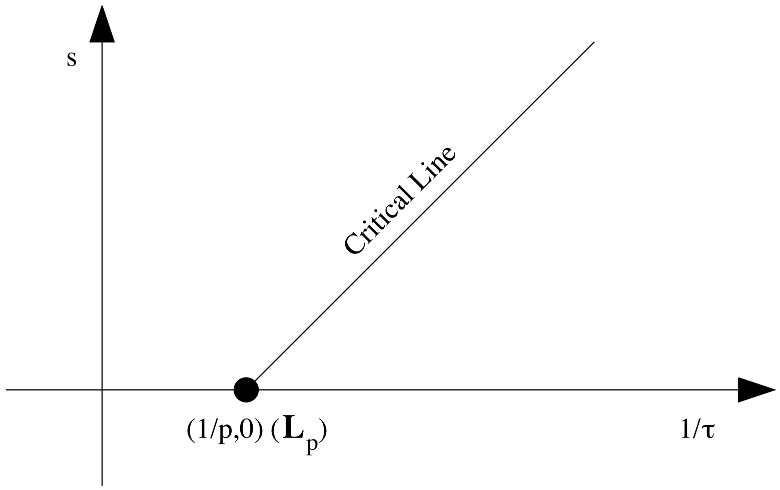

Figure 1.1: The critical line for nonlinear approximation in the case $d = 1$, $s = 1/\tau + 1/p$.

Thus, for these spaces, tree approximation has the same performance as n-term approximation. For spaces on the critical line this is no longer true. The construction of good tree approximations can be achieved quite simply from thresholding (see § 4).

1.2 Kolmogorov entropy

The inequality (1.11), and its extension to several space dimensions, has many interesting applications. We shall give two of them. The first one concerns the determination of the Kolmogorov entropy of function classes.

Let X be a metric space with distance function ρ. If $f \in X$ and $r > 0$, we let

$$\mathbf{B}(f,r) := \mathbf{B}(f,r)_X := \{g \in X : \rho(f,g) < r\}$$

denote the (open) ball of radius r about f. If $K \subset X$ is *compact*, then for each $\epsilon > 0$, there is a finite collection of balls $B(f_i, \epsilon)$, $i = 1, \ldots, n$, which cover K:

$$K \subset \bigcup_{i=1}^{n} \mathbf{B}(f_i, \epsilon).$$

The *covering number* $N_\epsilon(K) := N_\epsilon(K, X)$ is the smallest integer n for which there is such an ϵ-covering of K. The Kolmogorov *ϵ-entropy* of K is then by definition

$$H_\epsilon(K) := H_\epsilon(K, X) := \log N_\epsilon(K), \quad \epsilon > 0, \tag{1.12}$$

with log the logarithm to the base two.

We shall restrict our attention in this paper to the case $X = L_p(\Omega)$ where Ω is a Lipschitz domain in $\mathbb{R}^d$. We denote by $U_q^s(L_\tau(\Omega))$ the unit ball of the Besov space $B_q^s(L_\tau(\Omega))$. A fundamental result in approximation theory is the following.

Theorem 1.1 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ and let $1 \le p \le \infty$, and $s > d/\tau - d/p$. Then,*

$$H_\epsilon(U_q^s(L_\tau(\Omega)), L_p(\Omega)) \asymp \epsilon^{-d/s}, \tag{1.13}$$

with constants of equivalency depending only on s and $\delta := s - d/\tau + d/p$.

This theorem is well known when Ω is a cube in $I\!R^d$ (at least in the case $\tau \geq 1$). We shall be interested in the upper estimate in (1.13). The usual proofs of the upper estimate, see for example Chapter 13 of [17], utilize discretization and finite dimensional geometry. For $\tau \geq 1$ they can also be derived by interpolation arguments from the classical result of Birman and Solomjak on the entropy of Sobolev balls [3]. We shall give a new, more elementary proof of this upper estimate by relating Kolmogorov entropy to deterministic encoding.

To describe deterministic encoding, let $K \subset X$ again be a compact subset of $L_p(\Omega)$. An encoder for K consists of two mappings. The first is a mapping E from K into a set $\mathcal{B}$ of bitstreams. That is, the elements $B \in \mathcal{B}$ are sequences of zeros and ones. Thus, E assigns to each element $f \in K$ an element $E(f) \in \mathcal{B}$. The second mapping D associates to each $B \in \mathcal{B}$ an element $f_B \in L_p(\Omega)$. The mapping D decodes B. A codebook describes how a bitstream is converted to f_B.

Generally, $D(E(f)) \neq f$ and $\|f - D(E(f))\|_{L_p(\Omega)}$ measures the error that occurs in the encoding. The *distortion* of the encoding pair (D, E) on K is given by

$$d(K, E, D) := \sup_{f \in K} \|f - D(E(f))\|_{L_p(\Omega)}. \tag{1.14}$$

Given a compact set K of X, we let $\mathcal{B}_K := \{E(f) : \ f \in K\}$ denote the set of bitstreams $E(f)$ that arise in encoding the elements of K. We also let

$$M(K, E, D) := \max\{\#(E(f)) : f \in K\} \tag{1.15}$$

denote the largest length of the bitstreams that appear when E encodes the elements of K. The efficiency of encoding is measured by the distortion for a given bit allocation. Thus, the optimal distortion rate is given by

$$d_n(K) := \inf_{D,E} d(K, E, D) \tag{1.16}$$

where the infimum is taken over all encoding decoding pairs D, E for which the bit allocation $M(K, E, D)$ is $\leq n$.

It is easy to see that the rate distortion theory for optimal encoding is equivalent to determining the Kolmogorov entropy. Indeed, each ϵ covering $B(f_j, \epsilon)$, $j = 1, \dots, N$, of K gives an encoding pair E, D in an obvious way. For each f, the encoder E selects an integer $j \in \{1, \dots, N\}$ such that $\mathbf{B}(f_j, \epsilon)$ contains f and maps f into the binary digits of j. The decoder D maps a bitstream B into the element f_j where j is the integer with the bits B. This encoding pair has distortion $< \epsilon$. By taking a minimal ϵ-cover of K with $N_\epsilon(K)$ balls, we obtain an encoding of K with distortion $< \epsilon$ using at most $\lceil \log(N_\epsilon(K)) \rceil$ bits, i.e.

$$M(K, E, D) \leq \lceil H_\epsilon(K, X)) \rceil.$$

Conversely, given any encoding pair E, D with distortion $< \epsilon$, the balls $\mathbf{B}(f_B, \epsilon)$, $B \in \mathcal{B}$, give an ϵ cover of K. So any such pair satisfies

$$H_\epsilon(K) \leq M(K, E, D). \tag{1.17}$$

In this sense the two problems of constructing optimal encoders and estimating Kolmogorov entropy are equivalent.

However, in the practice of encoding, one is interested in realizing the mappings E and D by fast algorithms. This is not true in general for the encoder derived from an ϵ-covering since finding f_j might not be a trivial task. Thus a more relevant goal is to design a practical encoder and decoder such that the corresponding distortion $\tilde{d}_n(K)$ has at least the same asymptotic behaviour as the optimal $d_n(K)$ when n tends to $+\infty$. Moreover it might be desirable that this optimality is achieved not only for a specific K but for many possible classes with the same encoding. For example, we might require that the rate distortion performance is in accordance with Theorem 1.1 above when $K = U_q^s(L_\tau(\Omega))$ for various choices of s, τ and q. We refer to such an encoder as "*universal*".

1.3 Organization of material

The rest of this paper is organized as follows. In § 2, we introduce wavelet decompositions and recall how the coefficients in these decompositions can be used to describe the Besov spaces. In § 3, we recall some fundamental results in nonlinear approximation theory. In § 4, we introduce tree approximation and prove the results (1.11) in the multivariate case. In § 5 and § 6, we use the tree structure to build a universal encoder for any prescribed L_p metric with $1 \leq p < \infty$ used to measure the error. In §we employ this encoder in order to prove the upper estimates for encoding (Theorem 7.1) and Kolmogorov entropy (Corollary 7.2) for various Besov balls. Finally, in § 8 we derive analogous results for the case $p = \infty$.

It should be pointed out that wavelet decompositions have already been used in [2] and [15] for proving upper bounds of Kolmogorov entropies. In [15] the bounds are in the L_2 metric and involve a logarithmic factor since the classes which are considered are slightly different than the $U_q^s(L_\tau(\Omega))$. In [2], wavelets are used to prove upper bounds for Kolmogorov entropies by means of approximation procedures which are also universal with respect to the error metric L_p, $1 \leq p \leq \infty$. However, no explicit encoding strategies are presented there. The specificity of our approach is that the tree-structured encoding technique which is proved to be optimal is essentially close to practical algorithms such as given in [19] and [18] and that they are universal for a prescribed L_p metric. Note that some optimal or near optimal rate/distorsion bounds in the L^2 metric for wavelet-based encoders are also proved in [5] and [16]. Our approach also provides entropy bounds for the sets that precisely consist of those functions which can be approximated at a certain rate with a tree structure. These sets are larger than the Besov balls that are usually considered, and they are very natural in the context of image compression since they comply with the idea that the important coefficients produced by edges are naturally organized in a tree structure.

2 Wavelet decompositions and Besov classes

In this and the next section, we shall set forth the notation used throughout this paper, and we recall some known results on wavelet decompositions and nonlinear approximation which are related to the topics of this paper. We assume that the reader is familiar with the basics of wavelet theory (see [10]).

The results of this paper hold for quite general wavelet decompositions. However, we shall restrict oursleves to the compactly supported biorthogonal wavelets as introduced by Cohen, Daubechies, and Feauveau [4]. These are general enough to also include the orthogonal wavelets of compact support as introduced by Daubechies [9]. A good reference for these bases and their properties is Chapter 8 of the monograph of Daubechies [10].

The construction of biorthogonal wavelets begins with two compactly supported univariate scaling functions ϕ and $\tilde{\phi}$ whose shifts are in duality,

$$\int_{I\!R} \phi(x-k)\tilde{\phi}(x-k')\,dx = \delta(k-k'), \quad k,k' \in Z\!\!\!Z,$$

with δ the Kronecker delta. Associated to each of the scaling functions are mother wavelets ψ and $\tilde{\psi}$.

These functions can be used to generate a wavelet basis for the $L_p(I\!R^d)$ spaces as follows. We define $\psi^0 := \phi$, $\psi^1 := \psi$. Let V' denote the collection of vertices of the unit cube $[0,1]^d$ and V the nonzero vertices. For each vertex $v = (v_1, \dots, v_d) \in V'$, we define the multivariate function

$$\psi^v(x_1,\dots,x_d) := \psi^{v_1}(x_1)\cdots\psi^{v_d}(x_d), \ \tilde{\psi}^v(x_1,\dots,x_d) := \psi^{v_1}(x_1)\cdots\psi^{v_d}(x_d).$$

The collection of functions

$$\psi_I^v, \quad I \in \mathcal{D}, \ v \in V,$$

are a Riesz basis for $L_2(I\!R^d)$ (in the orthogonal case they form a complete orthonormal basis for $L_2(I\!R^d)$). They are an unconditional basis for $L_p(I\!R^d)$, $1 < p < \infty$. Each function f which is locally integrable on $I\!R^d$ has the wavelet expansion

$$f = \sum_{I\in\mathcal{D}} \sum_{v\in V} a_I^v(f)\psi_I^v, \quad a_I^v(f) := \langle f, \tilde{\psi}_I^v \rangle. \tag{2.1}$$

We can also start the wavelet decomposition at any dyadic level. For example, starting at dyadic level 0, we obtain

$$f = \sum_{I\in\mathcal{D}_0} \sum_{v\in V'} a_I^v(f)\psi_I^v + \sum_{j=1}^{\infty} \sum_{I\in\mathcal{D}_j} \sum_{v\in V} a_I^v(f)\psi_I^v. \tag{2.2}$$

It can be convenient, in the characterizations of Besov spaces, to choose different normalizations for the wavelets and coefficients appearing in the decompositions (2.1),(2.2). In (2.1), (2.2), we have normalized in $L_2(I\!R^d)$; we can also normalize in $L_p(I\!R^d)$, $0 < p \le \infty$, by taking

$$\psi_{I,p}^v := |I|^{-1/p+1/2}\psi_I^v, \quad I \in \mathcal{D}, \ v \in V. \tag{2.3}$$

Then, we can rewrite (2.2) as

$$f = \sum_{I\in\mathcal{D}_0} \sum_{v\in V'} a_{I,p}^v(f)\psi_{I,p}^v + \sum_{j=1}^{\infty} \sum_{I\in\mathcal{D}_j} \sum_{v\in V} a_{I,p}^v(f)\psi_{I,p}^v, \tag{2.4}$$

where

$$a_{I,p}^v(f) := \langle f, \tilde{\psi}_{I,p'}^v \rangle,$$

with $1/p + 1/p' = 1$.

For simplicity of notation, we shall combine all terms associated with a dyadic cube I in one expression:

$$A_I(f) := \begin{cases} \sum_{v\in V'} a_{I,p}^v(f)\psi_{I,p}^v, & I \in \mathcal{D}_0, \\ \sum_{v\in V} a_{I,p}^v(f)\psi_{I,p}^v, & I \in \mathcal{D}_j,\ j \geq 1. \end{cases} \tag{2.5}$$

Note that the definition of $A_I(f)$ does not depend on p and that

$$\|A_I(f)\|_{L_p(\mathbb{R}^d)} \asymp a_{I,p}(f) := \begin{cases} (\sum_{v\in V'} |a_{I,p}^v(f)|^p)^{1/p}, & I \in \mathcal{D}_0, \\ (\sum_{v\in V} |a_{I,p}^v(f)|^p)^{1/p}, & I \in \mathcal{D}_j,\ j \geq 1. \end{cases} \tag{2.6}$$

It is easy to go from one normalization to another. For example, for any $0 < p, q \leq \infty$, we have

$$\psi_{I,p}^v = |I|^{1/q-1/p}\psi_{I,q}^v, \quad a_{I,p}(f) = |I|^{1/p-1/q}a_{I,q}(f). \tag{2.7}$$

Note that we can compute the $L_p(\mathbb{R}^d)$ norms of single scale wavelet sums $S_j(f) := \sum_{I\in\mathcal{D}_j} A_I(f)$ from a fixed dyadic level j. Namely,

$$\|S_j(f)\|_{L_p(\mathbb{R}^d)} \asymp \|(a_{I,p}(f))_{I\in\mathcal{D}_j}\|_{\ell_p}, \quad 0 < p \leq \infty, \tag{2.8}$$

with the constants of equivalency depending only on p when p is small.

Many function spaces can be described by wavelet coefficients. In particular, such characterizations hold for the Besov spaces $B_q^s(L_\tau(\mathbb{R}^d))$, $s > 0$, $0 < \tau, q \leq \infty$. We shall only need the case when $B_q^s(L_\tau(\mathbb{R}^d))$ is compactly embedded in $L_1(\mathbb{R}^d)$ which means that $s > d/\tau - d$. We choose a univariate biorthogonal wavelet pair such that ψ has smoothness C^r, and $\tilde{\psi}$ has at least r vanishing moments with $r > s$. The Besov space $B_q^s(L_\tau(\mathbb{R}^d))$ can then be defined as the set of all functions f that are locally in $L_1(\mathbb{R}^d)$ and for which

$$\|f\|_{B_q^s(L_\tau(\mathbb{R}^d))} := \begin{cases} \left(\sum_{j=0}^\infty 2^{jsq}\left(\sum_{I\in\mathcal{D}_j} a_{I,\tau}(f)^\tau\right)^{q/\tau}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j\geq 0} 2^{js}\left(\sum_{I\in\mathcal{D}_j} a_{I,\tau}(f)^\tau\right)^{1/\tau}, & q = \infty, \end{cases} \tag{2.9}$$

is finite and with the usual change if $\tau = \infty$. The quasi-norm in (2.9) is equivalent to the other quasi-norms used to define Besov spaces in terms of moduli of smoothness or Fourier transforms.

In most applications, the functions of interest are not defined on $\mathbb{R}^d$ but rather on a bounded domain $\Omega \subset \mathbb{R}^d$. We shall assume that Ω is a Lipschitz domain (for a definition see e.g. Adams [1]). The Besov spaces $B_q^s(L_\tau(\Omega))$ for such domains are usually defined by moduli of smoothness but they can also be described by wavelet decompositions similar to (2.9).

To see this, we use the fact that any such function f has an extension $\mathcal{E}f$ to all of $\mathbb{R}^d$ which satisfies (see [14])

$$\|\mathcal{E}f\|_{B_q^s(L_\tau(\mathbb{R}^d))} \leq C\|f\|_{B_q^s(L_\tau(\Omega))},$$

with the constant C independent of f. In going further, we simply denote $\mathcal{E}f$ by f.

Since we now have a function f defined on all of $\mathbb{R}^d$, we can apply the characterization (2.9). In (2.9), we only have to include those ψ_I^v which do not vanish identically on Ω. For $j = 0, 1, \ldots$, we denote by $\mathcal{D}_j(\Omega)$ the collection of all dyadic cubes $I \in \mathcal{D}_j$ such that, for some $v \in V'$, ψ_I^v does not vanish identically on Ω. We further set $\mathcal{D}_+(\Omega) := \cup_{j \geq 0} \mathcal{D}_j(\Omega)$.

In analogy with (2.9), we have the following quasi-norm for the Besov space $B_q^s(L_\tau(\Omega))$:

$$\|f\|_{B_q^s(L_\tau(\Omega))} := \begin{cases} \left(\sum_{j=0}^{\infty} 2^{jsq} \|(a_{I,\tau}(f))_{I \in \mathcal{D}_j(\Omega)}\|_{\ell_\tau}^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j \geq 0} 2^{js} \|(a_{I,\tau}(f))_{I \in \mathcal{D}_j(\Omega)}\|_{\ell_\tau}, & q = \infty. \end{cases} \tag{2.10}$$

For simple domains (e.g. polyhedra, or piecewise smooth domains) one can also directly construct wavelet bases on Ω that satisfy (2.10).

We close this section with the following observation.

Remark 2.1 *Let $\tau > (s/d + 1/p)^{-1}$. Then the unit ball $U_q^s(L_\tau(\Omega))$ of $B_q^s(L_\tau(\Omega))$, is a compact subset of $L_p(\Omega)$.*

Proof: We define $\mu := \min(p, \tau)$ and introduce the *discrepancy*

$$\delta := s - \frac{d}{\mu} + \frac{d}{p} > 0. \tag{2.11}$$

Then, it follows from (2.10), (2.7) and (2.8) that for each $j = 0, 1, \ldots$,

$$\begin{aligned} 2^{j\delta} \|\sum_{I \in \mathcal{D}_j(\Omega)} A_I(f)\|_{L_p} &\asymp 2^{j\delta} \left(\sum_{I \in \mathcal{D}_j(\Omega)} a_{I,p}(f)^p\right)^{1/p} \leq 2^{j\delta} \left(\sum_{I \in \mathcal{D}_j(\Omega)} a_{I,p}(f)^\mu\right)^{1/\mu} \\ &= 2^{j\delta} \left(\sum_{I \in \mathcal{D}_j(\Omega)} 2^{-d(\frac{\mu}{p}-1)j} a_{I,\mu}(f)^\mu\right)^{1/\mu} \\ &= 2^{sj} \left(\sum_{I \in \mathcal{D}_j(\Omega)} a_{I,\mu}(f)^\mu\right)^{1/\mu} \\ &\leq \|f\|_{B_\infty^s(L_\mu(\Omega))} \leq \|f\|_{B_q^s(L_\mu(\Omega))} \leq \|f\|_{B_q^s(L_\tau(\Omega))}, \end{aligned} \tag{2.12}$$

for any $0 < q \leq \infty$. This implies the estimate

$$\|f - \sum_{j=0}^{J} \sum_{I \in \mathcal{D}_j(\Omega)} A_I(f)\|_{L_p} \leq \sum_{j > J} \|\sum_{I \in \mathcal{D}_j(\Omega)} A_I(f)\|_{L_p} \leq C 2^{-J\delta} \|f\|_{B_q^s(L_\tau(\Omega))}, \tag{2.13}$$

which shows the compactness of $U_q^s(L_\tau(\Omega))$. □

3 Nonlinear approximation

In this section, we shall recall some facts about nonlinear approximation that will serve as an orientation for the results on tree approximation presented in the next section. A general reference for the results of this section is [11]. We fix a domain Ω and consider the approximation of functions in $L_p(\Omega)$. We begin by describing what is known as *n term approximation.*

Let Σ_n be defined as the set of all functions S that satisfy the condition

$$S = \sum_{I\in\Lambda} A_I(S), \quad \#\Lambda \le n, \tag{3.1}$$

with A_I defined as in (2.5). We shall consider approximation in the space $L_p(\Omega)$ by the elements of Σ_n. Given $f \in L_p(\Omega)$, $1 \le p \le \infty$, we define (as in (1.3))

$$\sigma_n(f)_p := \inf_{S\in\Sigma_n} \|f - S\|_{L_p(\Omega)}, \quad n = 0, 1. \dots . \tag{3.2}$$

Note that by definition $\sigma_0(f)_p := \|f\|_{L_p(\Omega)}$.

We can describe the functions f for which $\sigma_n(f)_p$ has a prescribed asymptotic behavior as $n \to \infty$. For $1 \le p \le \infty$, $0 < q \le \infty$, and $s > 0$, we define the approximation class $\mathcal{A}^s_q(L_p(\Omega))$ to be the set of all $f \in L_p(\Omega)$ such that

$$\|f\|_{\mathcal{A}^s_q(L_p(\Omega))} := \begin{cases} \left(\sum_{n=0}^\infty [(n+1)^s \sigma_n(f)_p]^q \frac{1}{n+1}\right)^{1/q}, & 0 < q < \infty \\ \sup_{n\ge0}(n+1)^s\sigma_n(f)_p, & q = \infty, \end{cases} \tag{3.3}$$

is finite. From the monotonicity of $\sigma_n(f)_p$, it follow that (3.3) is equivalent to

$$\|f\|_{\mathcal{A}^s_q(L_p(\Omega))} \asymp \begin{cases} \left(\sum_{j\ge-1} [2^{js}\sigma_{2^j}(f)_p]^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j\ge-1} 2^{js}\sigma_{2^j}(f)_p, & q = \infty, \end{cases} \tag{3.4}$$

where for the purposes of this formula we define $\sigma_{1/2}(f)_p = \sigma_0(f)_p$.

It is possible to characterize the spaces $\mathcal{A}^s_q(L_p(\Omega))$ in several ways: in terms of interpolation spaces; in terms of wavelet coefficients; and in terms of smoothness spaces (Besov spaces). For spaces X, Y we denote by $(X, Y)_{\theta,q}$ the interpolation spaces generated by the real method of interpolation (K-functional) with parameters $0 < \theta < 1$, $0 < q \le \infty$. We shall denote by $\ell_{\mu,q}$ the Lorentz space of sequences $(c_I)_{I\in\mathcal{D}_+}$ indexed on dyadic intervals (for a definition of this space see [13] or [11]).

Theorem 3.1 ([12] *and* [6]) *Let* $1 < p < \infty$ *and let* $\psi, \tilde\psi$ *be a biorthogonal wavelet pair where* ψ *has smoothness of order* r *and* $\tilde\psi$ *has at least* r *vanishing moments. Then, the following characterizations of* $\mathcal{A}^s_q(L_p(\Omega))$ *hold:*
(i) *For each* $0 < s < r$, *we have that a function* f *is in* $A^{s/d}_q(L_p(\Omega))$ *if and only if the sequence* $(a_{I,p}(f))_{I\in\mathcal{D}_+(\Omega)}$ *defined by* (2.6) *is in the Lorentz sequence space* $\ell_{\tau,q}$ *with* $1/\tau := s/d + 1/p$ *and*

$$\|f\|_{\mathcal{A}^{s/d}_q(L_p(\Omega))} \asymp \|f\|_{L_p(\Omega)} + \|(a_{I,p}(f))_{I\in\mathcal{D}_+(\Omega)}\|_{\ell_{\tau,q}}. \tag{3.5}$$

(ii) *For each* $0 < s < r$ *and* $0 < q \le \infty$,

$$\mathcal{A}^{s/d}_q(L_p(\Omega)) = (L_p(\Omega), B^r_\mu(L_\mu(\Omega)))_{s/r,q}, \tag{3.6}$$

with equivalent norms, where $1/\mu = r/d + 1/p$,
(iii) *In the special case* $0 < s < r$ *and* $q = \tau = (s/d + 1/p)^{-1}$. *we have*

$$\mathcal{A}^{s/d}_\tau(L_p(\Omega)) = B^s_\tau(L_\tau(\Omega)). \tag{3.7}$$

where $1/\tau = s/d + 1/p$.

Thus, we have in (i) a characterization of the approximation spaces in terms of the decay of the wavelet coefficients, while in (ii) this space is characterized in terms of interpolation between Besov spaces. Characterization (iii) shows that in the special case $q = \tau = (s/d + 1/p)^{-1}$ the approximation space is identical with a Besov space.

As we already mentioned, there is a simple and useful geometrical interpretation of this theorem given in Figure 1: Theorem 3.1 says that the approximation space $\mathcal{A}^{s/d}_\tau(L_p)$ corresponds to the point $(1/\tau, s)$ of smoothness s on the critical line for nonlinear approximation in $L_p(\Omega)$, or equivalently on the line segment connecting $(1/p, 0)$ (corresponding to $L_p(\Omega)$) to $(1/\mu, r)$ (corresponding to $B^r_\mu(L_\mu(\Omega))$).

For any compact subset K of $L_p(\Omega)$, we let

$$\sigma_n(K)_p := \sup_{f\in K} \sigma_n(f)_p \tag{3.8}$$

be the error of n-term approximation for this class. The main inference we wish to retain from Theorem 3.1 is the following. For any point $(1/\tau, s)$, $0 < s < r$, lying above the critical line of nonlinear approximation and any of the Besov classes $B^s_q(L_\tau(\Omega))$ and its unit ball $U^s_q(L_\tau(\Omega))$, we have

$$\sigma_n(U^s_q(L_\tau(\Omega)) \le Cn^{-s/d}, \quad n = 1, 2, \dots, \tag{3.9}$$

with the constant C independent of n. This inequality also holds for the Besov spaces on the critical line provided $q \le (s/d + 1/p)^{-1}$.

We have already noted in the introduction that a way of constructing near best n-term approximations to a function $f \in L_p(\Omega)$ is to retain the n terms in the wavelet expansion of f which have the largest $L_p(\Omega)$ norms. We shall not formulate this result explicitly (see [11]) since we shall not need it. However, we shall need the following closely related result of Temlyakov (see [20] or [11]).

Theorem 3.2 *Let $1 \le p < \infty$ and let $\Lambda \subset \mathcal{D}_+(\Omega)$ be any finite set. If S is a function of the form $S = \sum_{I\in\Lambda} A_{I,p}(S)$, then*

$$\|S\|_{L_p(\Omega)} \le C_p \max_{I\in\Lambda} a_{I,p}(S)(\#\Lambda)^{1/p} \tag{3.10}$$

with the constant C_p depending only on p. Also, for any such set Λ and any $1 < p \le \infty$, we have

$$C'_p \min_{I\in\Lambda} a_{I,p}(S)(\#\Lambda)^{1/p} \le \|S\|_{L_p(\Omega)} \tag{3.11}$$

where again the constant depends only on p.

Note that in the case $p = 1$, the inequality (3.10) (as stated and proved in [6]) is given for the Hardy space H_1 in place of L_1 but then (3.10) follows because the L_1 norm can be bounded by the H_1 norm. Similarly, (3.11) holds for the space BMO from which one derives the case $p = \infty$.

4 Tree approximation

We turn now to the main topic of this paper which is tree approximation. Dyadic cubes I in $\mathbb{R}^d$ have one parent (the smallest dyadic cube which properly contains I) and 2^d children (the largest dyadic cubes strictly contained in I). By a tree $\mathcal{T}$ we shall mean a set of dyadic cubes from $\mathcal{D}_+(\Omega)$ with the property: if $|I| < 1$ and $I \in \mathcal{T}$, then its parent is also in $\mathcal{T}$. We denote by Σ_n^t the collection of all functions that satisfy

$$S = \sum_{I \in \mathcal{T}} A_I(S), \quad \#\mathcal{T} \leq n, \tag{4.1}$$

with $\mathcal{T}$ a tree. If $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, then we recall (1.9) and define the error of tree approximation by

$$t_n(f)_p := \inf_{S \in \Sigma_n^t} \|f - S\|_{L_p(\Omega)}. \tag{4.2}$$

More generally for a compact subset $K \subset L_p(\Omega)$, we set

$$t_n(K)_p := \sup_{f \in K} t_n(f)_p. \tag{4.3}$$

4.1 The case $p < \infty$

In the following we fix the L_p metric in which the error is measured and we shall assume first that $p < \infty$. There is a simple and constructive way of generating tree approximations of a given function $f \in L_p(\Omega)$ by thresholding its wavelet coefficients. For each $\eta > 0$, we let

$$\Lambda(f, \eta) := \{I \in \mathcal{D}_+(\Omega) : a_{I,p}(f) \geq \eta\}. \tag{4.4}$$

Defining now $\mathcal{T}(f, \eta)$ as the smallest tree containing $\Lambda(f, \eta)$ we note that

$$\mathcal{T}(f, \eta) \subseteq \mathcal{T}(f, \eta'), \quad \eta' \leq \eta, \tag{4.5}$$

and that these sets depend on p. With each tree $\mathcal{T}(f, \eta)$ we associate now the approximant

$$S(f, \eta) := \sum_{I \in \mathcal{T}(f,\eta)} A_I(f). \tag{4.6}$$

It will be convenient to associate a family of spaces with this construction by bounding the cardinality of the trees $\mathcal{T}(f, \eta)$ in terms of the threshold η. To this end, note first that for $1 < p \leq \infty$ one has by the inequality (3.11) in Theorem 3.2

$$C_p'\eta\left(\#(\mathcal{T}(f, \eta))\right)^{1/p} \leq \|S(f, \eta)\|_{L_p(\Omega)}.$$

Thus defining $\mathcal{B}_\lambda(L_p(\Omega))$ as the set of those $f \in L_p(\Omega)$ for which there exists a constant $C(f)$ such that

$$\#(\mathcal{T}(f, \eta)) \leq C(f)\eta^{-\lambda}, \tag{4.7}$$

for all $\eta > 0$, one obtains a strict subset of $L_p(\Omega)$ only when $0 < \lambda < p$. It is easy to see that $\mathcal{B}_\lambda(L_p(\Omega))$ constitutes indeed a linear space: if f and g are in $\mathcal{B}_\lambda(L_p(\Omega))$, we simply remark that $\mathcal{T}(f + g, \eta) \subset \mathcal{T}(f, \eta/2) \cup \mathcal{T}(g, \eta/2)$ so that we have $\#(\mathcal{T}(f + g, \eta)) \leq \#(\mathcal{T}(f, \eta/2)) + \#(\mathcal{T}(g, \eta/2))$. Moreover, we define a quasinorm $\|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} := C^*(f)^{1/\lambda}$ where $C^*(f)$ is the smallest constant such that (4.7) holds.

The next theorem will examine the approximation properties of $S(f, \eta)$.

Theorem 4.1 *Let $1 \le p < \infty$, and $0 < \lambda < p$. Then we have*

$$\|f - S(f,\eta)\|_{L_p(\Omega)} \le c_1 \|f\|_{\mathcal{B}_\lambda(L_p(\Omega))}^{\lambda/p} \eta^{1-\lambda/p}, \tag{4.8}$$

where c_1 depends only on λ if λ is close to p. Moreover, let $(1/\tau, s)$ be a point above the critical line for nonlinear approximation in L_p, i.e., s and τ should satisfy $s > d/\tau - d/p$. Then, if $0 < q \le \infty$, $B_q^s(L_\tau(\Omega))$ is continuously embedded in $\mathcal{B}_\lambda(L_p(\Omega))$ with $\lambda := \frac{d}{s+d/p}$:

$$\|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} \le c_2 \|f\|_{B_q^s(L_\tau(\Omega))}, \tag{4.9}$$

with c_2 depending only δ when δ is close to zero and on the size of the support of φ and ψ.

Proof: Let $f \in \mathcal{B}_\lambda(L_p(\Omega))$ and let $M := \|f\|_{\mathcal{B}_\lambda(L_p(\Omega))}$. To prove (4.8), we note that for each cube I not in $\mathcal{T}(f, 2^{-j}\eta)$, we have $a_{I,p}(f) \le 2^{-j}\eta$. Let

$$\Sigma_j := \sum_{I \in \mathcal{T}(f, 2^{-j-1}\eta) \setminus \mathcal{T}(f, 2^{-j}\eta)} A_I(f).$$

Then, using (3.10) and (4.7), we deduce that

$$\|\Sigma_j\|_{L_p(\Omega)} \le C 2^{-j}\eta [\#(\mathcal{T}(f, 2^{-j-1}\eta))]^{1/p} \le C 2^{-j}\eta [M^\lambda 2^{j\lambda} \eta^{-\lambda}]^{1/p}. \tag{4.10}$$

Therefore,

$$\|f - S(f,\eta)\|_{L_p(\Omega)} \le \sum_{j=0}^{\infty} \|\Sigma_j\|_{L_p(\Omega)} \le C M^{\lambda/p} \eta^{1-\lambda/p} \sum_{j=0}^{\infty} 2^{-j(1-\lambda/p)} \le C M^{\lambda/p} \eta^{1-\lambda/p}, \tag{4.11}$$

where we have used that $\lambda < p$.

In order to prove (4.9), we also define $\Lambda_j(f,\eta) := \Lambda(f,\eta) \cap \mathcal{D}_j(\Omega)$. For $f \in B_q^s(L_\tau(\Omega))$, let $\tilde{M} = \|f\|_{B_q^s(L_\tau(\Omega))}$. Then the estimates (2.12) at the end of § 2 provide

$$\#(\Lambda_j(f,\eta))\eta^\tau \le \sum_{I \in \mathcal{D}_j(\Omega))} a_{I,p}(f)^\tau \le \tilde{M}^\tau 2^{-j\delta\tau}, \tag{4.12}$$

where δ is defined by (2.11). In order to exploit this estimate for bounding the cardinality of $\mathcal{T}(f,\eta)$ we define $\mathcal{T}_j(f,\eta) := \mathcal{T}(f,\eta) \cap \mathcal{D}_j(\Omega)$. Note next that a cube I is in $\mathcal{T}_j(f,\eta)$, $j \ge 0$, if and only if there is a cube $I' \subseteq I$ such that $I' \in \Lambda(f,\eta)$. In fact, when $I \notin \Lambda_j(f,\eta)$ then I must be an ancestor of some $I' \in \Lambda(f,\eta)$. Since any such cube I' belongs to at most one cube $I \in \mathcal{T}_j(f,\eta)$, we have, from (4.12),

$$\#\mathcal{T}_j(f,\eta) \le C \min(2^{jd}, \tilde{M}^\tau \sum_{k \ge j} 2^{-k\delta\tau}\eta^{-\tau}) \le C \min(2^{jd}, \tilde{M}^\tau 2^{-j\delta\tau}\eta^{-\tau}). \tag{4.13}$$

Here we have used the fact that $\mathcal{D}_j(\Omega)$ contains at most $C2^{jd}$ cubes because the wavelets have compact support and Ω is a bounded domain. In order to sum (4.13) over all $j \ge 0$, we observe first that the turnover level J, i.e., the smallest integer for which

$$2^{Jd} \ge \tilde{M}^\tau \sum_{k \ge J} 2^{-k\delta\tau} \eta^{-\tau}, \tag{4.14}$$

is given by $J = \left\lceil \frac{\lambda}{d} \log_2 \left(\frac{\tilde{M}}{\eta} \right) \right\rceil_+$. Thus the sum of (4.13) over all $j \geq 0$ is bounded by $C \left(\left(\frac{\tilde{M}}{\eta} \right)^\lambda + \left(\frac{\tilde{M}}{\eta} \right)^\tau \sum_{j=J} 2^{-j\delta\tau} \right)$ which yields

$$\#\mathcal{T}(f, \eta) \leq C\tilde{M}^\lambda \eta^{-\lambda}. \tag{4.15}$$

This establishes $M \leq C\tilde{M}$, confirming (4.9). □

We can use Theorem 4.1 to estimate the error of tree approximation.

Corollary 4.2 *Let $1 \leq p < \infty$ and $0 < \lambda < p$. Then, for each $n = 1, 2, \ldots$ and each $f \in \mathcal{B}_\lambda(L_p(\Omega))$, we have*

$$t_n(f)_p \leq c_1 \|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} n^{-(1/\lambda - 1/p)}, \tag{4.16}$$

with c_1 the constant in (4.8). *Moreover, let $(1/\tau, s)$ be a point above the critical line for nonlinear approximation in L_p, i.e., s and τ should satisfy $s > d/\tau - d/p$. Then, if $0 < q \leq \infty$, for each $n = 1, 2, \ldots$ and each $f \in B^s_q(L_\tau(\Omega))$, we have*

$$t_n(f)_p \leq c_3 \|f\|_{B^s_q(L_\tau(\Omega)} n^{-s/d}, \tag{4.17}$$

where $c_3 := c_1 c_2$ and c_1, c_2 are the constants in (4.8), (4.9), *respectively.*

Proof: Let $f \in \mathcal{B}_\lambda(L_p(\Omega))$. Given a positive integer n, we take η such that

$$\|f\|^\lambda_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{-\lambda} = n. \tag{4.18}$$

Then $S(f, \eta) \in \Sigma^t_n$, and (4.8) yields

$$t_n(f)_p \leq \|f - S(f, \eta)\|_{L_p(\Omega)} \leq c_1 \|f\|^{\lambda/p}_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{1-\lambda/p}.$$

Writing

$$\|f\|^{\lambda/p}_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{1-\lambda/p} = \|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} \left(\|f\|^\lambda_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{-\lambda} \right)^{\frac{1}{p} - \frac{1}{\lambda}}, \tag{4.19}$$

provides, in view of (4.18), the first estimate (4.16). Using (4.9) to bound the first factor on the right hand side of (4.19) yields again, on account of (4.18)

$$t_n(f)_p \leq c_1 c_2 \|f\|_{B^s_q(L_\tau(\Omega))} n^{-(1/\lambda - 1/p)}. \tag{4.20}$$

The second estimate (4.17) follows now because $\lambda := \frac{d}{s+d/p}$ means $1/\lambda - 1/p = s/d$. □

4.2 The case $p = \infty$

The constants in the estimates (4.10) and (4.11) in the proof of the above Theorem 4.1 depend on p as p tends to infinity (because the inequality (3.10) in Theorem 3.2 was used) and on $p - \lambda$ when λ tends to p. Consequently the constants c_1, c_2 deteriorate as $p - \lambda$ (resp. $\delta = s - d/\tau + d/p$) tends to zero and also when p increases to $+\infty$. In fact, Theorem 4.1 and Corollary 4.2 do not hold for $p = \infty$. To tackle the $p = \infty$ case, we have to overcome several technical difficulties. We have to compensate for the fact that we can no longer resort to Temlyakov's inequality. The main idea is to change the definition of $\Lambda(f, \eta)$ by introducing a dependence of the threshold on the level of scale. Moreover, recall from (4.13) that complete coarse scales up to a certain turnover level J can be included without spoiling the complexity of the trees. The reader who is not interested in these details should skip the rest of this section.

For f a function in $L_\infty(\Omega)$ and $\eta > 0$, we define, in analogy with (4.14), J to be the smallest positive integer such that

$$2^{Jd} \geq \sum_{j>J} \sum_{l \geq j} \#(\Lambda_l(f, \eta(l-J)^{-2})). \tag{4.21}$$

Whenever such a J exists, we define a modified set $\tilde{\Lambda}(f, \eta)$ by

$$\tilde{\Lambda}(f, \eta) := \{I \in \mathcal{D}_j(\Omega) \ ; \ j \leq J\} \cup \{I \in \Lambda_j(f, \eta(j-J)^{-2}) \ ; \ j > J\}; \tag{4.22}$$

If there is no J for which (4.21) holds, we simply set $\tilde{\Lambda}(f, \eta) := \mathcal{D}(\Omega)$. $\tilde{\Lambda}(f, \eta)$ thus includes all cubes with scales $j \leq J$, as well as those with scales j larger than J for which $|a_{I,\infty}(f)| \geq \eta(j-J)^{-2}$. As before we define $\tilde{\mathcal{T}}(f, \eta)$ as the smallest tree containing $\tilde{\Lambda}(f, \eta)$ and set

$$\tilde{S}(f, \eta) := \sum_{I \in \tilde{\mathcal{T}}(f,\eta)} A_I(f). \tag{4.23}$$

This construction gives again rise to a class of functions in $L_\infty(\Omega)$ as follows: for $0 < \lambda < \infty$, we say that f is in $\tilde{\mathcal{B}}_\lambda(L_\infty(\Omega))$ if and only if there exists a constant $C(f)$ such that

$$\#(\tilde{\mathcal{T}}(f, \eta)) \leq C(f) \eta^{-\lambda}, \tag{4.24}$$

for all $\eta > 0$. For the sake of convenience, we introduce a succinct notation for the r.h.s. of (4.21) in the case where $\tilde{\mathcal{T}}(f, \eta)$ is finite:

$$R(f, \eta, J) := \sum_{j>J} \sum_{l \geq j} \#(\Lambda_l(f, \eta(l-J)^{-2})). \tag{4.25}$$

We will make use of the following simple facts about the quantities $R(f, \eta, J)$.

Remark 4.3 *If* (4.25) *holds then for fixed f and η, $R(f, \eta, J)$ decreases as J increases. Moreover, for fixed f and J the quantities $R(f, \eta, J)$ increase as η decreases.*

The first statement in Remark 4.3 says that (4.25) already implies the existence of a turnover scale J satisfying (4.21) and hence the finiteness of $\tilde{\mathcal{T}}(f, \eta)$ so that $\tilde{S}(f, \eta)$ is well-defined. Moreover, (3.11) together with (2.12) imply a geometric decay of the quantities $a_{I,\infty}(f)$ in scale as soon as f belongs to any space $B^s_q(L_\tau(\Omega))$ above the critical line, i.e., $\frac{1}{\tau} < \frac{s}{d}$. Therefore for any $\eta > 0$ the set $\Lambda_l(f, \eta(l-J)^{-2})$ will be empty for l sufficiently large, so that (4.25) will indeed hold anywhere above the critical line for approximation in $L_\infty(\Omega)$.

The second statement in Remark 4.3 implies that the turnover scale J increases when η decreases.

We can now formulate the following adaptation of Theorem 4.1.

Theorem 4.4 *Assuming that (4.25) holds we have*

$$\|f - \tilde{S}(f,\eta)\|_{L_\infty(\Omega)} \le \tilde{c}_1 \eta, \tag{4.26}$$

where $\tilde{c}_1$ depends only on the support of φ and ψ. Moreover, let $(1/\tau, s)$ be a point above the critical line for nonlinear approximation in $L_\infty(\Omega)$, i.e., s and τ should satisfy $s > d/\tau$. Then if $0 < q \le \infty$, $B^s_q(L_\tau(\Omega))$ is embedded in $\mathcal{B}_\lambda(L_\infty(\Omega))$, with $\lambda := d/s$, in the sense that any f in $B^s_q(L_\tau(\Omega))$ satifies (4.24) with

$$C(f) \le \tilde{c}_2 \|f\|^\lambda_{B^s_q(L_\tau(\Omega))}, \tag{4.27}$$

where $\tilde{c}_2$ depends only on $s - d/\tau$ when this quantity becomes close to zero.

Proof: The error estimate (4.26) is immediate since we have

$$\begin{aligned}
\|f - \tilde{S}(f,\eta)\|_{L_\infty(\Omega)} &\le \|\sum_{I \notin \tilde{\Lambda}(f,\eta)} A_I(f)\|_{L_\infty(\Omega)} \\
&\le \sum_{j>J} \|\sum_{I \in \mathcal{D}_j(\Omega) \setminus \Lambda_j(f,\eta(j-J)^{-2})} A_I(f)\|_{L_\infty(\Omega)} \\
&\le \eta \sum_{j>J} (j-J)^{-2} \|\sum_{I \in \mathcal{D}_j(\Omega)} \psi_{I,\infty}\|_{L_\infty(\Omega)} = \tilde{c}_1 \eta.
\end{aligned}$$

In order to prove (4.27), we follow the same reasoning as in the proof of Theorem 4.1 (see (4.13), (4.15)) and remark first that

$$\#(\tilde{\mathcal{T}}(f,\eta)) \le 2^{Jd} + \sum_{j>J} \sum_{l \ge j} \#(\Lambda_l(f,\eta(l-J)^{-2})) \le 2^{Jd+1}. \tag{4.28}$$

On the other hand, for $f \in B^s_q(L_\tau(\Omega))$, with $s > d/\tau$, let $\delta = s - d/\tau$ and $\tilde{M} = \|f\|_{B^s_q(L_\tau(\Omega))}$. We then have by the definition of J in (4.21) and by (4.12)

$$\begin{aligned}
2^{(J-1)d} &\le \sum_{j \ge J} \sum_{l \ge j} \#(\Lambda_l(f,\eta(l-J+1)^{-2})) \\
&\le \sum_{j \ge J} \sum_{l \ge j} 2^{-l\delta\tau} \tilde{M}^\tau \eta^{-\tau} (l-J+1)^{2\tau} \\
&\le C\tilde{M}^\tau \eta^{-\tau} \sum_{j \ge J} (j-J+1)^{2\tau} 2^{-j\delta\tau} \\
&\le C\tilde{M}^\tau \eta^{-\tau} 2^{-J\delta\tau},
\end{aligned}$$

so that we obtain

$$2^{J(d+\delta\tau)} \le C\tilde{M}^\tau \eta^{-\tau}. \tag{4.29}$$

Combining this last estimate with (4.28), we deduce

$$\#(\tilde{\mathcal{T}}(f,\eta)) \le C[\tilde{M}^\tau \eta^{-\tau}]^{\frac{d}{d+\delta/\tau}} = \tilde{c}_2 \tilde{M}^\lambda \eta^{-\lambda}, \tag{4.30}$$

which concludes the proof. □

5 A tree based wavelet decomposition

The results of the previous section now give rise to a new wavelet decomposition based on trees. First we treat the case $1 \le p < \infty$. For $f \in L_p(\Omega)$ and each $k = 0, 1, \ldots$, we define the trees

$$\mathcal{T}_k(f) := \mathcal{T}(f, 2^{-k}) \tag{5.1}$$

where we adhere to the notation of the previous section and let

$$\Sigma_k(f) := \sum_{I \in \mathcal{T}_k(f)} A_I(f). \tag{5.2}$$

Bearing in mind that, according to (4.5), $\mathcal{T}_{k-1}(f) \subset \mathcal{T}_k(f)$, for all $k \ge 1$, we introduce the layers

$$\mathcal{L}_0(f) := \mathcal{T}_0(f), \quad \mathcal{L}_k(f) := \mathcal{T}_k(f) \setminus \mathcal{T}_{k-1}(f), \quad k \in I\!N,$$

corresponding to the wavelet coeffcients grouped by size, and set

$$\Delta_0(f) := \Sigma_0(f), \quad \Delta_k(f) := \Sigma_k(f) - \Sigma_{k-1}(f) = \sum_{I \in \mathcal{L}_k(f)} A_I(f), \quad k \ge 1. \tag{5.3}$$

Then, each $f \in L_p(\Omega)$ has the decomposition

$$f = \sum_{k \ge 0} \Delta_k(f). \tag{5.4}$$

In the case $p = \infty$, we modify the definition of the Δ_k as follows. We wish to employ the modified trees $\tilde{\mathcal{T}}(f, \eta)$ appearing in the definition (4.23) of $\tilde{S}(f, \eta)$. However, since we can no longer guarantee that the trees $\tilde{\mathcal{T}}(f, 2^{-k})$ are nested we note that the union of trees is a tree and set

$$\mathcal{T}_k(f) := \bigcup_{0 \le j \le k} \tilde{\mathcal{T}}(f, 2^{-j}). \tag{5.5}$$

In this case $\mathcal{L}_k(f)$ in (5.3) takes the form

$$\mathcal{L}_k(f) = \mathcal{T}_k(f) \setminus \mathcal{T}_{k-1}(f) = \tilde{\mathcal{T}}(f, 2^{-k}) \setminus \left(\bigcup_{0 \le j < k} \tilde{\mathcal{T}}(f, 2^{-j}) \right).$$

When $1 < p < \infty$ the unconditionality of the wavelet basis implies that the series in (5.4) converges in $L_p(\Omega)$ whenever $f \in L_p(\Omega)$. For $p = 1, \infty$ the strong convergence of the partial sums in (5.4) for f in any Besov space on the left of the respective critical line is ensured by Theorems 4.1 and 4.4, provided that for $p = \infty$ the modified tree approximations are used.

6 A universal encoding pair

In this section, we shall use the tree decomposition (5.4) to construct encoding pairs for L_p functions. We treat first the case $1 \le p < \infty$ in this section and defer the modifications necessary to obtain encoders for the full range $1 \le p \le \infty$ to a later section. The encoder

E to be described below assigns to each $f \in L_p(\Omega)$ an *infinite* bitstream which completely determines f. The encoder E is progressive in the following sense. Defining encoders E_N by associating with f a certain finite number of lead bits of $E(f)$, adding new bits gives additional information about f and increases the accuracy in approximating f (decreases the distortion).

The encoder

To begin the discussion, we fix p with $1 \le p < \infty$. For $f \in L_p(\Omega)$, the bitstream $E(f)$ will take the following form

$$\begin{aligned}&L(f), P_0(f), S_0(f), B_{0,0}(f), P_1(f), S_1(f), B_{0,1}(f), B_{1,0}(f), \ldots,\\ &P_N(f), S_N(f), B_{0,N}(f), B_{1,N-1}(f), \ldots, B_{N,0}(f), \ldots,\end{aligned} \tag{6.1}$$

where

- $L(f)$ is a bitstream that gives the *size* of the largest wavelet coefficient of f;
- each bitstream $P_k(f)$ gives the *positions* of the cubes in $\mathcal{L}_k(f)$;
- each bitstream $S_k(f)$ gives the *signs* of the coefficients $a^v_{I,p}(f)$, $I \in \mathcal{L}_k(f)$;
- each bitstream $B_{k,N-k}(f)$ adds a certain bit to each of the wavelet coefficients $a^v_{I,p}(f)$ for the $I \in \mathcal{L}_k(f)$, $0 \le k \le N$.

We now describe these bitstreams in more detail.

The bitstream $L(f)$**:** *Let κ be the integer such that the absolute value of the largest wavelet coefficient $a^v_{I,p}(f)$ of f appearing in* (5.4) *is in* $[2^{-\kappa}, 2^{-\kappa+1})$*. The first bit in $L(f)$ is a zero respectively one if $\kappa > 0$, respectively $\kappa \le 0$. It is followed by $|\kappa|$ ones while the last bit is zero indicating the termination of $L(f)$. Thus in the case $\kappa = 0$, $L(f)$ is the bitstream consisting of two zeros.*

We next discuss the bitstreams $P_k(f)$ which identify the positions of the cubes appearing in $\mathcal{L}_k(f)$. Here essential use will be made of the tree structure to guarantee efficient encoding of these positions.

There is a *natural ordering* of dyadic cubes that will always be referred to below. Each dyadic cube is of the form $2^{-j}(k + [0,1]^d)$, with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. The ordering of these cubes is determined by the *lexicographical* ordering of the corresponding $(d+1)$-tuples (j, k).

We shall use the following general result about encoding trees and their growth. According to the present context a tree is always understood to be a 2^d-tree, i.e., a node always branches into 2^d children.

Lemma 6.1 (i) *Given any finite tree $\mathcal{T}$, its positions can be encoded, in their natural order, with a bitstream P consisting of at most $m_0(1 + \#(\mathcal{T} \cap \mathcal{D}_0(\Omega))) + 2^d \#(\mathcal{T})$ bits where $m_0 := \lceil \log M_0 \rceil$ and $M_0 := \#\mathcal{D}_0(\Omega) + 1$.*

(ii) *If $\mathcal{T}$ contains the smaller tree $\mathcal{T}'$ and the positions of $\mathcal{T}'$ are known then the positions of $\mathcal{T} \setminus \mathcal{T}'$ can be encoded, in their natural order, with a bitstream P consisting of at most $m_0(1 + \#((\mathcal{T} \setminus \mathcal{T}') \cap \mathcal{D}_0(\Omega))) + 2^d(\#(\mathcal{T} \setminus \mathcal{T}')$ bits.*

(iii) *In case* (i) *or* (ii)*, if the bitstream P is embedded in a larger bitstream, then whenever the position of the first bit of P is known, we can identify the termination of P (i.e. the position of the last bit of P).*

Proof: We take the natural ordering of the cubes in $\mathcal{D}_0(\Omega)$. The number M_0 of these cubes is part of the codebook known to the decoder. We can identify each cube $I \in \mathcal{D}_0(\Omega)$ with a bit stream consisting of m_0 bits. We do not use the bitstream consisting of all zeros in this identification because this will be used to indicate the termination of this bitstream. The first bits of our encoding will identify the cubes in $\mathcal{D}_0(\Omega) \cap \mathcal{T}$ as follows. If all cubes from $\mathcal{D}_0(\Omega)$ are in $\mathcal{T}$ then we send the bitstream consisting of m_0 zeros terminating the encoding of cubes in $\mathcal{D}_0(\Omega) \cap \mathcal{T}$. Otherwise, we send the bitstreams associated with each of the cubes $I \in \mathcal{T} \cap \mathcal{D}_0(\Omega)$, in their natural order, and terminate with the bitstream consisting of m_0 zeros.

We next identify the cubes in $\mathcal{T} \cap \mathcal{D}_1(\Omega)$. Each such cube is a child of a cube from $\mathcal{T} \cap \mathcal{D}_0(\Omega)$. If $I \in \mathcal{T} \cap \mathcal{D}_0(\Omega)$, then to each of its children we assign a zero if the child is not in $\mathcal{T}$ and a one if the child is in $\mathcal{T}$. We arrange these bits according to the natural ordering of $\mathcal{T} \cap \mathcal{D}_0(\Omega)$ and then according to the natural ordering of the children. This bitstream will use $2^d\#(\mathcal{T}_0 \cap \mathcal{D}_0(\Omega))$ bits and will identify all cubes in $\mathcal{T} \cap \mathcal{D}_1(\Omega)$. We can repeat this process to identify all of the cubes in $\mathcal{T} \cap \mathcal{D}_2(\Omega)$ by using $2^d\#(\mathcal{T} \cap \mathcal{D}_1(\Omega))$ bits. If we continue in this way we shall eventually encode all cubes in $\mathcal{T}$ and arrive at (i).

The proof of (ii) is almost identical to (i).

Note also that the encoding will terminate with a sequence of 2^d zeros which will also serve to identify the completion of the encoding. Thus, property (iii) is also valid. $\square$

The bitstreams $P_k(f)$, $k \geq 0$. *These bitstreams are given by Lemma 6.1 and identify the positions of the cubes in $\mathcal{L}_k(f)$ for $k \geq 0$ with $\mathcal{T}_{-1}(f) := \emptyset$. Notice that some of these bitstreams may be empty. This occurs when $\kappa > 0$. Recall that the value of κ is identified by the lead bits $L(f)$ and so is known. Let $\kappa_0 := \max(\kappa, 0)$. From the lemma, we know that each of the $P_k(f)$, $k \geq \kappa_0$, consist of at most $m_0(1 + \#((\mathcal{L}_k(f))) \cap \mathcal{D}_0(\Omega)) + 2^d(\#(\mathcal{L}_k(f)))$ bits.*

We next describe the encoding of the signs of the wavelet coefficients. We take the natural ordering of the set V' of vertices of the unit cube. This in turn induces an ordering of the set V of nonzero vertices.

The bitstreams $S_k(f)$, $k \geq 0$. *These bitstreams give the signs of the wavelet coefficients. Let $k \geq 0$ and let $I \in \mathcal{L}_k(f)$. If $v \in V$ ($v \in V'$ in the case $I \in \mathcal{D}_0(\Omega)$), we assign the coefficient $a^v_{I,p}(f)$ the bit zero if this coefficient is nonnegative and one if this coefficient is negative. The bitstream $S_j(f)$ is this sequence of zeros and ones ordered according to the natural ordering of the cubes $I \in \mathcal{L}_k(f)$ and subsequently the natural ordering of the vertices. Each of the $S_k(f)$, $k \geq 0$, consists of $2^d(\#(\mathcal{L}_k(f)))$ bits.*

We next discuss how we encode coefficients. Any real number a has a binary representation

$$\sum_{j=-\infty}^{\infty} b_j(a)2^{-j}$$

with each $b_j(a) \in \{0,1\}$. In the case a has two representations (i.e., a is a binary rational) we choose the representation with a finite number of ones. First consider the encoding of the coefficients in $\mathcal{T}_0(f)$ which is a little different from the general case of encoding the coefficients in $\mathcal{L}_k(f)$, $k \geq 1$. Recall the integer κ given in the lead bits $L(f)$. $\mathcal{T}_0(f)$ will be nonempty if $\kappa \leq 0$. We know that each coefficient $a = a^v_{I,p}(f)$, $I \in \mathcal{T}_0(f)$, $v \in V'$, satisfies $b_j(a) = 0$, $j \leq \kappa$.

The bitstream $B_{0,0}(f)$. *In compliance with the natural ordering of the cubes* $I \in \mathcal{T}_0(f)$ *given by* $P_0(f)$, *and in compliance with the natural ordering of* V', *we send the bits* $b_j(a^v_{I,p}(f))$, $j = \kappa, \dots, 0$, $I \in \mathcal{T}_0(f)$, $v \in V$ ($v \in V'$ *in the case* $I \in \mathcal{D}_0(\Omega)$). *This bitstream will consist of at most* $2^d(|\kappa|+1)\#\mathcal{T}_0(f)$ *bits.*

We now describe the bitstreams $B_{k,n-k}$, $k = 0, \dots, n$ for $n \geq 1$. A coefficient $a^v_{I,p}(f)$ corresponding to $I \in \mathcal{L}_n(f)$ and $v \in V$ ($v \in V'$ if $I \in \mathcal{D}_0(\Omega)$) satisfies $|a^v_{I,p}(f)| < 2^{-n+1}$. Thus, $b_\ell(a^v_{I,p}(f)) = 0$, $\ell < n$. Hence sending a single bit $b_n(a^v_{I,p}(f))$ for $I \in \mathcal{L}_n(f)$ reduces the quantization error to 2^{-n} for those cubes. In addition the accuracy of the coefficients $a^v_{I,p}(f)$ for $I \in \mathcal{L}_k(f)$, $k < n$, has to be updated to the level 2^{-n}. By induction this requires a single additional bit $b_n(a^v_{I,p}(f))$ for each such coefficient. Therefore for each $0 \leq k \leq n$ the bitstream $B_{k,n-k}(f)$ consists of the bits $b_n(a^v_{I,p}(f))$ for $I \in \mathcal{L}_k(f)$ ordered according to $P_k(f)$ and the natural ordering of V (respectively V').

The bitstream $B_{k,j}(f)$**,** $j,k \geq 0, j+k > 0$**:** *In compliance with the natural ordering of the cubes* $I \in \mathcal{L}_k(f)$ *given by* $P_k(f)$, *and in compliance with the natural ordering of* V ($v \in V'$ *in the case* $I \in \mathcal{D}_0(\Omega)$), *we send the bits* $b_{j+k}(a^v_I(f))$. *This bitstream will consist of at most* $2^d(\#(\mathcal{L}_k(f)))$ *bits.*

This completes the description of the encoder E. For each $N \geq 0$, we define the encoder E_N which assigns to $f \in L_p(\Omega)$ the first portion of the bitstream for $E(f)$:

$$L(f), \dots, P_N(f), S_N(f), B_{0,N}(f), \dots . B_{N,0}(f), \tag{6.2}$$

In particular, when the number κ encoded by $L(f)$ exceeds N the following rule will apply. When successively reading in the bits in $L(f)$, the encoder realizes the case $\kappa > N$ when the first bit is a zero and a one appears at position $N+2$. In this case the encoding terminates and the bitstream $E_N(f)$ consists of one zero followed by $N+1$ ones. We further define $\mathcal{B}_N$ to be the bitstreams $E_N(f)$. While each bitstream $E_N(f)$ is finite, the collection $\mathcal{B}_N$ is infinite. However, when we restrict f to come from a compact set U, we will obtain a finite set $\mathcal{B}_N(U)$.

The decoder

Let us now describe the decoder D_N associated to E_N. Let B be any bitstream from $\mathcal{B}_N$.

Decoding κ: If the first two bits in $L(f)$ are zero then we know that $\kappa = 0$. Otherwise, the first bit of B is zero or one which identifies the sign of the number κ. Next comes a sequence of ones followed by a zero if $\kappa \leq N$. In this case the number of ones determine κ (i.e., $|\kappa|$ is equal to this number of ones). If the first bit is zero and the $N + 2$nd bit is one the decoder knows that $\kappa > N$ and hence that all wavelet coefficients have absolute value below 2^{-N}. The decoder then assigns the approximant 0.

Decoding the lead tree: Recall that $\kappa_0 := \max(\kappa, 0)$. Next comes a sequence of zeros and ones which identifies the cubes I in $\mathcal{T}_{\kappa_0}(B)$. Recall from Lemma 6.1 that we know when this sequence terminates. Next comes for each $I \in \mathcal{T}_{\kappa_0}(B)$, in their natural order, a sequence of zeros and ones which gives for each $I \in \mathcal{T}_{\kappa_0}(B)$, $v \in V$ ($v \in V'$, in case $I \in \mathcal{D}_0(\Omega)$) bits $b(j, I, v, B)$, where $j \leq \kappa_0$ if $\kappa_0 = 0$ and $j = \kappa_0$ if $\kappa_0 > 0$.

Progressive reconstruction of the trees $\mathcal{T}_k(B)$. Each new subsequent bitstream identifies the new cubes in $\mathcal{L}_k(B)$, sends one additional bit $b_k(a^v_{I,p}(f))$ for each of the old cubes $I \in \mathcal{L}_j(B)$, $v \in V$ ($v \in V'$ if $I \in \mathcal{D}_0(\Omega)$), $j < k$, and one bit $b_k(a^v_{I,p}(f))$ for the new cubes $I \in \mathcal{L}_k(B)$. In totality, the bitstream B determines a nested sequence of trees $\mathcal{T}_k(B)$, $k = 0, \ldots, N$, and for each $I \in \mathcal{T}_N$, $v \in V$ ($v \in V'$, in case $I \in \mathcal{D}_0(\Omega)$), and $j \leq N$, a number $b(j, I, v, B) \in \{0, 1\}$. By definition, these numbers are zero if $j < k$ in the case $I \in \mathcal{L}_k(B)$, $k = 1, 2, \ldots, N$ ($j \leq \kappa_0$ in the case $k = 0$). The decoder D_N uses this information to construct an element $S_N(B)$ from $L_p(\Omega)$ as follows.

For each $I \in \mathcal{T}_N(B)$, and each $v \in V$ ($v \in V'$ in case $I \in \mathcal{D}_0(\Omega)$), we define

$$a^v_{I,p,N}(B) := \sum_{j \leq N} b(j, I, v, B) 2^{-j}. \tag{6.3}$$

and

$$A^N_I(B) := \begin{cases} \sum_{v \in V'} a^v_{I,p,N}(B) \psi^v_{I,p}, & I \in \mathcal{D}_0, \\ \sum_{v \in V} a^v_{I,p,N}(B) \psi^v_{I,p}, & I \in \mathcal{D}_j, \ j \geq 1. \end{cases} \tag{6.4}$$

It follows that in the case $B = E(f)$ one has

$$|a^v_{I,p}(f) - a^v_{I,p,N}(E(f))| \leq 2^{-N}. \tag{6.5}$$

We define

$$\Delta^N_k(B) := \sum_{I \in \mathcal{L}_k(B)} A^N_I(B) \tag{6.6}$$

and

$$\tilde{S}_N(B) := \sum_{k=0}^{N} \Delta^N_k(B). \tag{6.7}$$

The decoder D_N maps B into $\tilde{S}_N(B) \in L_p(\Omega)$.

7 Performance of the encoders E_N on compact sets $K \subset L_p(\Omega)$, $1 \leq p < \infty$

We next examine the distortion of the encoding E_N, D_N on compact sets which are unit balls of Besov spaces. Again we treat first only the case $1 \leq p < \infty$.

Theorem 7.1 *Let $1 \le p < \infty$, and let $0 < \lambda < p$. If $U := U(\mathcal{B}_\lambda(L_p(\Omega)))$, we have*

$$M(U, E_N, D_N) \le c_4 2^{\lambda N}, \tag{7.1}$$

and

$$d(U, E_N, D_N) \le c_5 2^{-N\lambda s/d} \tag{7.2}$$

with $s := \frac{d}{\lambda} - \frac{d}{p}$ and the constants c_4, c_5 depending only on p and $p - \lambda$.

Moreover, for $0 < q \le \infty$ and $(1/\tau, s)$ above the critical line for nonlinear approximaton in L_p, i.e., $\delta := s - d/\tau + d/p > 0$, the same estimate holds for $U := U(B^s_q(L_\tau(\Omega))$, with $\lambda := \frac{d}{s+d/p}$ and the constants c_4, c_5 depending only on p, τ, and the discrepancy δ.

Proof: Let $f \in U := U(\mathcal{B}_\lambda(L_p(\Omega)))$. Then, f has all wavelet coefficients ≤ 1 in absolute value. Hence $\kappa \ge 0$. As noted earlier if $\kappa > N$, then $E_N(f)$ consists of $N + 2$ bits. So the number of bits n_L in $L(f)$ satisfies

$$n_L \le N + 2. \tag{7.3}$$

If $f \in U$, we know from (4.7) that $\#(\mathcal{T}_N(f)) = \#(\mathcal{T}(f, 2^{-N}) \le 2^{\lambda N}$. This means that the number n_P of bits in all of the bitstreams $P_k(f)$, $k = 0, \dots, N$ will satisfy

$$n_P \le m_0 \#(\mathcal{D}_0(\Omega) + 1) + 2^d \#\mathcal{T}_N(f) \le m_0 \#(\mathcal{D}_0(\Omega) + 1) + 2^d 2^{\lambda N}. \tag{7.4}$$

The total number of bits n_S appearing in the S_k, $k = 0, \dots, N$, satisfies

$$n_S \le 2^d \#\mathcal{T}_N(f) \le 2^d 2^{\lambda N} \tag{7.5}$$

because there are at most 2^d coefficients associated with each $I \in \mathcal{T}_N(f)$. For each coefficient $a^v_{I,p}(f)$, $I \in \mathcal{L}_k(f)$, $k = 0, \dots, N$, we will send at most $(N - k + 1)$ bits. Hence, using the estimate $\#(\mathcal{T}_k(f)) \le 2^{k\lambda}$ ensured by (4.7), we find that the total number of bits n_B in all of the sequences $B_{j,k}$, $0 \le j, k \le N$ will satisfy

$$n_C \le \sum_{k=0}^{N} (N + 1 - k) 2^d 2^{k\lambda} \le C 2^{\lambda N} \tag{7.6}$$

with C depending only on λ and d. Hence the total number of bits used in $E_N(f)$ does not exceed

$$n_L + n_P + n_S + n_C \le C(N + 2^{N\lambda}) \le c_4 2^{N\lambda}. \tag{7.7}$$

This completes the proof of (7.1).

Let

$$S_N(f) := S(f, 2^{-N})$$

be the function of (4.6). We have shown in Theorem 4.1 (see (4.8)) that

$$\|f - S_N(f)\|_{L_p(\Omega)} \le c_1 2^{-N(1-\lambda/p)}. \tag{7.8}$$

On the other hand, from Temlyakov's inequality (3.10) and (6.5), we have

$$\|S_N(f) - \tilde{S}_N(E(f))\|_{L_p(\Omega)} \le C 2^{-N} (\#\mathcal{T}_N(f))^{1/p} \le C 2^{-N} 2^{\lambda N/p}. \tag{7.9}$$

Therefore,

$$\|f - \tilde{S}_N(E(f))\|_{L_p(\Omega)} \leq c_5 2^{-N(1-\lambda/p)}.$$

Since $1 - \lambda/p = \lambda s/d$, this proves (7.2).

In the case where $U := U(B^s_q(L_\tau(\Omega))$, similar estimates are obtained by using (4.9) in order to majorize $\#(\mathcal{T}_N(f))$ and $\|f - S_N(f)\|_{L_p(\Omega)}$ as above (see Corollary 4.2). □

As a corollary of Theorem 7.1, we obtain upper estimates for the Kolmogorov entropy of the balls U.

Corollary 7.2 *Let $1 \leq p < \infty$, $0 < q \leq \infty$ and let the point $(1/\tau, s)$ be above the critical line for nonlinear approximaton in L_p. If $U := U(B^s_q(L_\tau(\Omega))$, then we have:*

$$H_\epsilon(U) \leq c_6 \epsilon^{-d/s}, \quad \epsilon > 0 \tag{7.10}$$

with the constant c_6 depending only on p, τ, and the discrepancy $\delta := s - d/\tau + d/p$.

Proof: Let N be the smallest integer such that $c_5 2^{-N\lambda s/d} < \epsilon$ with c_5 the constant in Theorem 7.1. We have shown in Theorem 7.1 that the encoding pair E_N, D_N has distortion $\leq c_5 2^{-N\lambda s/d} < \epsilon$ and $M(U, E_N, D_N) \leq c_4 2^{\lambda N} = c_4 \epsilon^{-d/s}$. Hence the corollary follows from (1.17). □

8 The case $p = \infty$

We wish to describe next how to modify the above encoders so as to obtain Theorem 7.1 and Corollary 7.2 also for $p = \infty$.

To this end, we simply use the modified trees $\mathcal{T}_k(f)$ from (5.5). The bitstreams $L(f), P_k(f), S_k(f)$ are then defined in the same way as described in § 6. The only further modification concerns the bitstreams $B_{k,j}(f)$. The reason is that, since for $p = \infty$ Temlyakov's inequality (3.10) is no longer applicable, a somewhat higher accuracy for the quantization is needed for the estimation of the quantization error. In fact, the main obstruction caused by the L_∞-norm is that locally wavelets from many levels may overlap. Therefore we will exploit the decay of wavelet coefficients required by (4.22). To this end, recall the turnover level J_k for the tree $\mathcal{T}_k(f)$ defined by (4.21). Now from the definition of $\tilde{\mathcal{T}}(f, \eta)$ we know that $I \in \mathcal{D}_j(\Omega) \cap \mathcal{L}_k(f)$ implies that $j > J_{k-1}$ and that

$$|a^v_{I,\infty}(f)| \leq 2^{-k+1-\ell(j-J_{k-1})}, \tag{8.1}$$

where

$$\ell(j - J) := \lfloor 2 \log_2(j - J) \rfloor.$$

Modified $B_{k,n-k}(f)$**:** *In compliance with the natural ordering of the cubes $I \in \mathcal{L}_k(f)$ and in compliance with the natural ordering of the $v \in V$ ($v \in V'$ if $I \in \mathcal{D}_0(\Omega)$) we send for each $I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)$, $k = 1, \dots, n$, $j = J_{k-1} + 1, \dots,$ the two bits $b_l(a^v_{I,\infty}(f))$, $l = \ell(j - J_{k-1}) + 2n - k + 1, \ell(j - J_{k-1}) + 2n - k$. Analogous modifications apply to $B_{0,0}(f)$.*

Hence when decoding, (6.3) is replaced for $I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)$ by

$$a^v_{I,\infty,N}(B) := \sum_{r \le \ell(j-J_{k-1})+2N-k+1} b(r, I, v, B) 2^{-r}, \tag{8.2}$$

so that now for $k = 1, \dots, N$, $I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)$,

$$|a^v_{I,\infty}(f) - a^v_{I,\infty,N}(E(f))| \le 2^{-(\ell(j-J_{k-1})+2N-k)}. \tag{8.3}$$

The counterparts of the above results read then as follows.

Theorem 8.1 *Let $\lambda < \infty$. If $U := U(\mathcal{B}_\lambda(L_\infty(\Omega)))$, we have*

$$M(U, E_N, D_N) \le c_7 2^{\lambda N}, \tag{8.4}$$

and

$$d(U, E_N, D_N) \le c_5 2^{-N\lambda s/d} \tag{8.5}$$

with $s := d/\lambda$ and the constants c_4, c_5 depending only on λ.

Moreover, for $0 < q \le \infty$ and $(1/\tau, s)$ above the critical line for nonlinear approximaton in L_∞, i.e., $\delta := s - d/\tau > 0$, the same estimate holds for $U := U(B^s_q(L_\tau(\Omega))$, with $\lambda := \frac{d}{s}$ and the constants c_4, c_5 depending only on p, τ, and the discrepancy $\delta = s - d/\tau$. Furthermore, we have:

$$H_\epsilon(U) \le c_6 \epsilon^{-d/s}, \quad \epsilon > 0 \tag{8.6}$$

with the constant c_6 depending only on τ, and the discrepancy $\delta := s - d/\tau$.

Proof: We essentially follow the arguments in the proofs of Theorem 7.1 and Corollary 7.2. The estimates (7.4) and (7.5) for n_P and n_S remain the same. The estimate (7.6) is replaced now by

$$n_C \le C \sum_{k=0}^{N} 2(N-k+1) 2^d 2^{\lambda k} \le C\, 2^{\lambda N}, \tag{8.7}$$

so that the total number of bits $n_L + n_P + n_S + n_C$ in the modified $E(f)$ still satisfies (7.7).

The approximation error (7.8) of the form

$$\|f - S_N(f)\|_{L_\infty(\Omega)} \le c_1 2^{-N} \tag{8.8}$$

follows now from Theorem 4.4.

Only the estimation of the quantization error requires a different argument because, as mentioned above, Temlyakov's inequality (3.10) is no longer applicable. This will be compensated by the higher accuracy provided by the modification of the bitstreams $B_{k,j}(f)$ and the additional decay of wavelet coefficients required by (4.22) in the definition of the trees $\tilde{\mathcal{T}}(f, \eta)$.

We will use the fact that for each fixed level $j \ge 0$ only a uniformly bounded finite number of terms $A_I(f) - A^N_I(B)$, $I \in \mathcal{D}_j$, are simultaneously nonzero at any given point in Ω. As before let $B := E(f)$ and note that

$$\begin{aligned} \|S_N(f) - \tilde{S}_N(B)\|_{L_\infty(\Omega)} \le & \sum_{I \in \mathcal{T}_0(f)} \|A_I(f) - A^N_I(B)\|_{L_\infty(\Omega)} \\ & + \sum_{k=1}^{N} \sum_{j=J_{k-1}+1}^{\infty} \Big\| \sum_{I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)} \big(A_I(f) - A^N_I(B)\big) \Big\|_{L_\infty(\Omega)}. \end{aligned} \tag{8.9}$$

By (8.3) the first sum on the right hand side of (8.9) is clearly bounded by $C\,2^{-N}$. The second sum is, in view of (8.3), bounded by

$$C\,2^{-N}\sum_{k=1}^{N}\sum_{j=J_{k-1}+1}^{\infty} 2^{-\ell(j-J_{k-1})-(N-k)} \leq C\,2^{-N},$$

which provides the desired counterpart to (7.9):

$$\|S_N(f)-\tilde{S}_N(B)\|_{L_\infty(\Omega)} \leq C\,2^{-N}. \tag{8.10}$$

The rest of the proof is the same as before in § 7. □

4.4 (3.10) does bit count, (7.9).

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