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Initial-boundary value problems for linear dispersive evolution equations on the half-line

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Initial-Boundary Value Problems for Linear Dispersive Evolution Equations on the Half-Line

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1. Introduction

A general approach to solving boundary value problems for two dimensional linear and integrable nonlinear PDEs was announced in [2] and further developed in [3,4]. This method can be applied to *linear PDEs with constant coefficients* and to *integrable nonlinear PDEs*. It involves (a) Formulating the given PDE as the compatibility condition of two linear eigenvalue equations, which we refer to as a Lax pair. (b) Performing the *simultaneous* spectral analysis of these two equations.

In this paper we show that this method can be rigorously implemented for the solution of the following class of initial-boundary value problems:

$$\partial_t q(x,t) + i\sum_{j=0}^{n} \alpha_j D^j q(x,t) = 0\,, \qquad 0<x<\infty\,, \quad 0<t<T\,, \tag{1.1a}$$

$$q(x,0) = q_0(x)\,, \quad 0<x<\infty\,, \tag{1.1b}$$

$$D^l q(0,t) = f_l(t)\,, \quad 0<t<T\,, \quad 0\le l\le N-1\,, \tag{1.1c}$$

where $\alpha_1,\ldots,\alpha_n$ $(n\ge 2)$ are real numbers, $D=-i\partial_x$, T is a positive number, and N is a positive integer. The initial data $q_0(x)$ and the solution $q(x,t)$ (and their derivatives) are assumed to have some decay as $x\to\infty$.

We will denote by $\omega(k)$ the polynomial defined by

$$\omega(k) = \sum_{j=0}^{n} \alpha_j k^j\,. \tag{1.2}$$

Then equation (1.1a) can be written concisely as $\partial_t q + i\omega(D)q = 0$.

This method can be used to:

(a) Show that the above IBV problem is well posed for

$$N = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd and } \alpha_n > 0 \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd and } \alpha_n < 0 \end{cases} . \tag{1.3}$$

(b) Derive an explicit representation of the solution $q(x,t)$ of (1.1a), in the Ehrenpreis form (cf. [1])

$$q(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{i[kx-\omega(k)t]}\hat{q}_0(k)\,dk + \int_{\partial\mathcal{D}_+} e^{i[kx-\omega(k)t]}\hat{Q}(k)\,dk\right], \tag{1.4}$$

where

$$\hat{q}_0(k) = \int_0^{\infty} e^{-ikx} q_0(x)\,dx\,, \quad \operatorname{Im} k \le 0\,, \tag{1.5}$$

$$\hat{Q}(k) = \sum_{j=1}^{n} \alpha_j\left(\hat{Q}_{j-1}(k) + k\hat{Q}_{j-2}(k) + \cdots + k^{j-2}\hat{Q}_1(k) + k^{j-1}\hat{Q}_0(k)\right), \tag{1.6a}$$

$$\hat{Q}_j(k) = \int_0^{T} e^{i\omega(k)t} D^j q(0,t)dt\,, \quad j = 0,1,\cdots,n-1\,,\ k\in\mathbb{C}\,, \tag{1.6b}$$

and the oriented contour $\partial\mathcal{D}_+$ is the boundary of the domain $\mathcal{D}_+$ defined by

$$\mathcal{D}_+ = \{k\in\mathbb{C} :\ \operatorname{Im}\omega(k) > 0\,,\ \operatorname{Im} k > 0\}\,. \tag{1.7}$$

The orientation of $\partial\mathcal{D}_+$ is such that $\mathcal{D}_+$ is on the left-hand side of the increasing direction of $\partial\mathcal{D}_+$.

(c) Determine the global relation satisfied by the initial and boundary values of $q(x,t)$ (cf. (1.10)). These relations together with the given boundary conditions can be used to construct $\hat{Q}(k)$ (up to terms whose integrals along $\partial\mathcal{D}_+$ vanish) through the solution of a system of linear algebraic equations and to prove existence of solution for the IBV problem (1.1).

For later reference we define

$$\mathcal{D}_- = \{k\in\mathbb{C} :\ \operatorname{Im}\omega(k) > 0\,,\ \operatorname{Im} k < 0\}\,, \tag{1.8}$$

$$\mathcal{D} = \mathcal{D}_+\cup\mathcal{D}_- = \{k\in\mathbb{C} :\ \operatorname{Im}\omega(k) > 0\}\,. \tag{1.9}$$

Remark 1.1. Examples of $\mathcal{D}_\pm$ for several differential operators can be found in appendix A.1.

Since $\operatorname{Im}\omega(k)$ for $k = k_R + ik_I$ is a harmonic function in (x, y), the set $\mathbb{C} \setminus \partial\mathcal{D}$ is the union of disjoint unbounded simply connected open sets. Moreover, as $k \to \infty$ the variety $\partial\mathcal{D}$ approaches the variety $\operatorname{Im}(k+\alpha)^n = 0$ asymptotically, where $\alpha = \alpha_{n-1}/(n\alpha_n)$. These two observations imply the following lemma immediately.

Lemma 1.1. *The components of $\mathcal{D}$ are simply connected and unbounded. Outside the curve defined by $|\omega(k)| = R$ for $R > 0$ sufficiently large, $\partial\mathcal{D}$ is the union of smooth disjoint simple contours that approach the rays of the variety $\operatorname{Im}(k+\alpha)^n = 0$ asymptotically as $k \to \infty$, where $\alpha = \alpha_{n-1}/(n\alpha_n)$. Moreover, $\mathcal{D}_R = \{k \in \mathcal{D} : |\omega(k)| > R\}$ has n components, and $\mathcal{D}_{R,+} = \{k \in \mathcal{D}_+ : |\omega(k)| > R\}$ has N components, where N is given by (1.3).*

We will denote the components of $\mathcal{D}_{R,+}$ by $\mathcal{D}_{R,1}, \ldots, \mathcal{D}_{R,N}$, in the counterclockwise direction, and the components of $\mathcal{D}_R$ in the lower half-plane by $\mathcal{D}_{R,N+1}, \ldots, \mathcal{D}_{R,n}$, also in the counterclockwise direction.

The following results will be derived in Sections 2–5.

Proposition 1.1. (Representation of solutions of (1.1*a*)) *Assume that $q(x,t)$ is a sufficiently smooth (up to the boundary) solution of (1.1a) that also has sufficient decay as $x \to \infty$, uniformly in $0 \le t \le T$. Then $q(x,t)$ is given by equation (1.4). Furthermore, the boundary values of $q(x,t)$ satisfy the following global relation on the closure of the lower half-plane:*

$$\hat{Q}(k) = e^{i\omega(k)T}\hat{q}_T(k) - \hat{q}_0(k)\,, \quad k \in \overline{\mathbb{C}_-}\,, \tag{1.10}$$

where $\hat{Q}(k)$ is defined by equation (1.6), $\hat{q}_0(k)$ denotes the Fourier transform (of the trivial extension) of $q_0(x) = q(x,0)$ (cf. (1.5)), and $\hat{q}_T(k)$ denotes the Fourier transform (of the trivial extension) of $q(x,T)$.

Proposition 1.2. (Representation of solutions of (1.1)) *Assume that $q(x,t)$ is a sufficiently smooth (up to the boundary) solution of (1.1) that also has sufficient decay as $x \to \infty$, uniformly in $0 \le t \le T$. Then for $(x,t) \in \big([0,\infty) \times [0,T]\big) \setminus \{(0,T)\}$, $q(x,t)$ is given by*

$$q(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{i[kx-\omega(k)t]}\hat{q}_0(k)\,dk + \int_{\partial\mathcal{D}_{R,+}} e^{i[kx-\omega(k)t]}\tilde{Q}(k)\,dk\right], \tag{1.11}$$

where $R > 0$ is a sufficiently large number, $\hat{q}_0(k)$ is the Fourier transform (of the trivial extension) of $q_0(x)$,

$$\tilde{Q}(k) = \sum_{j=1}^{n} \omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,, \tag{1.12}$$

$$\omega_j(k) = \alpha_n k^j + \alpha_{n-1}k^{j-1} + \cdots + \alpha_{n-j} \qquad \text{for} \quad 0 \le j \le n-1\,, \tag{1.13}$$

and $\tilde{Q}_j(k)$ for $0 \le j \le n-1$ are obtained from the initial and boundary data in the following way.

For $0 \le j \le N-1$, the function $\tilde{Q}_j(k)$ is given by

$$\tilde{Q}_j(k) = \int_0^T e^{i\omega(k)t} f_j(t)\,dt \qquad \text{for} \quad k \in \mathbb{C}\,, \tag{1.14}$$

and $\tilde{Q}_N(k), \dots, \tilde{Q}_{n-1}(k)$ *for* $k \in \overline{\mathcal{D}_{R,m}}$, $1 \le m \le N$, *are given through the unique solution of the following* $(n-N)\times(n-N)$ *system of linear equations:*

$$\sum_{j=N+1}^{n} \omega_{n-j}\big(\lambda_{l,m}(k)\big)\hat{Q}_{j-1}(k) = -\hat{q}_0\big(\lambda_{l,m}(k)\big) - \sum_{j=1}^{N}\omega_{n-j}\big(\lambda_{l,m}(k)\big)\hat{Q}_{j-1}(k)\,, \tag{1.15}$$

where $N+1 \le l \le n$ *and* $\lambda_{l,m} : \overline{\mathcal{D}_{R,m}} \longrightarrow \overline{\mathcal{D}_{R,l}}$ *is the biholomorphic map defined by*

$$\omega\big(\lambda_{l,m}(k)\big) = \omega(k) \qquad \forall\, k \in \overline{\mathcal{D}_{R,m}}\,. \tag{1.16}$$

Remark 1.2. The explicit solution of the system of equations in (1.15) for several illustrative examples are discussed in appendix A.2.

Theorem 1.1. (Existence and uniqueness of solution for smooth data)
Assume that :

q_0 *is a* C^∞ *function on* $[0,\infty)$ *that is rapidly decreasing as* $x \to \infty$, *i.e.* (1.17)
$\lim_{x\to\infty} x^m D^l q_0(x) = 0$ *for any nonnegative integers* m *and* l.

f_l *is a* C^∞ *function on* $[0,T]$ *for* $0 \le l \le N-1$. (1.18)

The functions q_0, f_l *and* (1.1*a*) *are compatible at* $x=0$ *and* $t=0$ *to all orders.* (1.19)

Then the IBV problem (1.1) *has a unique solution* $q(x,t)$ *such that* $t \longrightarrow q(\cdot,t)$ *is a* C^∞ *map from* $[0,T]$ *into* $\mathscr{S}\big([0,\infty)\big)$, *the Schwartz space of smooth functions on* $[0,\infty)$ *that decrease rapidly as* $x \to \infty$.

Theorem 1.2. (Existence and uniqueness of weak solution for Sobolev data)
Assume that :

q_0 *belongs to the Sobolev space* $H^{\tilde{n}}(0,\infty)$, *where* $\tilde{n}$ *is the smallest integer* $\ge n/2$. (1.20)

f_l *belongs to the Sobolev space* $H^{\frac{1}{2}+\frac{(2\tilde{n}-2l-1)}{2n}}(0,T)$ *for* $0 \le l \le N-1$. (1.21)

$f_l(0) = D^l q_0(0)$ *for* $0 \le l \le N-1\,.$ (1.22)

Then there is a unique function $q(x,t)$ *with the following properties*:

The map $t \longrightarrow q(\cdot,t)$ *is a continuous map from* $[0,T]$ *into* $H^{\tilde{n}}(0,\infty)$. (1.23)

$q(x,t)$ *satisfies the initial and boundary conditions* (1.1*b*)–(1.1*c*). (1.24)

Given any $\phi \in C_c^\infty(\mathbb{R})$ *such that* $D^j\phi(0) = 0$ *for* $0 \le j \le n-\tilde{n}-1$, *the function* $\big(q(\cdot,t),\phi\big)_{L_2(0,\infty)}$ *is differentiable on* $(0,T)$, *and*

$$\frac{d}{dt}\big(q(\cdot,t),\phi\big)_{L_2(0,\infty)} = \big(q(\cdot,t), i\omega(D)\phi\big)_{L_2(0,\infty)} + \sum_{j=n-\tilde{n}}^{n-1}\big[\omega_{n-j-1}(D)q(0,t)\big]\overline{D^j\phi(0)} \tag{1.25}$$

for $0 < t < T$, *where* ω *and* ω_j *are defined by* (1.2) *and* (1.13).

Moreover this unique weak solution defined by (1.23)–(1.25) *has the property that*

$$x \longrightarrow D^j q(x,\cdot) \text{ is a continuous map from } [0,\infty) \text{ into } H^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}}(0,T) \tag{1.26}$$

for $0 \le j \le n-1$.

Remark 1.3. Let $q(x,t)$ satisfy the forced version of (1.1), i.e. the right-hand side of (1.1a) is replaced by $f(x,t)$. It can be shown that $q(x,t)$ admits the explicit representation (1.4) where $\hat{Q}(k)$ contains the additional term $\int_0^T \int_0^\infty e^{-ikx'+i\omega(k)t'} f(x',t')\,dx'dt'$.

Given a nonlinear *non-integrable* PDE, the nonlinear terms can be considered as a forcing of the associated linear system. Then, at least for sufficiently small data or for sufficiently small time, it should be possible to prove well-posedness for the nonlinear PDE.

Remark 1.4. It is possible to extend the above results to other types of boundary conditions. Some examples are discussed in appendix A.3.

2. The Lax Pair Formulation

In this section we will derive Proposition 1.1 and Proposition 1.2 under the assumption that $q(x,t)$ is a smooth solution of (1.1a) with sufficient decay as $x \to \infty$.

First we note that equation (1.1a) is the compatibility condition of the following pair of equations,[1]

$$\partial_x \mu(x,t,k) - ik\mu(x,t,k) = q(x,t)\,, \tag{2.1a}$$

$$\partial_t \mu(x,t,k) + i\omega(k)\mu(x,t,k) = -q_*(x,t,k)\,, \tag{2.1b}$$

where $\omega(k)$ is defined by (1.2) and

$$q_*(x,t,k) = \sum_{j=1}^{n} \alpha_j \left(D^{j-1} + kD^{j-2} + k^2 D^{j-3} + \cdots + k^{j-1}\right) q(x,t)\,. \tag{2.2}$$

Indeed, equation (1.1a) is the compatibility condition of equation (2.1a) and of

$$\partial_t \mu + i\omega(D)\mu = 0\,. \tag{2.3}$$

Equation (2.1b) is obtained by using equation (2.1a) to eliminate the $D^j\mu$'s in equation (2.3). Note that q_* is related to $\hat{Q}$ by

$$\hat{Q}(k) = \int_0^T e^{i\omega(k)t} q_*(0,t)\,dt\,. \tag{2.4}$$

[1] In analogy with the theory of integrable nonlinear evolution equations and in honor of P.D. Lax, we call equations (2.1) a Lax pair.

We will construct a solution μ which satisfies both equations defining the Lax pair and which is sectionally holomorphic. It has the form

$$\mu = \begin{cases} \mu_2, & k \in \mathcal{D}_+ \\ \mu_3, & k \in \mathcal{E}_+ = \{k \in \mathbb{C}, \operatorname{Im}\omega(k) < 0, \operatorname{Im} k > 0\} \\ \mu_4, & \operatorname{Im} k < 0, \end{cases} \tag{2.5}$$

where the functions μ_2, μ_3 and μ_4 will be defined below.

Let $\zeta = (x,t)$. The domain $0 \le x \le \infty$, $0 \le t \le T$ is a polygon in the ζ-plane with corners $\zeta_1 = (\infty, T)$, $\zeta_2 = (0,T)$, $\zeta_3 = (0,0)$ and $\zeta_4 = (\infty, 0)$ (cf. Figure 2.1).

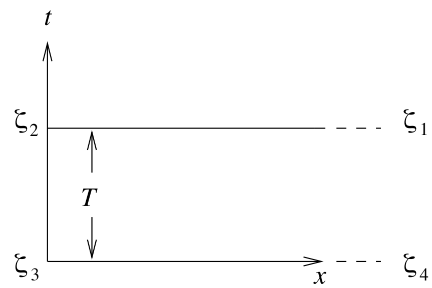

Figure 2.1

Equation (2.1) can be written in the form

$$\left(\mu e^{-ikx+i\omega(k)t}\right)_x = e^{-ikx+i\omega(k)t} q, \tag{2.6a}$$

$$\left(\mu e^{-ikx+i\omega(k)t}\right)_t = -e^{-ikx+i\omega(k)t} q_* . \tag{2.6b}$$

Let $\zeta_\dagger$ be an arbitrary point in the polygon and let $\int_{\zeta_\dagger}^{\zeta}$ denote the line integral from $\zeta_\dagger$ to $\zeta = (x,t)$. Then the function

$$\mu_\dagger(x,t,k) = \int_{\zeta_\dagger}^{\zeta} e^{ik(x-x')-i\omega(k)(t-t')} \left[q(x',t')\,dx' - q_*(x',t',k)\,dt'\right] \tag{2.7}$$

is a particular solution of (2.6). Furthermore the definition of $\mu_\dagger$ is independent of the path from $\zeta_\dagger$ to ζ. We must now choose the point $\zeta_\dagger$ in such a way that this function is holomorphic in k.

It is shown in [3] that for a polygon there exists a canonical way of choosing the $\zeta_\dagger$'s, namely they are the corners of the polygon. For the polygon in Figure 2.1, we therefore define μ_j by (2.7) where $\zeta_\dagger = \zeta_j$.

In particular, we have $\mu_1 = \mu_4$ (since $q_*(\infty, t) = 0$), and

$$\mu_4(x,t,k) = \int_\infty^x e^{ik(x-x')} q(x',t)\,dx' \tag{2.8}$$

is holomorphic for $\operatorname{Im} k < 0$ and bounded for $\operatorname{Im} k \le 0$. Splitting the integral $\int_{\zeta_3}^{\zeta}$ into one along the t-axis and one parallel to the x-axis we find

$$\mu_3(x,t,k) = -e^{ikx} \int_0^t e^{-i\omega(k)(t-t')} q_*(0,t',k)\,dt' + \int_0^x e^{ik(x-x')} q(x',t)\,dx', \tag{2.9}$$

which is an entire function of k and bounded for $k \in \overline{\mathcal{E}_+}$. Similarly

$$\mu_2(x,t,k) = -e^{ikx} \int_T^t e^{-i\omega(k)(t-t')} q_*(0,t',k)\, dt' + \int_0^x e^{ik(x-x')} q(x',t)\, dx' \qquad (2.10)$$

is an entire function of k and bounded for $k \in \overline{\mathcal{D}_+}$.

Using the representation (2.7) the "jump" of μ can be computed in terms of line integrals along the boundary of the polygon; for example $\mu_2 - \mu_3 = \int_{\zeta_2}^{\zeta_3}$. Hence we have, by (1.5) and (2.4),

$$\begin{aligned} \mu_2 - \mu_3 &= e^{i[kx-\omega(k)t]} \hat{Q}(k)\,, && k \in L\,, && (2.11a)\\ \mu_2 - \mu_4 &= e^{i[kx-\omega(k)t]} (\hat{Q}(k) + \hat{q}_0(k))\,, && k \in l_1\,, && (2.11b)\\ \mu_3 - \mu_4 &= e^{i[kx-\omega(k)t]} \hat{q}_0(k)\,, && k \in l_2\,, && (2.11c) \end{aligned}$$

where $L = \{k \in \mathbb{C} : \operatorname{Im}\omega(k) = 0\,, \operatorname{Im} k > 0\}$, l_1 denotes the part of the real axis that is a part of the boundary of $\mathcal{D}_+$, and l_2 denotes the part of the real axis that is a part of the boundary of $\mathcal{E}_+$. Note that l_1 or l_2 can be empty.

Equations (2.5), (2.8)–(2.10) and integration by parts imply

$$\mu(x,t,k) = -\frac{q(x,t)}{ik} + o(1/|k|)\,, \quad k \to \infty\,, \qquad (2.12)$$

which together with (2.11) define an elementary scalar Riemann-Hilbert problem (cf. [7]), whose unique solution is given by

$$\mu(x,t,k) = \frac{1}{2\pi i} \left[\int_{\mathbb{R}} e^{i[zx-\omega(z)t]} \hat{q}_0(z) \frac{dz}{z-k} + \int_{\partial\mathcal{D}_+} e^{i[zx-\omega(z)t]} \hat{Q}(z) \frac{dz}{z-k} \right]. \qquad (2.13)$$

Equations (2.12) and (2.13) imply equation (1.4).

It follows from (2.6) and Green's theorem that the integral of $\exp(-ikx+i\omega(k)t)(q dx - q_* dt)$ along the boundary of the polygon vanishes. This yields the important global relation

$$\int_\infty^0 e^{-ikx+i\omega(k)T} q(x,T)\, dx - \int_T^0 e^{i\omega(k)t} q_*(0,t)\, dt + \int_0^\infty e^{-ikx} q(x,0)\, dx = 0 \quad \text{for } k \in \overline{\mathbb{C}_-}$$

and (1.10) follows in view of (2.4). The derivation of Proposition 1.1 is complete.

Next we turn to the derivation of Proposition 1.2. Recall that (cf. Lemma 1.1), for $R > 0$ sufficiently large, $\mathcal{D}_R = \{k \in \mathcal{D} : |\omega(k)| > R\}$ has n components, N (cf. (1.3)) of which are in $\mathbb{C}_+$, labeled counterclockwise by $\mathcal{D}_{R,1}, \mathcal{D}_{R,2}, \ldots, \mathcal{D}_{R,N}$, and the other components $\mathcal{D}_{R,N+1}, \ldots, \mathcal{D}_{R,n}$ are in $\mathbb{C}_-$.

We can rewrite (1.4) by Cauchy's theorem as

$$q(x,t) = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} e^{i[kx-\omega(k)t]} \hat{q}_0(k)\, dk + \int_{\partial\mathcal{D}_{R,+}} e^{i[kx-\omega(k)t]} \hat{Q}(k)\, dk \right]. \qquad (2.14)$$

We may also assume that

$$\omega'(k) \neq 0 \qquad \text{for} \quad |\omega(k)| > \frac{R}{2}\,. \tag{2.15}$$

It follows from (2.15) that (1.16) defines a map $\lambda_{l,m}$ which is biholomorphic from a neighborhood of $\overline{\mathcal{D}_{R,m}}$ to a neighborhood of $\overline{\mathcal{D}_{R,l}}$. Note that (1.6) and (1.16) imply that

$$\hat{Q}_j(\lambda_{l,m}(k)) = \hat{Q}_j(k) \qquad \text{for} \quad 0 \le j \le n-1\,, 1 \le m, l \le n\,. \tag{2.16}$$

Moreover, it follows easily from (1.16) that

$$\lambda_{j,l}\big[\lambda_{l,m}(k)\big] = \lambda_{j,m}(k) \qquad \text{for} \quad k \in \overline{\mathcal{D}_{R,m}}\,, 1 \le m, l, j \le n\,, \tag{2.17}$$

and

$$\lambda_{l,m}(k) = e^{i(l-m)\frac{2\pi}{n}} k + O\,(1) \qquad \text{as} \quad k \to \infty \quad \text{in} \quad \overline{\mathcal{D}_{R,m}}\,. \tag{2.18}$$

Using (1.13) we can rewrite (1.6) as

$$\hat{Q}(k) = \sum_{j=1}^{n} \omega_{n-j}(k)\hat{Q}_{j-1}(k) = \sum_{j=1}^{N} \omega_{n-j}(k)\hat{Q}_{j-1}(k) + \sum_{j=1}^{n-N} \omega_{j-1}(k)\hat{Q}_{n-j}(k)\,. \tag{2.19}$$

Let $m \in \{1, 2, \ldots, N\}$. From (1.10), (2.16) and (2.19) we find

$$\begin{aligned}\sum_{j=1}^{n-N} \omega_{j-1}\big(\lambda_{l,m}(k)\big)\hat{Q}_{n-j}(k) = -\hat{q}_0\big(\lambda_{l,m}(k)\big) - \sum_{j=1}^{N} \omega_{n-j}\big(\lambda_{l,m}(k)\big)\hat{Q}_{j-1}(k) \\ + e^{i\omega(k)T}\,\hat{q}_T\big(\lambda_{l,m}(k)\big)\,,\end{aligned} \tag{2.20}$$

for $N+1 \le l \le n$ and $k \in \overline{\mathcal{D}_{R,m}}$.

The system (2.20) is uniquely solvable for $\hat{Q}_N(k), \ldots, \hat{Q}_{n-1}(k)$. In fact, using Cramer's rule, (2.16) and (2.20) we have the explicit formula

$$\begin{aligned}\hat{Q}_{n-j}(k) = \left[\frac{\det B_j\big(\lambda_{N+1,m}(k), \lambda_{N+2,m}(k), \ldots, \lambda_{n,m}(k)\big)}{\det A\big(\lambda_{N+1,m}(k), \lambda_{N+2,m}(k), \ldots, \lambda_{n,m}(k)\big)}\right] \\ + e^{i\omega(k)T}\left[\frac{\det C_j\big(\lambda_{N+1,m}(k), \lambda_{N+2,m}(k), \ldots, \lambda_{n,m}(k)\big)}{\det A\big(\lambda_{N+1,m}(k), \lambda_{N+2,m}(k), \ldots, \lambda_{n,m}(k)\big)}\right]\end{aligned} \tag{2.21}$$

for $1 \le j \le n-N$ and $k \in \overline{\mathcal{D}_{R,m}}$, where $A(z_1, z_2, \ldots, z_{n-N})$ is the $(n-N) \times (n-N)$ matrix defined by

$$A(z_1, z_2, \ldots, z_{n-N}) = \begin{bmatrix} \omega_0(z_1) & \omega_1(z_1) & \cdots & \omega_{n-N-1}(z_1) \\ \omega_0(z_2) & \omega_1(z_2) & \cdots & \omega_{n-N-1}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_0(z_{n-N}) & \omega_1(z_{n-N}) & \cdots & \omega_{n-N-1}(z_{n-N}) \end{bmatrix}, \tag{2.22}$$

and the matrices $B_j(z_1, z_2, \ldots, z_{n-N})$ and $C_j(z_1, z_2, \ldots, z_{n-N})$ are obtained by replacing the j^{th} column of $A(z_1, z_2, \ldots, z_{n-N})$ by the columns

$$-\begin{bmatrix} \hat{q}_0(z_1) + \sum_{s=1}^N \omega_{n-s}(z_1)\hat{Q}_{s-1}(z_1) \\ \hat{q}_0(z_2) + \sum_{s=1}^N \omega_{n-s}(z_2)\hat{Q}_{s-1}(z_2) \\ \vdots \\ \hat{q}_0(z_{n-N}) + \sum_{s=1}^N \omega_{n-s}(z_{n-N})\hat{Q}_{s-1}(z_{n-N}) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{q}_T(z_1) \\ \hat{q}_T(z_2) \\ \vdots \\ \hat{q}_T(z_{n-N}) \end{bmatrix} \tag{2.23}$$

respectively. Note that (1.13) and (2.22) imply

$$\det A(z_1, z_2, \ldots, z_{n-N}) = (\alpha_n)^{n-N} \prod_{1 \le l < j \le n-N} (z_j - z_l)\,. \tag{2.24}$$

Therefore the denominators in (2.21) are always nonzero, since the numbers $\lambda_{l,m}(k)$ for $N+1 \le l \le n$ are distinct when $k \in \overline{\mathcal{D}_{R,m}}$, under the assumption (2.15).

It is clear from (1.6b), (2.18) and (2.22)–(2.23) that both terms on the right-hand side of (2.21) are holomorphic in $\mathcal{D}_{R,m}$, and have at least $O(|k|^{-j})$ decay as $k \to \infty$. In particular, we have by Jordan's lemma (cf. [6])

$$\int_{\partial \mathcal{D}_{R,m}} e^{i[kx-\omega(k)t]}\omega_{j-1}(k)e^{i\omega(k)T} \left[\frac{\det C_j\big(\lambda_{m,N+1}(k), \lambda_{m,N+2}(k), \ldots, \lambda_{m,n}(k)\big)}{\det A\big(\lambda_{m,N+1}(k), \lambda_{m,N+2}(k), \ldots, \lambda_{m,n}(k)\big)} \right] dk = 0$$

for $1 \le j \le n-N$ and $(x,t) \in \big([0,\infty) \times [0,T]\big) \setminus \{(0,T)\}$. Thus, in view of (2.19), the contribution of the second term on the right-hand side of (2.21) to the second integral on the right-hand side of (2.14) is 0.

We conclude that in (2.14) one can replace the functions $\hat{Q}_{n-j}(k)$, for $1 \le j \le n-N$, $k \in \partial\mathcal{D}_{R,m}$ and $1 \le m \le N$, by the first term on the right-hand side of (2.21). In other words, the solution $q(x,t)$ of (1.1a) can be expressed in terms of the initial data $q(x,0)$ and the N boundary data $q(0,t), Dq(0,t), \ldots, D^{N-1}q(0,t)$ via (1.11)–(1.16). This completes the derivation of Proposition 1.2.

Remark 2.1. Observe that for a given t_* between 0 and T, the values of $q(x,t_*)$ does not depend on the boundary data beyond t_*. This of course follows by applying the representation formula (1.4) to $[0,\infty) \times [0,t_*]$ instead of $[0,\infty) \times [0,T]$. Alternatively we can see it from (1.4) for $[0,\infty) \times [0,T]$ as follows.

We have

$$\hat{Q}_j(k) = \int_0^{t_*} e^{i\omega(k)t}Q_j(t)\,dt + \int_{t_*}^T e^{i\omega(k)t}Q_j(t)\,dt\,. \tag{2.25}$$

Since the function

$$e^{i[kx-\omega(k)t_*]}\int_{t_*}^T e^{i\omega(k)t}Q_j(t)\,dt = e^{ikx}\int_{t_*}^T e^{i\omega(k)(t-t_*)}Q_j(t)\,dt$$

is holomorphic on $\mathcal{D}$ and has sufficient decay at ∞, the only nonzero contribution of $\hat{Q}_j(k)$ in (1.4) comes from the first term on the right-hand side of (2.25).

3. The Spectral Map

Motivated by the discussion in Section 2, we introduce in this section the spectral map $\mathbf{S}$ and study some of its properties. The spectral map will be used to relate the boundary data to the spectral data in the case where $q_0 = 0$.

Definition 3.1. Let Ω be an open subset of $\mathbb{C}$. The space $\mathscr{H}(\bar{\Omega})$ consists of all the functions that are holomorphic in a neighborhood of $\bar{\Omega}$.

Definition 3.2. Let T be a positive number and $R > 0$ be large enough so that (2.15) is satisfied. The map $\mathbf{S}_T : [L_1(0,T)]^N \longrightarrow [\mathscr{H}(\overline{\mathcal{D}_{R,+}})]^n$ is defined as follows. Given $(f_0, f_1, \ldots, f_{N-1}) \in [L_1(0,T)]^N$,

$$\mathbf{S}_T(f_0, f_1, \ldots, f_{N-1}) = (g_0, g_1, \ldots, g_{n-1}), \tag{3.1}$$

where for $0 \le j \le N-1$ the function g_j is defined by

$$g_j(k) = \int_0^T e^{i\omega(k)t} f_j(t)\, dt \qquad \text{for} \quad k \in \mathbb{C}, \tag{3.2}$$

and for $1 \le j \le N$ the function g_{n-j} is defined on $\overline{\mathcal{D}_{R,m}}$ $(1 \le m \le N)$ by

$$g_{n-j}(k) = \frac{\det B_j\big(\lambda_{N+1,m}(k), \lambda_{N+2,m}(k), \ldots, \lambda_{n,m}(k)\big)}{\det A\big(\lambda_{N+1,m}(k), \lambda_{N+2,m}(k), \ldots, \lambda_{n,m}(k)\big)}. \tag{3.3}$$

Here $A(z_1, z_2, \ldots, z_{n-N})$ is defined by (2.22), the matrix $B_j(z_1, z_2, \ldots, z_{n-N})$ is obtained by replacing the j^{th} column of $A(z_1, z_2, \ldots, z_{n-N})$ by the column

$$-\begin{bmatrix} \sum_{s=1}^N \omega_{n-s}(z_1) g_{s-1}(z_1) \\ \sum_{s=1}^N \omega_{n-s}(z_2) g_{s-1}(z_2) \\ \vdots \\ \sum_{s=1}^N \omega_{n-s}(z_{n-N}) g_{s-1}(z_{n-N}) \end{bmatrix}, \tag{3.4}$$

and the polynomials $\omega_j(k)$ are defined as in (1.13).

Clearly $\mathbf{S}_T$ is a linear map. We shall write (3.1) concisely as $\mathbf{S}_T \mathbf{f} = \mathbf{g}$.

Lemma 3.1. *Let $\mathbf{g} = \mathbf{S}_T \mathbf{f}$. Then*

$$\sum_{j=1}^n \omega_{n-j}\big(\lambda_{l,m}(k)\big) g_{j-1}(k) = 0 \qquad \text{for } k \in \overline{\mathcal{D}_{R,m}},\ 1 \le m \le N,\ N+1 \le l \le n, \tag{3.5}$$

and $\mathbf{g}$ is invariant under the transformations $\lambda_{l,m}$, i.e.,

$$\mathbf{g}(\lambda_{l,m}(k)) = \mathbf{g}(k), \qquad k \in \overline{\mathcal{D}_{R,m}},\ 1 \le m, l \le N. \tag{3.6}$$

Proof. Note that (1.16) and (3.2) imply

$$g_j\big(\lambda_{l,m}(k)\big) = g_j(k) \qquad \text{for} \quad k \in \mathbb{C}\,,\ 0 \le j \le N-1\,,\ 1 \le m, l \le n\,. \tag{3.7}$$

Equations (3.5) and (3.6) follow immediately from (3.3), (3.4), (3.7) and (2.17). ∎

Lemma 3.2. *Let* $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1}) \in [C^\infty(0,T)]^N$ *such that* f_j *vanishes to all orders at* 0 *and* T *for* $0 \le j \le N-1$, *and* $(g_0, g_1, \ldots, g_{n-1}) = \mathbf{S}_T\mathbf{f}$. *Then the functions* $k^\alpha(d^\beta g_j/dk^\beta)(k)$ *are bounded on* $\overline{\mathcal{D}_{R,+}}$ *for* $0 \le j \le n-1$ *and* $\alpha, \beta = 0, 1, 2, \ldots$.

Proof. The boundedness of $k^\alpha(d^\beta g_j/dk^\beta)(k)$ for $0 \le j \le N-1$ follows immediately from (3.2) and integration by parts. The boundedness of $k^\alpha(d^\beta g_j/dk^\beta)(k)$ for $N \le j \le n-1$ then follows from (2.18), (2.22), (2.24), (3.3) and (3.4). ∎

In Section 5 we will consider boundary data in the fractional order Sobolev space $H^s(0,T)$, which is the restriction of the Sobolev space $H^s(\mathbb{R})$ to the interval $(0,T)$ and is equipped with the standard quotient norm (cf. [5]). The space $H^s_0(0,T)$ is the closure of $C^\infty_c(0,T)$ in $H^s(0,T)$. The following lemma (cf. [5]) describes the relation between these spaces when $\frac{1}{2} < s < 1$.

Lemma 3.3. *Let* $u \in H^s(0,T)$ *and* $(1/2) < s < 1$. *Then* $u \in H^s_0(0,T)$ *if and only if* $u(0) = u(T) = 0$. *Moreover, if* $u \in H^s_0(0,T)$, *then its trivial extension (denoted by* $\tilde{u}$*) belongs to* $H^s(\mathbb{R})$, *and*

$$\|\tilde{u}\|_{H^s(\mathbb{R})} \le C\|u\|_{H^s(0,T)}\,.$$

For boundary data f_j in the fractional order Sobolev spaces, it is necessary to characterize $\mathbf{S}_T\mathbf{f}$ in terms of its integrability on $\partial\mathcal{D}_{R,+}$.

Definition 3.3. Let $\mathscr{C}$ be a piecewise smooth curve (cf. [6]) in the k-plane. The space $L^s_2(\mathscr{C})$ consists of all the functions $g(k)$ that satisfy

$$\|g\|_{L^s_2(\mathscr{C})} = \|\,(1+|k|^s)g(k)\|_{L_2(\mathscr{C},|dk|)} < \infty\,. \tag{3.8}$$

Lemma 3.4. *Let* $\tilde{n}$ *be the smallest integer* $\ge n/2$. *Suppose that* $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1})$,

$$f_j \in H_0^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}}(0,T) \qquad \textit{for} \quad 0 \le j \le N-1\,,$$

and $(g_0, g_2, \ldots, g_{n-1}) = \mathbf{S}_T\mathbf{f}$. *Then there exists a positive constant* C *such that*

$$\|\omega_{n-j}g_{j-1}\|_{L_2^{\tilde{n}}(\partial\mathcal{D}_{R,+})} \le C\sum_{l=0}^{N-1}\|f_l\|_{H^{\frac{1}{2}+\frac{(2\tilde{n}-2l-1)}{2n}}(0,T)} \qquad \textit{for} \quad 1 \le j \le n\,. \tag{3.9}$$

Proof. Let $\tilde{f}_j$ be the trivial extension of f_j to $\mathbb{R}$. It follows from Lemma 3.3 that $\tilde{f}_j \in H^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}}(\mathbb{R})$ and

$$\int_{-\infty}^{\infty}(1+|\tau|^{1+\frac{(2\tilde{n}-2j-1)}{n}})|\mathscr{F}_j(\tau)|^2\,d\tau \le C\|f_j\|^2_{H^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}}(0,T)}\,, \tag{3.10}$$

where $\mathscr{F}_j$ is the Fourier transform of $\tilde{f}_j$.

Note that, by (3.2), $g_j(k) = \mathscr{F}_j(-\omega(k))$ for $0 \le j \le N-1$. Since $|\omega(k)| = R$ or $\operatorname{Im}\omega(k) = 0$ on $\partial\mathcal{D}_{R,+}$, it follows from (3.10) and a change of variables that, for $0 \le j \le N-1$,

$$\| |\omega(k)|^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}} |g_j(k)| |\omega'(k)|^{1/2} \|_{L_2(\partial\mathcal{D}_{R,+})} \le C \|f_j\|_{H^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}}(0,T)}, \tag{3.11}$$

which implies (3.9) for $1 \le j \le N$.

On the other hand, in view of (1.13), (2.18), (2.22), (2.24), (3.3), (3.4) and (3.7), we have

$$|k^{j-1} g_{n-j}(k)| \le C \sum_{\ell=1}^{n-N} \sum_{s=1}^{N} \left| \omega_{n-s}\big(\lambda_{N+\ell,m}(k)\big) g_{s-1}(k) \right| \tag{3.12}$$

for $k \in \partial\mathcal{D}_{R,m}$ and $1 \le m \le N$. The estimate (3.9) for $N+1 \le j \le n$ follows from (3.12), (2.18) and the estimates for $1 \le j \le N$ that have already been established. ∎

Since $\tilde{n} \ge 1$, we have immediately the following corollary to Lemma 3.4.

Corollary 3.1. *Under the assumptions of Lemma 3.4, the functions $\omega_{n-j}(k) g_{j-1}(k)$ for $1 \le j \le n$ belong to $L_1(\partial\mathcal{D}_{R,+}, |dk|)$.*

Let $\mathcal{C}_R$ (cf. Figure 3.1) be the contour (in the complex τ plane) that goes from ∞ to $-R$ along the negative real axis, then from $-R$ to R along the upper half of the circle $|\tau| = R$, and finally goes from R to ∞ along the positive real axis. Then $\tau = \omega(k)$ is a biholomorphic map between the closure of $\mathcal{D}_{R,m}$ and the closure of the domain Ω_R above $\mathcal{C}_R$ in the upper-half τ-plane, and its inverse will be denoted by $k = \vartheta_m(\tau)$.

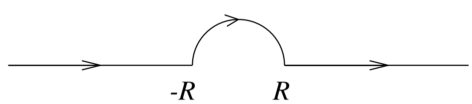

Figure 3.1

The following corollary to Lemma 3.4 follows immediately by the change of variable $\tau = \omega(k)$.

Corollary 3.2. *Under the assumptions of Lemma 3.4, the functions*

$$[\vartheta_m(\tau)]^l \omega_{n-j}\big(\vartheta_m(\tau)\big) g_{j-1}\big(\vartheta_m(\tau)\big) \Big[\omega'\big(\vartheta_m(\tau)\big)\Big]^{-1}$$

belongs to the space $L_2^{\frac{1}{2}+\frac{(2\tilde{n}-2l-1)}{2n}}(\partial\Omega_R)$ for $0 \le l \le n-1$, $1 \le j \le n$ and $1 \le m \le N$.

The relevancy of Corollary 3.2 is given by the next lemma.

Lemma 3.5. *Let $G \in L_2^s(\partial\Omega_R) \cap \mathscr{H}(\bar{\Omega}_R)$. Then the function $F(r) = \int_{\partial\Omega_R} e^{-irz} G(z)\, dz$ in the real variable r belongs to the Sobolev space $H^s(\mathbb{R})$.*

Proof. Let $z = z_R + iz_I$ and $\psi(z_R, z_I)$ be a smooth function with compact support that is identically 1 on an open set containing the circle defined by $|z| = R$. We can write

$$F(r) = \int_{\partial\Omega_R} e^{-irz}\psi(z_R, z_I)G(z)\,dz + \int_{\partial\Omega_R} e^{-irz}[1 - \psi(z_R, z_I)]G(z)\,dz\,. \tag{3.13}$$

Using integration by parts, the smoothness of $\psi(z_R, z_I)G(z)$ and its holomorphy at $-R$ and R, we see that the first integral on the right-hand side of (3.13) defines a function in the space $\mathscr{L}(\mathbb{R})$. Since the second integral on the right-hand side of (3.13) defines a function in $H^s(\mathbb{R})$, the lemma follows. □

4. The IBV Problem With Smooth Data

We consider in this section the IBV (1.1) with smooth data, where N is given by (1.3). The precise assumptions on the initial and boundary data are given in (1.17)–(1.19).

Let the numbers $a_{l,m} = (\partial_t^m D^l q)(0,0)$ be computed from $q_0(x)$ and (1.1a). The compatibility condition (1.19) means that $a_{l,m} = f_l^{(m)}(0)$ for $0 \le l \le N-1$ and for all nonnegative integers m.

Let $\mathscr{S}\big([0,\infty)\big)$ be the Schwartz space of smooth functions on $[0,\infty)$ that decrease rapidly as $x \to \infty$. It is the restriction of the Schwartz space $\mathscr{S}(\mathbb{R})$ to $[0,\infty)$, and is a Fréchet space under the metric defined by

$$d(f,g) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{2^{m+l}} \frac{\sup_{x\in[0,\infty)} |x^m D^l(f-g)(x)|}{1 + \sup_{x\in[0,\infty)} |x^m D^l(f-g)(x)|}\,.$$

Let $\tilde{q}_0$ be a Schwartz function that is an extension of q_0 to the whole real line and define, for $(x,t) \in [0,\infty) \times [0,T]$,

$$\tilde{q}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i[x\xi - \omega(\xi)t]} a(\xi)\,d\xi\,, \tag{4.1}$$

where $a(\xi)$ is the Fourier transform of $\tilde{q}_0(x)$. It is easy to see that $\tilde{q}$ is a solution of (1.1a) and $t \longrightarrow \tilde{q}(\cdot,t)$ is a C^∞ map from $[0,T]$ into $\mathscr{S}\big([0,\infty)\big)$. Therefore, by considering the difference $q - \tilde{q}$, it suffices for us to study the reduced IBV problem

$$q_t + i\omega(D)q = 0 \qquad 0 < x < \infty\,, \quad 0 < t < T\,, \tag{4.2a}$$
$$q(x,0) = 0 \qquad 0 < x < \infty\,, \tag{4.2b}$$
$$D^l q(0,t) = f_l(t) \qquad 0 < t < T\,, \quad 0 \le l \le N-1\,, \tag{4.2c}$$

under the assumption (1.18) and that

$$f_l(t) \text{ vanishes to all orders at } t = 0 \text{ for } 0 \le l \le N-1. \tag{4.3}$$

Motivated by Proposition 1.2 and the discussion in Remark 2.1, we first extend f_l to be a C^∞ function on $[0, T+1]$ such that

$$f_l(t) \text{ vanishes to all orders at } t = T+1 \text{ for } 0 \le l \le N-1. \tag{4.4}$$

Then we define, using the spectral map from Section 3,

$$(\tilde{Q}_0, \tilde{Q}_1, \ldots, \tilde{Q}_{n-1}) = \mathbf{S}_{T+1}(f_0, f_1, \ldots, f_{N-1}), \tag{4.5}$$

and (cf. (1.11)–(1.12))

$$q(x,t) = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{\partial \mathcal{D}_{R,+}} e^{i[kx - \omega(k)t]} \omega_{n-j}(k) \tilde{Q}_{j-1}(k)\, dk\,. \tag{4.6}$$

Our goal is to show that $q(x,t)$ satisfies (4.2).

First we note that, by (4.3), (4.4) and Lemma 3.2, the functions $(d^\beta \tilde{Q}_j / dk^\beta)(k)$, for $0 \le j \le n-1$ and $\beta = 0, 1, 2, \ldots$, are $O(|k|^{-\alpha})$ for any positive integer α as $k \to \infty$ in $\overline{\mathcal{D}_{R,+}}$. It follows immediately that $q(x,t)$ is a C^∞ function on $[0,\infty) \times [0,T]$ that satisfies (4.2a). Moreover, integration by parts implies that $t \longrightarrow q(\cdot, t)$ is a C^∞ map from $[0,T]$ into $\mathscr{S}\big([0,\infty)\big)$.

From (4.6) we obtain easily

$$q(x,0) = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{\partial \mathcal{D}_{R,+}} e^{ikx} \omega_{n-j}(k) \tilde{Q}_{j-1}(k)\, dk = 0 \qquad \forall\, x \ge 0\,, \tag{4.7}$$

since $\omega_{n-j}(k)\tilde{Q}_{j-1}(k)$ is holomorphic on $\mathcal{D}_{R,+}$ and $O(|k|^{-2})$ as $k \to \infty$ in $\overline{\mathcal{D}_{R,+}}$. So the initial condition (4.2b) is satisfied.

From Lemma 3.1 we have

$$\sum_{j=1}^{n} \omega_{n-j}\big(\lambda_{l,m}(k)\big) \tilde{Q}_{j-1}(k) = 0 \quad \text{for } k \in \overline{\mathcal{D}_{R,m}}\,, 1 \le m \le N\,, N+1 \le l \le n\,, \tag{4.8}$$

$$\tilde{Q}_j\big(\lambda_{l,m}(k)\big) = \tilde{Q}_j(k) \quad \text{for} \quad 1 \le j \le n-1\,,\ k \in \overline{\mathcal{D}_{R,m}}\,,\ 1 \le m, l \le N\,. \tag{4.9}$$

In fact, for $0 \le j \le N-1$,

$$\tilde{Q}_j(k) = \int_0^{T+1} e^{i\omega(k)t} f_j(t)\, dt \tag{4.10}$$

is defined for all $k \in \mathbb{C}$ and

$$\tilde{Q}_j(\lambda_{l,m}(k)) = \tilde{Q}_j(k) \qquad \text{for} \quad 0 \le j \le N-1\,,\ k \in \overline{\mathcal{D}_{R,m}}\,,\ 1 \le m, l \le n\,. \tag{4.11}$$

The relations (2.17) and (4.9) also make it possible to extend the functions $\tilde{Q}_j$, $N \le j \le n-1$, to the lower half-plane such that

$$\tilde{Q}_j\big(\lambda_{l,m}(k)\big) = \tilde{Q}_j(k) \quad \text{for} \quad N \le j \le n-1\,,\ k \in \overline{\mathcal{D}_{R,m}}\,,\ 1 \le m, l \le n\,. \tag{4.12}$$

Combining (4.8), (4.11) and (4.12) we see that

$$\sum_{j=1}^{n} \omega_{n-j}(k) \tilde{Q}_{j-1}(k) = 0 \quad \text{for } k \in \overline{\mathcal{D}_{R,l}} \quad \text{and} \quad N+1 \le l \le n\,.$$

Therefore, we can rewrite (4.6) as

$$q(x,t)=\frac{1}{2\pi}\sum_{j=1}^{n}\int_{\partial\mathcal{D}_R} e^{i[kx-\omega(k)t]}\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk\,. \tag{4.13}$$

It follows from (4.13) that

$$D^m q(0,t)=\frac{1}{2\pi}\sum_{j=1}^{n}\int_{\partial\mathcal{D}_R} e^{-i\omega(k)t}k^m\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk\,. \tag{4.14}$$

Let $\mathcal{C}_R$ be the contour depicted in Figure 3.1. For $\tau\in\mathcal{C}_R$ the equation $\omega(k)=\tau$ has n solutions $k_1(\tau),k_2(\tau),\ldots,k_n(\tau)$ such that $k_l(\tau)\in\partial\mathcal{D}_{R,l}$. Using the invariance relations (4.11) and (4.12) we find

$$\begin{aligned}&\int_{\partial\mathcal{D}_R} e^{-i\omega(k)t}k^m\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk\\ &=\int_{\mathcal{C}_R} e^{-it\tau}\left(\sum_{l=1}^{n}\frac{\big[k_l(\tau)\big]^m\omega_{n-j}\big(k_l(\tau)\big)}{\omega'\big(k_l(\tau)\big)}\right)\tilde{Q}_{j-1}\big(k_1(\tau)\big)\,d\tau\,.\end{aligned} \tag{4.15}$$

Note that, by the residue theorem (cf. [6]), we have

$$\sum_{l=1}^{n}\frac{\big[k_l(\tau)\big]^m\omega_{n-j}\big(k_l(\tau)\big)}{\omega'\big(k_l(\tau)\big)}=\lim_{r\to\infty}\frac{1}{2\pi i}\int_{C_r}\frac{z^m\omega_{n-j}(z)}{\omega(z)-\tau}\,dz\,, \tag{4.16}$$

where C_r is the counterclockwise oriented circle defined by $|z|=r$. Combining (1.2), (1.13) and (4.16) we see that, for $0\le m\le n-1$ and $1\le j\le n$,

$$\sum_{l=1}^{n}\frac{\big[k_l(\tau)\big]^m\omega_{n-j}\big(k_l(\tau)\big)}{\omega'\big(k_l(\tau)\big)}=\begin{cases}1 & \text{if } m=j-1\\ 0 & \text{if } m\neq j-1\end{cases}\,. \tag{4.17}$$

It then follows from (4.14), (4.15) and (4.17) that

$$D^m q(0,t)=\frac{1}{2\pi}\int_{\mathcal{C}_R} e^{-it\tau}\tilde{Q}_m\big(k_1(\tau)\big)\,d\tau\quad 0\le m\le n-1\,.$$

In particular, in view of (4.10) and the relation $\omega\big(k_1(\tau)\big)=\tau$, we have

$$D^m q(0,t)=\frac{1}{2\pi}\int_{\mathcal{C}_R} e^{-it\tau}\hat{f}_m(\tau)\,d\tau\quad\text{for}\quad 0\le m\le N-1\,, \tag{4.18}$$

where

$$\hat{f}_m(\tau)=\int_0^{T+1} e^{i\tau t}f_m(t)\,dt\,. \tag{4.19}$$

The Fourier inversion formula and (4.18)–(4.19) imply that $q(x,t)$ satisfies the boundary conditions in (4.2c).

We have thus established the existence part of Theorem 1.1. It remains only to establish the uniqueness part. Let q_1 and q_2 be two such solutions and $u = q_1 - q_2$. Then u is a solution of (1.1a) with homogeneous initial and boundary conditions. It follows from (1.1a), the homogeneous boundary conditions and integration by parts that, in the case of an even n, or an odd n with $\alpha_n > 0$ (so $N = (n+1)/2$), we have

$$\frac{d}{dt}\int_0^\infty |u(x,t)|^2\,dx = 0\,,$$

whereas in the case of an odd n with $\alpha_n < 0$ (so $N = (n-1)/2$), we have

$$\frac{d}{dt}\int_0^\infty |u(x,t)|^2\,dx - \alpha_n |D^N q(0,t)|^2 = 0\,.$$

Therefore, in general we have

$$\frac{d}{dt}\int_0^\infty |u(x,t)|^2\,dx \le 0 \quad \text{for} \quad 0 < t < T\,, \tag{4.20}$$

which together with the homogeneous initial condition imply $u = 0$.

5. The IBV Problem With Sobolev Data

We consider in this section again the IBV problem (1.1) but with the assumptions on the initial and boundary data given by (1.20)–(1.22). Note that the pointwise evaluations in (1.22) make sense because $N \le \tilde{n}$. More precisely, we are looking for a weak solution of (1.1) with the properties stated in (1.23)–(1.26).

Let $\tilde{q}_0 \in H^{\tilde{n}}(\mathbb{R})$ be an extension of q_0 and define $\tilde{q}(x,t)$ by (4.1), where $a(\xi)$ is the Fourier transform of $\tilde{q}_0(x)$.

Lemma 5.1. *The function $\tilde{q}$ has the following properties:*

(i) *The map $t \longrightarrow \tilde{q}(\cdot,t)$ is continuous from $\mathbb{R}$ into $H^{\tilde{n}}(\mathbb{R})$.*

(ii) *Given any $\phi \in C_c^\infty(\mathbb{R})$ such that $D^j\phi(0) = 0$ for $0 \le j \le n - \tilde{n} - 1$, the function $\big(\tilde{q}(\cdot,t),\phi\big)_{L_2(0,\infty)}$ is differentiable on $(0,T)$, and*

$$\frac{d}{dt}\big(\tilde{q}(\cdot,t),\phi\big)_{L_2(0,\infty)} = \big(\tilde{q}(\cdot,t), i\omega(D)\phi\big)_{L_2(0,\infty)} + \sum_{j=n-\tilde{n}}^{n-1}\big[\omega_{n-j-1}(D)\tilde{q}(0,t)\big]\overline{D^j\phi(0)}$$

for $0 < t < T$.

(iii) *The map $x \longrightarrow D^j\tilde{q}(x,\cdot)$ is continuous from $\mathbb{R}$ into $H^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}}(0,T)$ for $0 \le j \le n-1$.*

Proof. Properties (i) and (ii) follow easily from (4.1).

To prove (iii) we break (4.1) up into two integrals:

$$\tilde{q}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i[x\xi-\omega(\xi)t]}\psi(\xi)a(\xi)\,d\xi+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i[x\xi-\omega(\xi)t]}[1-\psi(\xi)]a(\xi)\,d\xi\,, \tag{5.1}$$

where $\psi\in C_c^\infty(\mathbb{R})$ is a smooth cut-off function that is identically 1 on the interval $[-L,L]$ and $|\omega(-L)|=|\omega(L)|=R$. Note that outside the interval $[-L,L]$ the function $\omega'(\xi)$ does not vanish (cf. (2.15)).

The first integral on the right-hand side of (5.1) defines a C^∞ function on $\mathbb{R}^2$ which clearly satisfies property (iii). We can rewrite the second integral as

$$\begin{aligned}&\frac{1}{2\pi}\int_{(-\infty,-L)\cup(L,\infty)}e^{i[x\xi-\omega(\xi)t]}[1-\psi(\xi)]\frac{a(\xi)}{\omega'(\xi)}\,\omega'(\xi)d\xi\\&=\frac{1}{2\pi}\int_{\omega(-\infty)}^{\omega(-L)}e^{i[x\xi_1(\eta)-\eta t]}[1-\psi(\xi_1(\eta))]\frac{a(\xi_1(\eta))}{\omega'(\xi_1(\eta))}\,d\eta\\&\qquad+\frac{1}{2\pi}\int_{\omega(L)}^{\omega(\infty)}e^{i[x\xi_2(\eta)-\eta t]}[1-\psi(\xi_2(\eta))]\frac{a(\xi_2(\eta))}{\omega'(\xi_2(\eta))}\,d\eta\,,\end{aligned} \tag{5.2}$$

where $\xi_1(\eta)$ and $\xi_2(\eta)$ are the inverses of $\eta=\omega(\xi)$ on the intervals $(-\infty,-L]$ and $[L,\infty)$ respectively.

We have

$$\begin{aligned}&\left|\int_{\omega(-\infty)}^{\omega(-L)}|\eta|^{1+\frac{(2\tilde{n}-2j-1)}{n}}|\xi_1(\eta)|^{2j}\left|\frac{a(\xi_1(\eta))}{\omega'(\xi_1(\eta))}\right|^2\,d\eta\right|\\&=\int_{-\infty}^{-L}|\omega(\xi)|^{1+\frac{(2\tilde{n}-2j-1)}{n}}|\xi|^{2j}\frac{|a(\xi)|^2}{|\omega'(\xi)|}\,d\xi\le C\int_{-\infty}^{-L}|\xi|^{2\tilde{n}}|a(\xi)|^2\,d\xi<\infty\,,\end{aligned}$$

and similarly,

$$\left|\int_{\omega(L)}^{\omega(\infty)}|\eta|^{1+\frac{(2\tilde{n}-2j-1)}{n}}|\xi_2(\eta)|^{2j}\left|\frac{a(\xi_2(\eta))}{\omega'(\xi_2(\eta))}\right|^2\,d\eta\right|<\infty\,.$$

Therefore the functions defined by the integrals on the right-hand side of (5.2) also satisfy property (iii). □

By considering the difference between q and $\tilde{q}$, it suffices to solve the reduced IBV problem (4.2) for a weak solution q that satisfies (1.23)–(1.26), under the assumption (1.21) and that

$$f_l(0)=0\quad\text{for}\quad 0\le l\le N-1\,. \tag{5.3}$$

Let f_l, $0\le l\le N-1$, be extended to $(0,T+1)$ such that

$$f_l\in H^{\frac{1}{2}+\frac{(2\tilde{n}-2l-1)}{2n}}(0,T+1) \tag{5.4}$$

and

$$f_l(t)=0\qquad\text{for}\quad t\ge T+(1/2)\,. \tag{5.5}$$

It follows from Lemma 3.3, (5.4)–(5.5) that, for $0 \le l \le N-1$, the extended function f_l satisfies

$$f_l \in H^{\frac{1}{2}+\frac{(2\tilde{n}-2l-1)}{2n}}(0,T+1)\,. \tag{5.6}$$

The construction of the weak solution for the reduced IBV problem is now identical with the construction in Section 4: We first define $\tilde{Q}_0,\dots,\tilde{Q}_{n-1}$ by (4.5) and then define $q(x,t)$ by (4.6).

From (5.6) and Corollary 3.1 we see that the integrals in (4.6) are well-defined, and property (1.26) follows from Corollary 3.2, Lemma 3.5 and a change of variables.

Next we consider the property (1.23). Let $\psi(k_R,k_I)$ be a smooth cut-off function in the real variables k_R and k_I such that $\psi(k_R,k_I)=1$ when $|\omega(k_R+ik_I)|<2R$ and $\psi(k_R,k_I)=0$ when $|\omega(k_R+ik_I)|>3R$. From (4.6) we have

$$\begin{aligned} q(x,t) = {}& \frac{1}{2\pi}\sum_{j=1}^{n}\int_{\partial\mathcal{D}_{R,+}} e^{i[kx-\omega(k)t]}\psi(k_R,k_I)\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk \\ & + \frac{1}{2\pi}\sum_{j=1}^{n}\int_{\partial\mathcal{D}_{R,+}} e^{i[kx-\omega(k)t]}[1-\psi(k_R,k_I)]\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk\,. \end{aligned} \tag{5.7}$$

Since $\psi(k_R,k_I)\omega_{n-j}(k)\tilde{Q}_{j-1}(k)$ is holomorphic at the corners of $\partial\mathcal{D}_{R.+}$, it follows from integration by parts that

$$t \longrightarrow \frac{1}{2\pi}\sum_{j=1}^{n}\int_{\partial\mathcal{D}_{R,+}} e^{i[kx-\omega(k)t]}\psi(k_R,k_I)\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk \quad \text{defines a}$$

$$C^\infty \text{ map from } \mathbb{R} \text{ into } \mathscr{L}\big([0,\infty)\big). \tag{5.8}$$

The second sum on the right-hand side of (5.7) can be written as

$$\frac{1}{2\pi}\sum_{j=1}^{n}\sum_{l=1}^{2N}\int_{\mathscr{C}_l} e^{i[kx-\omega(k)t]}[1-\psi(k_R,k_I)]\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk\,,$$

where the disjoint smooth contours $\mathscr{C}_l$ for $1\le l\le 2N$ come from the part of $\partial\mathcal{D}_{R,+}$ that is in the domain defined by $|\omega(k)|\ge 2R$.

Lemma 5.2. *Let $u\in L_2(0,\infty)$ and $v(k)=\int_0^\infty e^{ikx}u(x)\,dx$. Then $v(k)\in L_2(\mathscr{C}_l,|dk|)$ for $1\le l\le 2N$, and*

$$\|v\|_{L_2(\mathscr{C}_l,|dk|)} \le C\|u\|_{L_2(0,\infty)}\,. \tag{5.9}$$

Proof. If $\mathscr{C}_l$ is part of the real axis, then (5.9) follows from Plancherel's theorem.

Let $\mathscr{C}_l$ be a contour strictly inside the upper half k-plane. Without lost of generality we can take α_n (the leading coefficient of $\omega(k)$) to be positive, since the treatment for the case where α_n is negative is completely analogous. Furthermore we may suppose that $\mathscr{C}_l$ is defined by $k=g(\tau)$ for $\tau\in\mathbb{R}$ and $\tau>2R$, where $g(\tau)$ is one of the branches of the inverse of $\tau=\omega(k)$ such that

$$k = e^{i(j/n)\pi}(\tau/\alpha_n)^{1/n} - \alpha_{n-1}/(n\alpha_n) + O(\tau^{-1/n}) \quad \text{as} \quad \tau\to\infty\,, \tag{5.10}$$

and j is an integer satisfying $0 < (j/n) < 1$.

By a classical result (cf. [8], §11.7) the function $w(r) = v\big(re^{i(j/n)\pi}\big)$ belongs to $L_2(a,\infty)$, where $a = (2R/\alpha_n)^{1/n}$. Moreover, we have

$$\|w\|_{L_2(a,\infty)} \le C\|u\|_{L_2(0,\infty)}\,. \tag{5.11}$$

From (5.10) we see that the contour $\mathscr{C}_l$ can be parameterized by

$$k = h(r) = g(\alpha_n r^n) = re^{i(j/n)\pi)} - \alpha_{n-1}/(n\alpha_n) + O(r^{-1}) \quad \text{for} \quad a < r < \infty. \tag{5.12}$$

Using (5.12) and the definition of $v(k)$ we find

$$\begin{aligned} \big|v\big(h(r)\big) - w(r)\big| &\le \int_0^\infty \left|e^{ixh(r)} - e^{ixre^{i(j/n)\pi}}\right| |u(x)|\,dx \\ &\le C\|u\|_{L_2(0,\infty)} r^{-(3/2)} \qquad \text{for} \quad a < r < \infty\,. \end{aligned} \tag{5.13}$$

The estimate (5.9) follows from (5.11) and (5.13). □

By Lemma 5.2 and duality we immediately obtain the following corollary.

Corollary 5.1. *The map*

$$\eta \longrightarrow \int_{\mathscr{C}_l} e^{ikx}\eta(k)\,dk$$

is a bounded linear map from $L_2\big(\mathscr{C}_l, |dk|\big)$ *into* $L_2(0,\infty)$.

Combining (5.6), Lemma 3.4 and Corollary 5.1 we see that

$$t \longrightarrow \int_{\mathscr{C}_l} e^{i[kx-\omega(k)t]}[1-\psi(k_R,k_I)]\omega_{n-j}(k)\tilde{Q}_{j-1}(k)\,dk \quad \text{defines a continuous}$$
map from $\mathbb{R}$ into $H^{\tilde{n}}(0,\infty)$. (5.14)

Property (1.23) follows from (5.7), (5.8) and (5.14).

Let $\phi \in C_c^\infty(\mathbb{R})$ satisfy the assumptions in (1.25) and $\eta \in C_c^\infty(0,T)$. Since the space $C_c^\infty(0,T+1)$ is dense in the space $H_0^{\frac{1}{2}+\frac{(2\tilde{n}-2j-1)}{2n}}(0,T+1)$, Theorem 1.1 and a density argument imply that $q(x,t)$ satisfies (4.2b)–(4.2c), and

$$\begin{aligned} 0 = \int_0^T \Big[&\eta'(t)\big(q(\cdot,t),\phi(x)\big)_{L_2(0,\infty)} + \eta(t)\big(q(\cdot,t), i\omega(D)\phi(x)\big)_{L_2(0,\infty)} \\ &+ \eta(t)\sum_{j=n-\tilde{n}}^{n-1}\big[\omega_{n-j-1}(D)q(0,t)\big]\overline{D^j\phi(0)}\Big]\,dt\,. \end{aligned} \tag{5.15}$$

Property (1.25) follows immediately from (5.15).

We have thus proved the existence part of Theorem 1.2. It only remains to prove the uniqueness of the weak solution satisfying (1.23)–(1.25). Let q_1 and q_2 be two such weak solutions. Then we have

$$u = q_1 - q_2 \in C\big(0,T; H_0^{N-1}(0,\infty)\big) \cap C\big(0,T; H^{\tilde n}(0,\infty)\big)\,, \tag{5.16}$$

$$u(x,0) = 0\,. \tag{5.17}$$

Moreover, given $\phi \in C_c^\infty(\mathbb{R})$ such that $D^j\phi(0) = 0$ for $0 \le j \le n - \tilde n - 1$, we have

$$\frac{d}{dt}\big(u(\cdot,t),\phi\big)_{L_2(0,\infty)} = \big(u(\cdot,t), i\omega(D)\phi\big)_{L_2(0,\infty)} \tag{5.18a}$$

for $0 < t < T$ if $\tilde n = N$, and

$$\frac{d}{dt}\big(u(\cdot,t),\phi\big)_{L_2(0,\infty)} = \big(u(\cdot,t), i\omega(D)\phi\big)_{L_2(0,\infty)} + \alpha_n D^N u(0,t)\overline{D^N\phi(0)} \tag{5.18b}$$

for $0 < t < T$ if $\tilde n = N + 1$. We can then deduce (4.20) from (5.16) and (5.18) through mollification. Combining (4.20) and (5.17) we have $u = 0$.

Alternatively, we can prove the uniqueness of the weak solution using the existence of solution for the adjoint problem. Let $0 < t < T$. It follows easily from (1.25) and (5.17) that given any C^∞ map $\lambda : [0,t] \longrightarrow \mathscr{S}\big([0,\infty)\big)$ such that $D^j\lambda(0,t') = 0$ for $0 \le t' \le t$ and $0 \le j \le n - \tilde n - 1$, we have

$$\begin{aligned}\big(u(\cdot,t),\lambda(\cdot,t)\big)_{L_2(0,\infty)} = \int_0^t \bigg[&\big(u(\cdot,t'),\lambda_t(\cdot,t') + i\omega(D)\lambda(\cdot,t')\big)_{L_2(0,\infty)} \\ &+ \sum_{j=n-\tilde n}^{n-1} [\omega_{n-j-1}(D)u(0,t')]\overline{D^j\lambda(0,t')}\bigg]dt'\,.\end{aligned} \tag{5.19}$$

When n is even or when n is odd and $\alpha_n > 0$ (hence $\tilde n = N$ in both cases), Theorem 1.1 applied to the backward problem shows that given any $\phi \in \mathscr{S}\big([0,\infty)\big)$, there exists a C^∞ map $\lambda : [0,t] \longrightarrow \mathscr{S}\big([0,\infty)\big)$ such that $\lambda_t + i\omega(D)\lambda = 0$ on $[0,\infty) \times [0,t]$, $\lambda(\cdot,t) = \phi$, and $D^j\lambda(0,t') = 0$ for $0 \le j \le n - \tilde n - 1$ and $0 \le t' \le t$. When n is odd and $\alpha_n < 0$ (hence $\tilde n = N + 1$), we can further require $D^{n-\tilde n}\lambda(0,t') = 0$ for $0 \le t' \le t$. Since $D^j u(0,t) = 0$ for $0 \le j \le N - 1$ and $0 \le t \le T$, in all the cases we obtain from (5.19)

$$\big(u(\cdot,t),\phi\big)_{L_2(0,\infty)} = 0 \qquad \forall\, \phi \in \mathscr{S}\big([0,\infty)\big)\,.$$

It follows that $u(\cdot,t) = 0$ for $0 \le t \le T$.

Appendix

A.1. Illustration of $\mathcal{D}_+$ and $\mathcal{D}_-$

The domains D_+ and D_- for some simple examples are illustrated below. We will denote by $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) the smallest (resp. largest) integer greater (resp. less) than or equal to the real number x.

Example A.1.1. For $\omega(D) = D^n$, the domains $\mathcal{D}_\pm$ are defined by

$$\begin{aligned}\mathcal{D}_+ &= \{k \in \mathbb{C} : \operatorname{Im} k^n > 0 \,,\ \operatorname{Im} k > 0\} \\ &= \bigcup_{j=1}^{\lceil n/2 \rceil} \left\{ k : \ k = re^{i\theta} \,,\ \left(\frac{2j-2}{n}\right)\pi < \theta < \left(\frac{2j-1}{n}\right)\pi \,,\ 0 < r < \infty \right\} ,\end{aligned}$$

$$\begin{aligned}\mathcal{D}_- &= \{k \in \mathbb{C} : \operatorname{Im} k^n > 0 \,,\ \operatorname{Im} k < 0\} \\ &= \bigcup_{j=\lceil n/2 \rceil + 1}^{n} \left\{ k : \ k = re^{i\theta} \,,\ \left(\frac{2j-2}{n}\right)\pi < \theta < \left(\frac{2j-1}{n}\right)\pi \,,\ 0 < r < \infty \right\} .\end{aligned}$$

The cases for $n = 2, 3$ and 4 are depicted in Figure A.1.1.

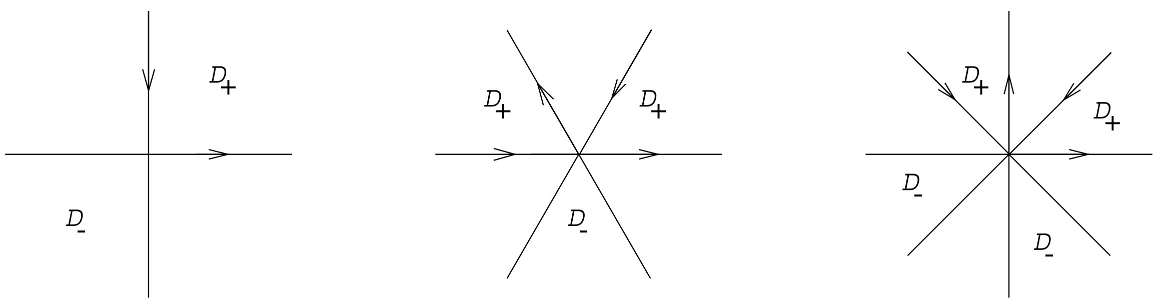

Figure A.1.1

Example A.1.2. For $\omega(D) = -D^n$, the domains $\mathcal{D}_\pm$ are defined by

$$\begin{aligned}\mathcal{D}_+ &= \{k \in \mathbb{C} : -\operatorname{Im} k^n > 0 \,,\ \operatorname{Im} k > 0\} \\ &= \bigcup_{j=1}^{\lfloor n/2 \rfloor} \left\{ k : \ k = re^{i\theta} \,,\ \left(\frac{2j-1}{n}\right)\pi < \theta < \left(\frac{2j}{n}\right)\pi \,,\ 0 < r < \infty \right\} ,\end{aligned}$$

$$\begin{aligned}\mathcal{D}_- &= \{k \in \mathbb{C} : -\operatorname{Im} k^n > 0 \,,\ \operatorname{Im} k > 0\} \\ &= \bigcup_{j=\lfloor n/2 \rfloor + 1}^{n} \left\{ k : \ k = re^{i\theta} \,,\ \left(\frac{2j-1}{n}\right)\pi < \theta < \left(\frac{2j}{n}\right)\pi \,,\ 0 < r < \infty \right\} .\end{aligned}$$

The cases for $n = 2, 3$ and 4 are depicted in Figure A.1.2.

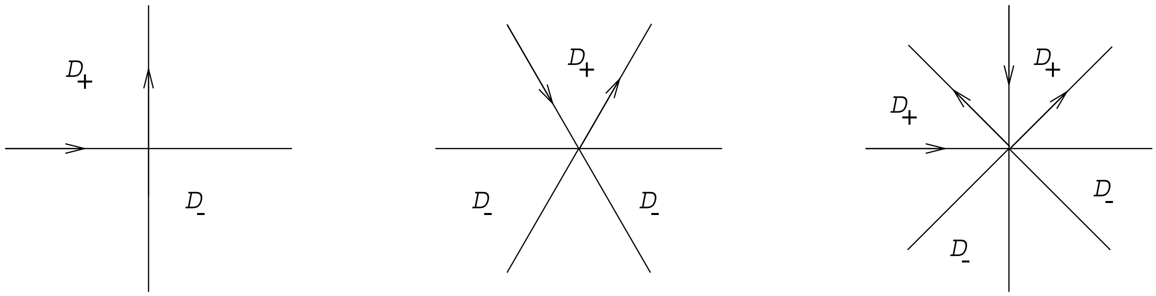

Figure A.1.2

Example A.1.3. For $\omega(D) = 2D^3 + D$, the domains $\mathcal{D}_\pm$ (cf. Figure A.1.3) are defined by

$$\begin{aligned}\mathcal{D}_+ &= \{k \in \mathbb{C} :\ \mathrm{Im}\,(2k^3 + k) > 0\,,\ \mathrm{Im}\,k > 0\}\\ &= \{k = (x+iy) :\ y > 0\,,\ 6x^2 - 2y^2 + 1 > 0\}\,,\end{aligned}$$

$$\begin{aligned}\mathcal{D}_- &= \{k \in \mathbb{C} :\ \mathrm{Im}\,(2k^3 + k) > 0\,,\ \mathrm{Im}\,k < 0\}\\ &= \{k = (x+iy) :\ y < 0\,,\ 6x^2 - 2y^2 + 1 < 0\}\,.\end{aligned}$$

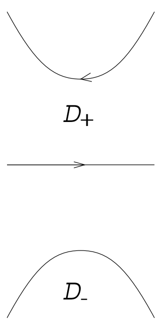

Figure A.1.3

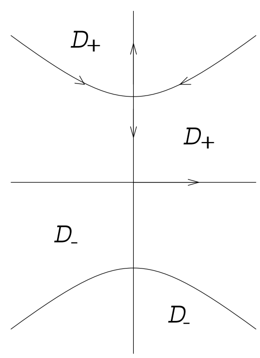

Figure A.1.4

Example A.1.4. For $\omega(D) = D^4 + D^2 + 1$, the domains $\mathcal{D}_\pm$ (cf. Figure A.1.4) are defined by

$$\begin{aligned}\mathcal{D}_+ &= \{k :\ \mathrm{Im}\,(k^4 + k^2 + 1) > 0\,,\ \mathrm{Im}\,k > 0\}\\ &= \{k = x+iy :\ y > 0, x > 0, 2x^2 - 2y^2 + 1 > 0\}\\ &\quad \cup \{k = x+iy :\ y > 0, x < 0, 2x^2 - 2y^2 + 1 < 0\}\,,\end{aligned}$$

$$\begin{aligned}\mathcal{D}_- &= \{k :\ \mathrm{Im}\,(k^4 + k^2 + 1) > 0\,,\ \mathrm{Im}\,k < 0\}\\ &= \{k = x+iy :\ y < 0, x > 0, 2x^2 - 2y^2 + 1 < 0\}\\ &\quad \cup \{k = x+iy :\ y < 0, x < 0, 2x^2 - 2y^2 + 1 > 0\}\,.\end{aligned}$$

A.2. Illustration of $\tilde{Q}(k)$

We now give the explicit form of the system of equations in (1.15) for $2 \leq n \leq 5$.

$\underline{n=2}$

In this case

$$\omega(k) = \alpha_2 k^2 + \alpha_1 k + \alpha_0 , \quad N = 1 , \quad l = 2 , \quad m = 1 .$$

Thus the system (1.15) is the single equation

$$\omega_1\big(\lambda_{2,1}(k)\big)\tilde{Q}_0(k) + \omega_0 \tilde{Q}_1(k) = -\hat{q}_0\big(\lambda_{2,1}(k)\big), \quad k \in \mathcal{D}_{R,1} , \tag{A.2.1}$$

where

$$\omega_0 = \alpha_2, \quad \omega_1(k) = \alpha_2 k + \alpha_1, \tag{A.2.2}$$

and $\lambda_{2,1}(k)$ is the solution of the equation $\omega(\lambda_{2,1}(k)) = \omega(k)$ satisfying

$$\lambda_{2,1}(k) \sim e^{i\pi} k, \quad k \to \infty \quad \text{in} \quad \mathcal{D}_{R,1} . \tag{A.2.3}$$

Given $\tilde{Q}_0(k)$, equation $(A.2.1)$ yields $\tilde{Q}_1(k)$ for $k \in D_1$.

$\underline{n=3}$

In this case

$$\omega(k) = \alpha_3 k^3 + \alpha_2 k^2 + \alpha_1 k + \alpha_0 .$$

There exist two subcases.

(i) $\alpha_3 > 0$

Then

$$N = 2 , \quad l = 3, \quad m = 1, 2 .$$

Thus the system (1.15) becomes a single equation

$$\omega_2\big(\lambda_{3,m}(k)\big)\tilde{Q}_0(k) + \omega_1\big(\lambda_{3,m}(k)\big)\tilde{Q}_1(k) + \omega_0 \tilde{Q}_2(k) = -\hat{q}_0\big(\lambda_{3,m}(k)\big) , \quad k \in \mathcal{D}_{R,m} , \tag{A.2.4}$$

where

$$\omega_0 = \alpha_3 , \ \omega_1(k) = \alpha_3 k + \alpha_2 , \ \omega_2(k) = \alpha_3 k^2 + \alpha_2 k + \alpha_1 , \tag{A.2.5}$$

and $\lambda_{3,m}(k)$ is the solution of the equation $\omega(\lambda_{3,m}(k)) = \omega(k)$ satisfying

$$\lambda_{3,1}(k) \sim e^{\frac{4i\pi}{3}} k , \quad k \in \mathcal{D}_{R,1} ; \quad \lambda_{3,2}(k) \sim e^{\frac{2i\pi}{3}} k , \quad k \in \mathcal{D}_{R,2} . \tag{A.2.6}$$

Given $\tilde{Q}_0$ and $\tilde{Q}_1$, equation $(A.2.4)$ yields $\tilde{Q}_2(k)$ for $k \in D_m$, $m = 1, 2$.

(ii) $\alpha_3 < 0$

Then

$$N = 1 , \quad l = 2, 3 , \quad m = 1 .$$

Thus the system (1.15) becomes

$$\begin{aligned}&\omega_2\big(\lambda_{2,1}(k)\big)\tilde{Q}_0(k)+\omega_1\big(\lambda_{2,1}(k)\big)\tilde{Q}_1(k)+\omega_0\tilde{Q}_2(k)=-\hat{q}_0\big(\lambda_{2,1}(k)\big)\,,\\&\omega_2\big(\lambda_{3,1}(k)\big)\tilde{Q}_0(k)+\omega_1\big(\lambda_{3,1}(k)\big)\tilde{Q}_1(k)+\omega_0\hat{Q}_2(k)=-\hat{q}_0\big(\lambda_{3,1}(k)\big)\,,\quad k\in\mathcal{D}_{R,1}\,,\end{aligned}\tag{A.2.7}$$

where ω_0, ω_1, ω_2 are given by the equations in $(A.2.5)$ and $\lambda_{l,1}(k)$ are the solutions of $\omega\big(\lambda_{l,1}(k)\big)=\omega(k)$ satisfying

$$\lambda_{2,1}(k)\sim e^{\frac{2i\pi}{3}}k\,,\quad \lambda_{3,1}(k)\sim e^{\frac{4i\pi}{3}}k\,,\quad k\in D_1\,.\tag{A.2.8}$$

Given $\tilde{Q}_0$, the system $(A.2.7)$ yields $\tilde{Q}_1(k)$ and $\tilde{Q}_2(k)$ for $k\in D_1$. In this respect we note that since ω_0 and ω_1 are defined by $(A.2.5)$, the matrix that needs to be inverted admits the following LU decomposition

$$\begin{bmatrix}\alpha_3 & \alpha_3\lambda_{2,1}(k)+\alpha_2\\ \alpha_3 & \alpha_3\lambda_{3,1}(k)+\alpha_2\end{bmatrix}=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}\alpha_3 & \alpha_3\lambda_{2,1}(k)+\alpha_2\\ 0 & \alpha_3\big(\lambda_{3,1}(k)-\lambda_{2,1}(k)\big)\end{bmatrix}.\tag{A.2.9}$$

<u>$n=4$</u>

In this case

$$\omega(k)=\alpha_4k^4+\alpha_3k^3+\alpha_2k^2+\alpha_1k+\alpha_0\,,\quad N=2\,,\quad l=3,4\,,\quad m=1,2\,.$$

Thus the system (1.15) becomes

$$\begin{aligned}&\omega_3\big(\lambda_{3,m}(k)\big)\tilde{Q}_0(k)+\omega_2\big(\lambda_{3,m}(k)\big)\tilde{Q}_1(k)+\omega_1\big(\lambda_{3,m}(k)\big)\tilde{Q}_2(k)+\omega_0\tilde{Q}_3(k)=-\hat{q}_0\big(\lambda_{3,m}(k)\big)\,,\\&\omega_4\big(\lambda_{4,m}(k)\big)\tilde{Q}_0(k)+\omega_2\big(\lambda_{4,m}(k)\big)\tilde{Q}_1(k)+\omega_1\big(\lambda_{4,m}(k)\big)\tilde{Q}_2(k)+\omega_0\tilde{Q}_3(k)=-\hat{q}_0\big(\lambda_{4,m}(k)\big)\,,\\&\qquad\text{for}\quad k\in\mathcal{D}_{R,m}\,,\end{aligned}\tag{A.2.10}$$

where

$$\begin{aligned}&\omega_0=\alpha_4\,, && \omega_1(k)=\alpha_4k+\alpha_3\,,\\&\omega_2(k)=\alpha_4k^2+\alpha_3k+\alpha_2\,, && \omega_3(k)=\alpha_4k^3+\alpha_3k^2+\alpha_2k+\alpha_1\,,\end{aligned}\tag{A.2.11}$$

and $\lambda_{l,m}(k)$ are the solutions of the equation $\omega(\lambda_{l,m}(k))=\omega(k)$ satisfying

$$\begin{aligned}&\lambda_{3,1}(k)\sim e^{i\pi}k\,, && \lambda_{4,1}(k)\sim e^{\frac{3i\pi}{2}}k\,, && k\in\mathcal{D}_{R,1}\,;\\&\lambda_{3,2}(k)\sim e^{\frac{i\pi}{2}}k\,, && \lambda_{4,2}(k)\sim e^{i\pi}k\,, && k\in\mathcal{D}_{R,2}\,.\end{aligned}\tag{A.2.12}$$

Given $\tilde{Q}_0$ and $\tilde{Q}_1$, the equations of $(A.2.10)$ yield $\tilde{Q}_2(k)$ and $\tilde{Q}_3(k)$ for $k\in D_m$, $m=1,2$. Since ω_0 and ω_1 are defined by $(A.2.11)$, the relevant matrix admits the following LU decomposition

$$\begin{bmatrix}\alpha_4 & \alpha_4\lambda_{3,m}(k)+\alpha_3\\ \alpha_4 & \alpha_4\lambda_{4,m}(k)+\alpha_3\end{bmatrix}=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}\alpha_4 & \alpha_4\lambda_{3,m}(k)+\alpha_3\\ 0 & \alpha_4\big(\lambda_{4,m}(k)-\lambda_{3,m}(k)\big)\end{bmatrix}.\tag{A.2.13}$$

$\underline{n=5}$

In this case

$$\omega(k) = \alpha_5 k^5 + \alpha_4 k^4 + \alpha_3 k^3 + \alpha_2 k^2 + \alpha_1 k + \alpha_0 \,.$$

There exist two subcases.

(i) $\alpha_5 > 0$

Then

$$N = 3\,, \quad l = 4, 5\,, \quad m = 1, 2, 3\,.$$

Thus the system (1.15) becomes

$$\begin{aligned} &\omega_4\big(\lambda_{4,m}(k)\big)\tilde{Q}_0(k) + \omega_3\big(\lambda_{4,m}(k)\big)\tilde{Q}_1(k) + \cdots + \omega_0 \tilde{Q}_4(k) = -\hat{q}_0\big(\lambda_{4,m}(k)\big)\,, \\ &\omega_4\big(\lambda_{5,m}(k)\big)\tilde{Q}_0(k) + \omega_3\big(\lambda_{5,m}(k)\big)\tilde{Q}_1(k) + \cdots + \omega_0 \tilde{Q}_4(k) = -\hat{q}_0\big(\lambda_{5,m}(k)\big)\,, \\ &\hspace{20em} k \in \mathcal{D}_{R,m}\,, \end{aligned} \quad (A.2.14)$$

where

$$\begin{aligned} &\omega_0 = \alpha_5\,, \quad \omega_1(k) = \alpha_5 k + \alpha_4\,, \quad \omega_2(k) = \alpha_5 k^2 + \alpha_4 k + \alpha_3\,, \\ &\omega_3(k) = \alpha_5 k^3 + \alpha_4 k^2 + \alpha_3 k + \alpha_2\,, \quad \omega_4(k) = \alpha_5 k^4 + \alpha_4 k^3 + \alpha_3 k^2 + \alpha_2 k + \alpha_1\,, \end{aligned} \quad (A.2.15)$$

and $\lambda_{l,m}(k)$ are the solutions of the equation $\omega(\lambda_{l,m}(k)) = \omega(k)$ satisfying

$$\begin{aligned} &\lambda_{4,1}(k) \sim e^{\frac{6i\pi}{5}} k\,, \qquad \lambda_{5,1}(k) \sim e^{\frac{8i\pi}{5}} k\,, \qquad k \in \mathcal{D}_{R,1}\,; \\ &\lambda_{4,2}(k) \sim e^{\frac{4i\pi}{5}} k\,, \qquad \lambda_{5,2}(k) \sim e^{\frac{6i\pi}{5}} k\,, \qquad k \in \mathcal{D}_{R,2}\,; \\ &\lambda_{4,3}(k) \sim e^{\frac{2i\pi}{5}} k\,, \qquad \lambda_{5,3}(k) \sim e^{\frac{4i\pi}{5}} k\,, \qquad k \in \mathcal{D}_{R,3}\,. \end{aligned} \quad (A.2.16)$$

Given $\tilde{Q}_0(k)$, $\tilde{Q}_1(k)$, $\tilde{Q}_2(k)$ the system (A.2.14) yields $\tilde{Q}_3(k)$ and $\tilde{Q}_4(k)$ for $k \in D_m$, $m = 1, 2, 3$. Since ω_0 and ω_1 are defined by $(A.2.15)$, the relevant LU decomposition is

$$\begin{bmatrix} \alpha_5 & \alpha_5 \lambda_{4,m}(k) + \alpha_4 \\ \alpha_5 & \alpha_5 \lambda_{5,m}(k) + \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_5 & \alpha_5 \lambda_{4,m}(k) + \alpha_4 \\ 0 & \alpha_5\big(\lambda_{5,m}(k) - \lambda_{4,m}(k)\big) \end{bmatrix}. \quad (A.2.17)$$

(ii) $\alpha_5 < 0$

Then

$$N = 2\,, \quad l = 3, 4, 5\,, \quad m = 1, 2\,.$$

Thus the system (1.15) becomes

$$\begin{aligned} &\omega_4\big(\lambda_{3,m}(k)\big)\tilde{Q}_0(k) + \cdots + \omega_0 \tilde{Q}_4(k) = -\hat{q}_0\big(\lambda_{3,m}(k)\big)\,, \\ &\omega_4\big(\lambda_{4,m}(k)\big)\tilde{Q}_0(k) + \cdots + \omega_0 \tilde{Q}_4(k) = -\hat{q}_0\big(\lambda_{4,m}(k)\big)\,, \\ &\omega_4\big(\lambda_{5,m}(k)\big)\tilde{Q}_0(k) + \cdots + \omega_0 \tilde{Q}_4(k) = -\hat{q}_0\big(\lambda_{5,m}(k)\big)\,, \quad k \in \mathcal{D}_{R,m}\,, \end{aligned} \quad (A.2.18)$$

where $\omega_0, \cdots, \omega_4$ are given by $(A.2.15)$ and $\lambda_{l,m}(k)$ satisfy

$$\begin{aligned} &\lambda_{3,1}(k) \sim e^{\frac{4i\pi}{5}} k\,, \qquad \lambda_{4,1}(k) \sim e^{\frac{6i\pi}{5}} k\,, \qquad \lambda_{5,1}(k) \sim e^{\frac{8i\pi}{5}} k, \qquad k \in \mathcal{D}_{R,1}\,; \\ &\lambda_{3,2}(k) \sim e^{\frac{2i\pi}{5}} k\,, \qquad \lambda_{4,2}(k) \sim e^{\frac{4i\pi}{5}} k\,, \qquad \lambda_{5,2}(k) \sim e^{\frac{6i\pi}{5}} k\,, \qquad k \in \mathcal{D}_{R,2}\,. \end{aligned} \quad (A.2.19)$$

Given $\tilde{Q}_0(k)$ and $\tilde{Q}_1(k)$ the system $(A.2.18)$ yields $\tilde{Q}_2(k)$, $\tilde{Q}_3(k)$, and $\tilde{Q}_4(k)$ for $k \in D_m$, $m = 1, 2$. Since ω_0, ω_1, ω_2 are defined by $(A.2.15)$, the matrix that needs to be inverted

$$\begin{bmatrix} \alpha_5 & \alpha_5\lambda_{3,m}(k) + \alpha_4 & \alpha_5\lambda_{3,m}^2(k) + \alpha_4\lambda_{3,m}(k) + \alpha_3 \\ \alpha_5 & \alpha_5\lambda_{4,m}(k) + \alpha_4 & \alpha_5\lambda_{4,m}^2(k) + \alpha_4\lambda_{4,m}(k) + \alpha_3 \\ \alpha_5 & \alpha_5\lambda_{5,m}(k) + \alpha_4 & \alpha_5\lambda_{5,m}^2(k) + \alpha_4\lambda_{5,m}(k) + \alpha_3 \end{bmatrix}$$

admits the LU decomposition where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{\lambda_{5,m}(k) - \lambda_{3,m}(k)}{\lambda_{4,m}(k) - \lambda_{3,m}(k)} & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} \alpha_5 & \alpha_5\lambda_{3,m}(k) + \alpha_4 & \alpha_5\lambda_{3,m}^2(k) + \alpha_4\lambda_{3,m}(k) + \alpha_3 \\ 0 & \alpha_5\big(\lambda_{4,m}(k) - \lambda_{3,m}(k)\big) & \alpha_5\big(\lambda_{4,m}^2(k) - \lambda_{3,m}^2(k)\big) + \alpha_4\big(\lambda_{4,m}(k) - \lambda_{3,m}(k)\big) \\ 0 & 0 & \alpha_5\big(\lambda_{5,m}(k) - \lambda_{3,m}(k)\big)\big(\lambda_{5,m}(k) - \lambda_{4,m}(k)\big) \end{bmatrix}.$$

We now give the explicit form of $\tilde{Q}(k)$ for Examples A.1.3 and A.1.4.

For Example A.1.3, given $\tilde{Q}_0(k)$ and $\tilde{Q}_1(k)$, equation $(A.2.4)$ yields

$$\tilde{Q}_2(k) = -\frac{1}{2}\left\{\hat{q}_0\big(\lambda_{3,m}(k)\big) + (2\lambda_{3,m}^2(k) + 1)\tilde{Q}_0(k) + 2\lambda_{3,m}(k)\tilde{Q}_1(k)\right\}, \qquad (A.2.20)$$

for $k \in \mathcal{D}_{R,m}$ and $m = 1, 2$, where

$$\lambda_{3,1}(k) = \frac{-k - i\sqrt{3k^2 + 2}}{2}, \quad \lambda_{3,2}(k) = \frac{-k + i\sqrt{3k^2 + 2}}{2}. \qquad (A.2.21)$$

Note that $\lambda_{3,1}(k) \sim e^{\frac{4i\pi}{3}}k$ and $\lambda_{3,2}(k) \sim e^{\frac{2i\pi}{3}}k$, as $k \to \infty$.

For Example A.1.4, given $\tilde{Q}_0(k)$ and $\tilde{Q}_1(k)$, using equation $(A.2.13)$, the equations of $(A.2.10)$ yield

$$\begin{aligned} \tilde{Q}_3(k) = {} & \frac{\lambda_{4,m}(k)\hat{q}_0\big(\lambda_{3,m}\big) - \lambda_{3,m}(k)\hat{q}_0\big(\lambda_{4,m}(k)\big)}{\lambda_{3,m}(k) - \lambda_{4,m}(k)} + \lambda_{3,m}(k)\lambda_{4,m}(k)\big(\lambda_{3,m}(k) + \lambda_{4,m}(k)\big)\tilde{Q}_0(k) \\ & + \big(\lambda_{3,m}(k)\lambda_{4,m}(k) - 1\big)\tilde{Q}_1(k), \\ \tilde{Q}_2(k) = {} & \frac{\hat{q}_0\big(\lambda_{4,m}(k)\big) - \hat{q}_0\big(\lambda_{3,m}(k)\big)}{\lambda_{3,m}(k) - \lambda_{4,m}(k)} - \big(\lambda_{3,m}^2(k) + \lambda_{4,m}^2(k) + \lambda_{3,m}(k)\lambda_{4,m}(k) + 1\big)\tilde{Q}_0(k) \\ & - \big(\lambda_{3,m}(k) + \lambda_{4,m}(k)\big)\tilde{Q}_1(k), \quad k \in \mathcal{D}_{R,m}, m = 1, 2, \end{aligned} \qquad (A.2.22)$$

where

$$\lambda_{3,1}(k) = \lambda_{4,2}(k) = e^{i\pi}k, \ \lambda_{3,2}(k) = e^{i\frac{\pi}{2}}\sqrt{k^2 + 1}, \ \lambda_{4,1}(k) = e^{i\frac{3\pi}{2}}\sqrt{k^2 + 1}. \qquad (A.2.23)$$

A.3. Other Types of Boundary Conditions

It is possible to extend the results of Theorems 1.1 and 1.2 to the case where the equations of (1.1b) are replaced by

$$F_l(q(0,t), \partial_x q(0,t), \cdots, \partial_x^{n-1} q(0,t)) = f_l(t)\,, \quad 0 < t < T\,, \quad 0 \le l \le N-1\,,$$

where F_l are linear functions of $\{\partial_x^j q(0,t)\}_0^{n-1}$. If these functions have *constant* coefficients, $\hat{Q}(k)$ can be obtained through the solution of a system of *algebraic* equations. For brevity of presentation we discuss in detail only a simple example.

Consider the following equation on the half-line:

$$q_t + q_{xxx} = 0\,, \qquad 0 < x < \infty\,, \quad 0 < t < T\,, \tag{A.3.1}$$

with initial condition

$$q(x,0) = q_0(x)\,, \qquad 0 < x < \infty\,. \tag{A.3.2}$$

In this case

$$\omega(k) = -k^3\,, \lambda_{2,1}(k) = \zeta k\,, \quad \lambda_{3,1}(k) = \zeta^2 k\,, \quad \zeta = e^{\frac{2i\pi}{3}}\,. \tag{A.3.3}$$

Using $\omega_0 = -1$, $\omega_1(k) = -k$, $\omega_2(k) = -k^2$, the equations of (2.20) become

$$\hat{Q}_2(k) + \zeta k \hat{Q}_1(k) + \zeta^2 k^2 \hat{Q}_0(k) = \hat{q}_0(\zeta k) - e^{-ik^3 T} \hat{q}_T(\zeta k)\,, \tag{A.3.4a}$$

$$\hat{Q}_2(k) + \zeta^2 k^2 \hat{Q}_1(k) + \zeta k^2 \hat{Q}_0(k) = \hat{q}_0(\zeta^2 k) - e^{-ik^3 T} \hat{q}_T(\zeta^2 k)\,, \quad k \in \mathcal{D}_+\,. \tag{A.3.4b}$$

Also, the definition of $\hat{Q}(k)$ (equation (1.6a)) implies

$$\hat{Q}_2(k) + k\hat{Q}_1(k) + k^2 \hat{Q}_0(k) = -\hat{Q}(k)\,. \tag{A.3.4c}$$

Using

$$1 + \zeta + \zeta^2 = 0\,,$$

equations $(A.3.4a) + (A.3.4.b) + (A.3.4c)$, $\zeta^2(A.3.4.a) + \zeta(A.3.4b) + (A.3.4c)$ and $\zeta(A.3.4a) + \zeta^2(A.3.4b) + (A.3.4c)$ become

$$3\hat{Q}_2(k) = -\hat{Q}(k) + \hat{q}_0(\zeta k) + \hat{q}_0(\zeta^2 k) - e^{-ik^3 T}(\hat{q}_T(\zeta k) + \hat{q}_T(\zeta^2 k))\,,$$

$$3k\hat{Q}_1(k) = -\hat{Q}(k) + \zeta \hat{q}_0(\zeta^2 k) + \zeta^2 \hat{q}_0(\zeta k) - e^{-ik^3 T}(\zeta \hat{q}_T(\zeta^2 k) + \zeta^2 \hat{q}_T(\zeta k))\,,$$

$$3k^2\hat{Q}_0(k) = -\hat{Q}(k) + \zeta^2 \hat{q}_0(\zeta^2 k) + \zeta \hat{q}_0(\zeta k) - e^{-ik^3 T}(\zeta^2 \hat{q}_T(\zeta^2 k) + \zeta \hat{q}_T(\zeta k))\,, \quad k \in \mathcal{D}_+\,, \tag{A.3.5}$$

These equations motivate the following result.

Proposition A.3.1. *Let $q(x,t)$ satisfy equations $(A.3.1)$–$(A.3.2)$, where $q_0 \in H^2(0,\infty)$, and any one of the following boundary conditions*

$$q(0,t) = f_0(t)\,, \quad 0 < t < T\,, \tag{A.3.6a}$$

$$q_x(0,t) = f_1(t)\,, \quad 0 < t < T\,, \tag{A.3.6b}$$

$$q_{xx}(0,t) = f_2(t)\,, \quad 0 < t < T\,, \tag{A.3.6c}$$

where $f_j \in H^{1-(j/3)}(0,T)$ *for* $0 \le j \le 2$. *Furthermore suppose that if* $(A.3.6a)$ *or* $(A.3.6b)$ *are valid then the following compatibility conditions are satisfied*

$$f_0(0) = q_0(0)\,, \tag{A.3.7a}$$
$$f_1(0) = q_0'(0)\,, \tag{A.3.7b}$$

respectively. The unique solution of this initial-boundary value problem is given by equation (1.4), *where for the cases* $(A.3.6a)$ *or* $(A.3.6b)$ *or* $(A.3.6c)$, $\hat{Q}(k)$ *is given by*

$$\hat{Q}(k) = -3k^2 \int_0^T e^{-ik^3t} f_0(t)\,dt + \zeta^2 \hat{q}_0(\zeta^2 k) + \zeta \hat{q}_0(\zeta k)\,, \tag{A.3.8a}$$
$$\hat{Q}(k) = 3ik \int_0^T e^{-ik^3t} f_1(t)\,dt + \zeta \hat{q}_0(\zeta^2 k) + \zeta^2 \hat{q}_0(\zeta k)\,, \tag{A.3.8b}$$
$$\hat{Q}(k) = 3 \int_0^T e^{-ik^3t} f_2(t)\,dt + \hat{q}_0(\zeta k) + \hat{q}_0(\zeta^2 k)\,. \tag{A.3.8c}$$

If a linear combination with constant coefficients of $\{\partial_x^j q(0,t)\}_0^2$ is given, it is also possible to obtain $\hat{Q}(k)$ explicitly. Consider for example the case

$$-iq_x(0,t) = \alpha q(0,t) + f(t)\,, \quad 0 < t < T\,, \tag{A.3.9}$$

where α is a constant and $f(t) \in H^{2/3}(0,T)$. Equation $(A.3.9)$ implies

$$\hat{Q}_1(k) = \alpha \hat{Q}_0(k) + \hat{f}(k)\,, \quad \hat{f}(k) = \int_0^T e^{-ik^3t} f(t)\,dt\,. \tag{A.3.10}$$

Eliminating the combination $\hat{Q}_1(k) - \alpha\hat{Q}_0(k)$ using $(A.3.5)$ we find

$$\begin{aligned} 3k^2 \hat{f}(k) = {} & (\alpha - k)\hat{Q}(k) + (k\zeta - \alpha\zeta^2)\hat{q}_0(\zeta^2 k) + (k\zeta^2 - \alpha\zeta)\hat{q}_0(\zeta k) \\ & + e^{-ik^3T}\{(\alpha\zeta^2 - k\zeta)\hat{q}_T(\zeta^2 k) + (\alpha\zeta - k\zeta^2)\hat{q}_T(\zeta k)\}\,. \end{aligned} \tag{A.3.11}$$

If α is not in $\overline{\mathcal{D}_+}$, equation $(A.3.11)$ yields $\hat{Q}(k)$ in terms of $\hat{f}(k)$ and $\hat{q}_0$, since the term involving $e^{-ik^3T}/(\alpha - k)$ is holomorphic and bounded in $\mathcal{D}_+$ and hence does not contribute to $q(x,t)$ given by (1.4). If $\alpha \in \mathcal{D}_+$, we can rewrite equation $(A.3.11)$ in such a way that the term involving $e^{-ik^3T}/(\alpha - k)$ again gives a zero contribution: Evaluating equation $(A.3.11)$ at $k = \alpha$, we find

$$3\alpha \hat{f}(\alpha) = (\zeta - \zeta^2)\Big[\hat{q}_0(\zeta^2\alpha) - \hat{q}_0(\zeta\alpha) + e^{-i\alpha^3T}\big(-\hat{q}_T(\zeta^2\alpha) + \hat{q}_T(\zeta\alpha)\big)\Big]\,. \tag{A.3.12}$$

In order to make the term involving $e^{-ik^3T}/(\alpha - k)$ holomorphic and bounded in D_+ we subtract the pole contribution, whose residue is given in terms of $\hat{f}(\alpha)$, $\hat{q}_0(\zeta^2\alpha)$ and $\hat{q}_0(\zeta\alpha)$. This motivates the following result.

Proposition A.3.2. *Let $q(x,t)$ satisfy equations (A.3.1)–(A.3.2), where $q_0 \in H^2(0,\infty)$, and equation (A.3.9). Furthermore, suppose that*

$$-iq_0'(0) = \alpha q_0(0) + f(0). \tag{A.3.13}$$

The unique solution of this IBV problem is given by equation (1.4), where $\hat{Q}(k)$ is computed as follows:

If α is not in D_+,

$$\hat{Q}(k) = \frac{1}{k-\alpha}\left\{-3k^2\hat{f}(k) + (k\zeta - \alpha\zeta^2)\hat{q}_0(\zeta^2 k) + (k\zeta^2 - \alpha\zeta)\hat{q}_0(\zeta k)\right\}; \tag{A.3.14}$$

if $\alpha \in \bar{D}_+$,

$$\begin{aligned}\hat{Q}(k) = \frac{1}{k-\alpha}\Big\{ &-3k^2\hat{f}(k) + 3\alpha^2\hat{f}(\alpha) + (k\zeta - \alpha\zeta^2)\hat{q}_0(\zeta^2 k) - (\alpha\zeta - \alpha\zeta^2)\hat{q}_0(\zeta^2\alpha) \\ &+ (k\zeta^2 - \alpha\zeta)\hat{q}_0(\zeta k) - (\alpha\zeta^2 - \alpha\zeta)\hat{q}_0(\zeta\alpha)\Big\};\end{aligned} \tag{A.3.15}$$

where

$$\hat{f}(k) = \int_0^T e^{-ik^3 t} f(t)\, dt. \tag{A.3.16}$$

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