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Remarks on Poincaré and Van Vleck Theorems

V.I. Buslaev

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Department of Mathematics
University of South Carolina

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V.I. Buslaev

Abstract

The paper [1] is dedicated to generalizations and applications of Poincare's theorem on recurrence equations with limit constant coefficients. In particular, applications in the theory of continued fractions, mainly to problems related with Van Vleck's theorem on regular C-fractions with limit constant coefficients are considered. Special attention is tributed to extremal properties of the singular sets of regular C-fractions and T-fractions with limit periodic coefficients.

0.1 Introduction. Preliminary definitions

Poincare's theorem on recurrence equations with limit constant coefficients is one of the most refined results of the theory. Such equations have found applications in Number Theory, Analysis and other fields of Mathematics. The well known Fibonacci numbers $f_{-1} = f_0 = 1$, $f_1 = 2$, $f_2 = 3$, $f_3 = 5$, $f_4 = 8, \ldots$ examplify solutions of the recurrence equation $f_n = f_{n-1} + f_{n-2}, \quad n = 1, 2, \ldots$, with the initial values $f_{-1} = f_0 = 1$. It is also well known that a continued fraction

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \ldots}} \tag{1}$$

can be considered as solution couple $\{A_n\}_{n=-1}^{\infty}$, $\{B_n\}_{n=-1}^{\infty}$ of the recurrence equation

$$w_n = b_n w_{n-1} + a_n w_{n-2}, \quad n = 1, 2, \ldots \tag{2}$$

with the initial values $A_{-1} = 1, A_0 = 0$ and $B_{-1} = 0, B_0 = 1$. Other applications of recurrence equations can be found in the theory of Pade approximations.

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Let us first consider the recurrence equation with constant coefficients

$$w_n = \alpha^1 w_{n-1} + \ldots + \alpha^k w_{n-k}, \quad n = 1, 2, \ldots \tag{3}$$

Here, the number k is called the length of recurrence equation. Given k initial values $w_{-k+1}, \ldots, w_0$, we can calculate all subsequent $w_1, w_2, \ldots$ using (3) stepwise. The sequence $\{w_n\}_{n=1-k}^{\infty}$ obtained in such a way is called the solution of the recurrence equation (3). The set of all solutions constitutes a k-dimensional linear vector space. Let us find its k linearly independent solutions. The algebraic polynomial $\lambda^k - \alpha^1\lambda^{k-1} - \ldots - \alpha^k$ is called characteristic polynomial of the recurrence equation (3). Let us factor out this polynomial: $\lambda^k - \alpha^1\lambda^{k-1} - \ldots - \alpha^k = \prod_{i=1}^{k}(\lambda - \lambda_i)$. It is easy to verify that for each $i = 1, \ldots, k$, the sequence $\{w_n\}_{n=1-k}^{\infty} = \{\lambda_i^n\}_{n=1-k}^{\infty}$ is a solution of the recurrence equation (3). If the roots of the characteristic polynomial are simple, these k solutions are linearly independent. In the case of multiple roots, we must complement the above set by the solutions of the form $\{n^j\lambda_i^n\}_{n=1-k}^{\infty}$, $j = 0, \ldots, r_i - 1$, where r_i is the multiplicity of the root λ_i. However, in the sequel we will consider only the case of simple roots. Moreover, we assume that all these roots are distinct in modulus:

$$A(1): \qquad |\lambda_1| < |\lambda_2| < \ldots < |\lambda_k|$$

(we will denote here and in the sequel by A( ) the corresponding assumption). Under assumption A(1), the general solution of recurrence equation (3) has the following form

$$\{w_n\}_{n=1-k}^{\infty} = C_1\{\lambda_1^n\}_{n=1-k}^{\infty} + \ldots + C_k\{\lambda_k^n\}_{n=1-k}^{\infty}$$

and $\lim_{n\to\infty} \lambda_{i-1}^n/\lambda_i^n = 0$, $i = 2, \ldots, k$. Hence, the following statement holds true.

Statement. *Consider the recurrence equation (3) and suppose that the assumption A(1) is fulfilled. Then for each solution $\{w_n\}_{n=1-k}^{\infty}$ of the recurrence equation (3), the limit $\lim_{n\to\infty} w_{n+1}/w_n$ exists and equals one of the roots of the characteristic polynomial.*

Which one of the roots it is ?

We can not give an a priori answer. It may be the largest in modulus root λ_k, if $C_k \neq 0$, it may also be λ_{k-1}, if $C_k = 0$ and $C_{k-1} \neq 0$, etc.

The second natural question arises, whether A(1) is an essential assumption. The answer is trivial: of course, it is, as illustrated by the following example.

Example. $w_n = w_{n-2}$, $n = 1, 2, \ldots$.

In this example $w_n = w_0$ for even n and $w_n = w_{-1}$ for odd n. Consequently, the limit $\lim_{n\to\infty} w_{n+1}/w_n$ does not exist. The reason is that the characteristic polynomial $\lambda^2 - 1 = (\lambda-1)(\lambda+1)$ has two roots equal in modulus.

0.2 Poincare's theorem.

It turns out that the above statement is true in more general situations. Consider a recurrence equation with variable coefficients

$$w_n = \alpha_n^1 w_{n-1} + \ldots + \alpha_n^k w_{n-k}, \quad n = 1, 2, \ldots \tag{4}$$

and suppose that these variable coefficients have limits

$$A(2): \quad \lim_{n\to\infty} \alpha_n^i = \alpha^i, \quad i = 1, \ldots, k\,.$$

As earlier, the polynomial $\lambda^k - \alpha^1\lambda^{k-1} - \ldots - \alpha^k = \prod_{i=1}^k(\lambda - \lambda_i)$ is called the characteristic polynomial of the recurrence equation (4). We can not find the explicit formulae for the general solution of (4). However, (and it is a remarkable fact) the statement still remains true.

Poincare's theorem (cf. [1]). *Consider the recurrence equation (4) and suppose that the assumptions A(2) and A(1) are fulfilled. Then for each solution $\{w_n\}_{n=1-k}^\infty$ of (4), either $w_n = 0$ for all sufficiently large $n \geq n_0$, or the limit $\lim_{n\to\infty} w_{n+1}/w_n$ exists and equals one of the roots of the characteristic polynomial.*

Let us return to the question of necessity of the assumption A(1). As we saw, A(1) is an essential assumption. But it turns out that it is possible to give such a generalization of Poincare's theorem that can be formulated omitting A(1); moreover, if A(1) is valid as an additional assumption, the generalization will coincide with Poincare's theorem.

Theorem 1 *(cf. [14]). Suppose that for the recurrence equation (4) the assumption A(2) is fulfilled. Then for each solution $\{w_n\}_{n=1-k}^\infty$ of (4) either $w_n = 0$ for all sufficiently large $n \geq n_0$, or*

1° *the upper limit $\limsup_{n\to\infty} |w_n|^{1/n}$ is equal to the modulus of one of the roots of the characteristic polynomial;*

2° *it is possible to construct a recurrence equation*

$$w_n = \beta_n^1 w_{n-1} + \ldots + \beta_n^l w_{n-l}, \quad n = 1, 2, \ldots,$$

where l is the number of the roots of the characteristic polynomial lying on the circumference $|\lambda| = \limsup_{n\to\infty} |w_n|^{1/n}$ (let us denote these roots by $\lambda_{p_1}, \ldots, \lambda_{p_l}$), such that

a) $\{w_n\}_{n=1-l}^\infty$ is its solution;

b) $\lim_{n\to\infty} \beta_n^i = \beta^i, \quad i = 1, \ldots, l$;

c) $\lambda^l - \beta^1\lambda^{l-1} - \ldots - \beta^l = \prod_{i=1}^l(\lambda - \lambda_{p_i})$.

It is easy to see that under additional assumption A(1) we always have $l = 1$. Thus, the above recurrence equation is of the form $w_n = \beta_n^1 w_{n-1}$ and by b) $\lim_{n\to\infty} w_n/w_{n-1} = \lim_{n\to\infty} \beta_n^1 = \beta^1$ and by c) $\beta^1 = \lambda$, where λ is the unique root of the characteristic polynomial lying on the circumference $|\lambda| = \limsup_{n\to\infty} |w_n|^{1/n}$.

0.3 Perron's theorem.

Consider the recurrence equation (4) with assumptions A(2) and A(1), like in Poincare's theorem. The following natural question arises.

Question. Is it true that for each root λ of the characteristic polynomial there exists a solution $\{w_n\}_{n=1-k}^{\infty}$ of (4) such that $\lim_{n\to\infty} w_{n+1}/w_n = \lambda$?

Without additional assumptions, the answer to this question is in the negative.

Example. $w_1 = w_0,\ w_n = w_{n-1} + w_{n-2}\,,\ n = 2, 3, \ldots$.

Each solution $\{w_n\}_{n=-1}^{\infty}$ of this recurrence equation is of the form $w_n = w_0 f_{n-1}\,,\quad n = 1, 2, \ldots$, where $f_{-1}, f_0, f_1, f_2, \ldots$ denotes the Fibonacci's sequence, see above. Hence the limit $\lim_{n\to\infty} w_{n+1}/w_n = \lim_{n\to\infty} f_n/f_{n-1} = (1+\sqrt{5})/2$ is the same for all solutions.

The reason is simple. The solutions do not depend on the initial value w_{-1} in this example, because the coefficient $\alpha_1^2 = 0$.

The recurrence equation (4) is called *non-degenerate* if

$$A(3): \quad \alpha_n^k \neq 0\,, \quad n = 1, 2, \ldots .$$

It means that we can solve (4) "backwards" for each $n = 1, 2, \ldots$, that is we can express w_{n-k} in terms of $w_{n-k+1}, \ldots, w_n$.

Perron's theorem (cf. [2]). *Consider the recurrence equation (4) and suppose that the assumptions A(2), A(1) and A(3) are fulfilled. Then for each root λ of the characteristic polynomial there exists a solution $\{w_n\}_{n=1-k}^{\infty}$ of (4) such that $\lim_{n\to\infty} w_{n+1}/w_n = \lambda$.*

Corollary. *Consider the recurrence equation (4) and suppose that the assumptions A(2), A(1) and A(3) are fulfilled. Then the general solution of (4) is of the form*

$$\{w_n\}_{n=1-k}^{\infty} = C_1\{w_n^1\}_{n=1-k}^{\infty} + \ldots + C_k\{w_n^k\}_{n=1-k}^{\infty}\,,$$

where $\{w_n^i\}_{n=1-k}^{\infty}$ is a solution of (4) such that $\lim_{n\to\infty} w_{n+1}^i/w_n^i = \lambda_i\,,\quad i = 1, \ldots, k$.

It is easy to see that the solution $\{w_n^1\}_{n=1-k}^{\infty}$ corresponding to the root λ_1 with minimal modulus is unique up to the constant multiplier. It is not easy to find the initial values corresponding to this unique solution. The original proof of Perron's theorem was very complicated. A much easier proof was given by Evgrafov in 1953 (cf. [3]). Another proof of Perron's theorem was given by Freiman in 1957 (cf. [4]). Freiman's proof is applicable also to systems of recurrence equations.

0.4 Vector version of Poincare–Perron's theorem.

Consider the system of recurrence equations

$$\left\{ \begin{array}{lll} w_n^1 & = & \alpha_n^{1,1} w_{n-1}^1 + \ldots + \alpha_n^{1,k} w_{n-1}^k \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_n^k & = & \alpha_n^{k,1} w_{n-1}^1 + \ldots + \alpha_n^{k,k} w_{n-1}^k \end{array} \right. , \quad n = 1, 2, \ldots \tag{5}$$

or in vector form $\vec{w}_n = \Psi_n \vec{w}_{n-1}$, where $\vec{w}_n := \begin{bmatrix} w_n^1 \\ \vdots \\ w_n^k \end{bmatrix}$, $\quad \Psi_n := \begin{bmatrix} \alpha_n^{1,1} & \ldots & \alpha_n^{1,k} \\ \ldots & \ldots & \ldots \\ \alpha_n^{k,1} & \ldots & \alpha_n^{k,k} \end{bmatrix}$.

Denote $\vec{w}_n := \begin{bmatrix} w_{n-k+1} \\ \vdots \\ w_n \end{bmatrix}$. Then (4) can be rewritten in the form (5) with

$$\Psi_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 1 \\ \alpha_n^k & \ldots & \ldots & \alpha_n^2 & \alpha_n^1 \end{bmatrix}.$$

The assumption A(2) can be rewritten in this case as

$$A(2'): \quad \lim_{n\to\infty} \Psi_n = \Psi\,.$$

The assumption A(1) saves its form, but here $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of the limit matrix Ψ. The assumption A(3) has the form

$$A(3'): \quad det \Psi_n \neq 0\,, \quad n = 1, 2, \ldots$$

Consequently, the classical Poincare and Perron theorems are particular cases (when matrices Ψ_n have special form, see above), of the following statement.

Poincare-Perron's theorem (vector version). *Consider the system of recurrence equations (5) and suppose that the assumptions $A(2')$, $A(1)$ are fulfilled. Then*

1° *for each solution $\{\vec{w}_n\}_{n=0}^\infty$ of the system (5) either $\vec{w}_n = \vec{0}$ for all sufficiently large $n \geq n_0$, or for some $m \in \{1, \ldots, k\}$ there exist limits $\lim_{n\to\infty} w_{n+1}^m / w_n^m = \lambda$, $\lim_{n\to\infty} \vec{w}_n / w_n^m = \vec{e}$, where λ is some eigenvalue of Ψ and $\vec{e}$ is the corresponding eigenvector ($e^m = 1$);*

2° *under the additional assumption $A(3')$, for each eigenvalue λ and the corresponding eigenvector $\vec{e}$ ($e^m = 1$) of the matrix Ψ there exists a solution $\{\vec{w}_n\}_{n=0}^\infty$ of (5) such that the limit equalities in the previous assertion are valid with this λ and $\vec{e}$.*

The proof of this theorem can be found in (cf. [4]) and in more general form in (cf. [15]).

Denote $\vec{w}_n = T\vec{u}_n$, where T is a matrix independent of n and transform Ψ to the diagonal form ($T^{-1}\Psi T = D$). Then we can assert that the case when the limit matrix is diagonal is a general case of this theorem. Furthermore, a part of nontrivial contents of the diagonal case of this theorem can be obtained under assumptions that are weaker than $A(2')$ and $A(1)$. The corresponding result was obtained in (cf. [15]). For simplicity, we formulate here this result only for $k = 2$. It is easy to see that in the diagonal case the assumptions $A(2')$ and $A(1)$ mean that $\lim_{n\to\infty} \alpha_n^{1,2} = \lim_{n\to\infty} \alpha_n^{2,1} = 0$ and $|\lim_{n\to\infty} \alpha_n^{1,1}| < |\lim_{n\to\infty} \alpha_n^{2,2}|$ that are stronger than the assumption

$$A(4): \quad |\alpha_n^{1,1}| + |\alpha_n^{1,2}| \leq q(|\alpha_n^{2,2}| - |\alpha_n^{2,1}|)\,, \quad n \geq n_0$$

with some $q < 1$. $A(4)$ does not require the existence of the limits. The following theorem is valid.

Theorem 2 *(cf. [15]). Consider the system of recurrence equations (5) with $k = 2$ and suppose that the assumptions $A(3')$ and $A(4)$ with $q < 1$ are fulfilled. Then*

1° *there exists a unique (up to a constant multiplier) non-trivial solution $\{\vec{v}_n\}_{n=0}^\infty = \{[v_n^1, v_n^2]\}_{n=0}^\infty$ of (5) such that $|v_n^1| \geq |v_n^2|$ for all $n = 0, 1, \ldots$ whereas for other solutions $\{\vec{u}_n\}_{n=0}^\infty = \{[u_n^1, u_n^2]\}_{n=0}^\infty$ of (5) $|u_n^1| < |u_n^2|$ for all sufficiently large $n \geq n_1$ (n_1 depends on the solution $\{\vec{u}_n\}_{n=0}^\infty$);*

2° *$|v_n^1/u_n^2| \leq q^{n-n_1} |v_{n_1}^1 / u_{n_1}^2|$, where $\{\vec{v}_n\}_{n=0}^\infty$, $\{\vec{u}_n\}_{n=0}^\infty$, n_1 mentioned in the assertion 1° (it means that the exceptional solution $\{\vec{v}_n\}_{n=0}^\infty$ has a minimal growth among all other solutions);*

3° *if $\lim_{n\to\infty} \alpha_n^{1,2} / (|\alpha_n^{2,2}| - |\alpha_n^{2,1}|) = 0$, then $\lim_{n\to\infty} u_n^1 / u_n^2 = 0$;*

4° *if $\lim_{n\to\infty} \alpha_n^{2,1} / \alpha_n^{2,2} = 0$, then $\lim_{n\to\infty} v_n^2 / v_n^1 = 0$.*

0.5 Generalization of classical Poincare and Perron theorems for the recurrence equation (2).

The following theorems suggest weakening of the assumptions $A(2)$ and $A(1)$ in classical Poincare and Perron theorems in the case $k = 2$, i.e. for the equation (2). Consider the roots λ_n^1 and λ_n^2 of polynomials $\lambda^2 - b_n\lambda - a_n$, $n = 1, 2, \ldots$. Note that the couple of assumptions $A(2)$ and $A(1)$ is equivalent to the requirement

$$A(5): \quad \lim_{n\to\infty} \lambda_n^1 = \lambda^1 , \quad \lim_{n\to\infty} \lambda_n^2 = \lambda^2 , \quad |\lambda^1| < |\lambda^2| .$$

Let us introduce the following assumptions, where each is weaker than $A(5)$ (in a modified notation)

$$A(6): \quad \lim_{n\to\infty} \lambda_n^1 = \lambda , \quad \limsup_{n\to\infty} \lambda_n^2 < |\lambda| ,$$

$$A(7): \quad \lim_{n\to\infty} \lambda_n^1 = \lambda , \quad \liminf_{n\to\infty} \lambda_n^2 > |\lambda| ,$$

$$A(8): \quad \limsup_{n\to\infty} (|b_n - \lambda| + \theta^{-1}|\lambda^2 - b_n\lambda - a_n|) < |\lambda| - \theta ,$$

$$A(9): \quad \liminf_{n\to\infty} (|b_n - \lambda| - \theta^{-1}|\lambda^2 - b_n\lambda - a_n|) > |\lambda| + \theta ,$$

where $\lambda \in \mathcal{C}$ and $\theta > 0$ are some fixed numbers.

It is easy to see that $A(5) \Longrightarrow A(6) \Longrightarrow A(8)$. Indeed, $A(5) \Longrightarrow A(6)$ is trivial (in modified notation) and $A(6)$ implies that $\lim_{n\to\infty} |\lambda^2 - b_n\lambda - a_n| = 0$ and $\limsup_{n\to\infty} |b_n - \lambda| = \limsup_{n\to\infty} |\lambda_n^1 + \lambda_n^2 - \lambda| = \limsup_{n\to\infty} |\lambda_n^2| < |\lambda|$ and obviously we can substitute $|\lambda|$ in the end of this chain by $|\lambda| - \theta$ for sufficiently small $\theta > 0$.

Analogously, $A(5) \Longrightarrow A(7) \Longrightarrow A(9)$.

It turns out that we can save a part of nontrivial contents of the classical Poincare and Perron theorems for the equation (2) substituting the assumption $A(5)$ by one of the new assumptions $A(6) - A(9)$. More precisely, the following statements are true.

Theorem 3 *(cf. [15]). Consider the recurrence equation (2) and suppose that $A(3)$ and $A(8)$ are fulfilled. Then*

1° *there exists a unique (up to a constant multiplier) non-trivial solution $\{v_n\}_{n=-1}^{\infty}$ of (2) of minimal growth, that is for each other solution $\{u_n\}_{n=-1}^{\infty}$ $\lim_{n\to\infty} v_n/u_n = 0$;*

2° *under assumption $A(6)$ (that is stronger than $A(8)$) for each solution $\{u_n\}_{n=-1}^{\infty}$, except for the indicated unique solution $\{v_n\}_{n=-1}^{\infty}$, the limit equality holds $\lim_{n\to\infty} u_{n+1}/u_n = \lambda$.*

Theorem 4 *(cf. [15]). Consider the recurrence equation (2) and suppose that $A(3)$ and $A(9)$ are fulfilled. Then*

1° *there exists a unique (up to a constant multiplier) non-trivial solution $\{v_n\}_{n=-1}^{\infty}$ of (2) of minimal growth, that is for each other solution $\{u_n\}_{n=-1}^{\infty}$ $\lim_{n\to\infty} \max\{|v_n|, |v_{n-1}|\}/(u_n - \lambda u_{n-1}) = 0$.*

2° *under assumption $A(7)$ (that is stronger than $A(9)$) for the exceptional solution $\{v_n\}_{n=-1}^{\infty}$ the limit equality holds $\lim_{n\to\infty} v_{n+1}/v_n = \lambda$.*

Let us remark that under assumption $A(9)$ with $\theta > |\lambda|$ we can remove the term λu_{n-1} from the limit equality $\lim_{n\to\infty} v_n/(u_n - \lambda u_{n-1}) = 0$. Also, it is possible to construct an example showing that the constant $\theta > |\lambda|$ in $A(9)$ can not be substituted by any constant $|\lambda| - \epsilon$ with $\epsilon > 0$, in order to remove this term λu_{n-1}.

The last two theorems have applications in the theory of convergence of continued fractions.

0.6 Generalization of classical Poincare's theorem for $k = \infty$.

Let us consider the recurrence equation

$$w_n = \alpha_n^1 w_{n-1} + \ldots + \alpha_n^n w_0\,, \quad n = 1, 2, \ldots\,,$$

where w_n depends on all previous $w_{n-1}, \ldots, w_0$. Moreover we will deal with relations

$$w_n = \sum_{|i|=1}^{\infty} \alpha_n^i w_{n-i} \quad , \quad n = 1, 2, \ldots\,, \tag{6}$$

where w_n depends on all previous $w_{n-1}, w_{n-2}, \ldots$ and all subsequent $w_{n+1}, w_{n+2}, \ldots$. Such relations appear for example in some converse problems of multipoint Pade approximations (cf. [14]). We need assumptions that would guarantee us the convergence of infinite series participating in (6). These series converge if we assume that

$$A(10): \quad \limsup_{n\to\infty} |w_n|^{1/n} = 1\,, \quad \limsup_{n\to\infty} |w_{-n}|^{1/n} \le 1\,,$$

$$A(11): \quad \alpha_n(\lambda) = 1 - \sum_{|i|=1}^{\infty} \alpha_n^i \lambda^{-i} \in H(E_\delta), \quad n = 1, 2, \ldots\,,$$

that is $\alpha_n(\lambda)$ are holomorphic functions in the ring $E_\delta = \{1 - \delta < |\lambda| < 1 + \delta\}$, where $\delta > 0$ does not depend on n. Finally, instead of $A(2)\,, i = \pm 1, \pm 2, \ldots$, of Poincare's theorem we need a stronger assumption

$$A(12): \quad \alpha_n(\lambda) \rightrightarrows \alpha(\lambda) = 1 - \sum_{|i|=1}^{\infty} \alpha^i \lambda^{-i} \quad , \quad \lambda \in E_\delta\,.$$

The following theorem is true.

Theorem 5 *(cf. [14]). Consider the relations (6) and suppose that* $A(10)$, $A(11)$ *and* $A(12)$ *are fulfilled. Then*
1° $\alpha(\lambda)$ *has at least one root on the circumference* $|\lambda| = 1$*;*
2° *it is possible to construct a recurrence equation*

$$w_n = \beta_n^1 w_{n-1} + \ldots + \beta_n^l w_{n-l}, \quad n = 1, 2, \ldots,$$

where l is the number of the roots of the characteristic polynomial lying on the circumference $|\lambda| = \limsup_{n\to\infty} |w_n|^{1/n}$ *(let us denote these roots by* $\lambda_{p_1}, \ldots, \lambda_{p_l}$*), such that*
a) $\{w_n\}_{n=1-l}^{\infty}$ *is its solution;*
b) $\lim_{n\to\infty} \beta_n^i = \beta^i, \quad i = 1, \ldots, l$*;*
c) $\lambda^l - \beta^1\lambda^{l-1} - \ldots - \beta^l = \prod_{i=1}^{l}(\lambda - \lambda_{p_i})$ *.*

0.7 Preliminary definitions and theorems of the theory of continued fractions.

Let us turn to the applications of the results related with Poincare's theorem to the theory of convergence of continued fractions. Consider a continued fraction (1). We will denote the continued fraction (1) by $K_{n=1}^{\infty}\frac{a_n}{b_n}$. A finite truncation $K_{i=1}^{n}\frac{a_i}{b_i} = A_n/B_n$ is called n-th convergent of (1). The continued fraction (1) converges, if the sequence A_n/B_n , $n = 1, 2, \ldots$ has a limit. It is easy to prove by induction that the sequences of nominators $\{A_n\}_{n=-1}^{\infty}$ and denominators $\{B_n\}_{n=-1}^{\infty}$ are both solutions of the recurrence equation (2) with initial values $A_{-1} = 1$, $A_0 = 0$ and $B_{-1} = 0$, $B_0 = 1$. Let us introduce linear fractional transformations $S_n(W) = \frac{a_n}{W+b_n}, \quad n = 1, 2, \ldots$. Using the recurrence equation (2) and initial values A_{-1}, A_0, B_{-1}, B_0 we have

$$\frac{A_{n-1}W + A_n}{B_{n-1}W + B_n} = \frac{A_{n-1}W + b_nA_{n-1} + a_nA_{n-2}}{B_{n-1}W + b_nB_{n-1} + a_nB_{n-2}} = \frac{A_{n-2}\frac{a_n}{W+b_n} + A_{n-1}}{B_{n-2}\frac{a_n}{W+b_n} + B_{n-1}} =$$

$$\frac{A_{n-2}S_n(W) + A_{n-1}}{B_{n-2}S_n(W) + B_{n-1}} = \ldots = \frac{A_{-1}S_1 \circ \ldots \circ S_n(W) + A_0}{B_{-1}S_1 \circ \ldots \circ S_n(W) + B_0} = S_1 \circ \ldots \circ S_n(W) \tag{7}$$

In particular, for $W = 0$ we have $A_n/B_n = S_1 \circ \ldots \circ S_n(0)$. It means that the convergence of (1) is equivalent to that of compositions of the linear fractional transformations $S_n(W) = \frac{a_n}{W+b_n}$ at the point $W = 0$. It is common to study the convergence of such compositions at other points $W \in \mathcal{C}$.

Two continued fractions $K_{n=1}^{\infty}\frac{a_n}{b_n}$ and $K_{n=1}^{\infty}\frac{a_n^*}{b_n^*}$ are called equivalent, if there exists a sequence $r_0 = 1, r_1, r_2, \ldots$ of complex numbers different from 0 and such that $a_n^* = r_{n-1}r_n a_n$, $b_n^* = r_n b_n$, $n = 1, 2, \ldots$. It is easy to see that $\{r_1 \ldots r_n A_n\}_{n=1}^{\infty}$ and $\{r_1 \ldots r_n B_n\}_{n=1}^{\infty}$ are both solutions of the recurrence equation

$$w_n^* = b_n^* w_{n-1}^* + a_n^* w_{n-2}^*\,, \quad n = 1, 2, \ldots$$

with the initial values $A_{-1}^* = 1$, $A_0^* = 0$ and $B_{-1}^* = 0$, $B_0^* = 1$. This means that $\{A_n^*\}_{n=1}^{\infty} = \{r_1 \ldots r_n A_n\}_{n=1}^{\infty}$ and $\{B_n^*\}_{n=1}^{\infty} = \{r_1 \ldots r_n B_n\}_{n=1}^{\infty}$. Thus, $A_n^*/B_n^* = A_n/B_n$, that is equivalent fractions converge or diverge simultaniosly.

Historically, the oldest criterion of convergence of continued fraction (1) with complex variables was given by Worpitsky.

Worpitsky's criterion (cf. [5]). *If* $|a_n| \leq 1/4, \quad n = 1, 2, \ldots,$ *then the continued fraction* $K_{n=1}^{\infty}\frac{a_n}{1}$ *converges.*

Worpitsky's criterion remained obscure for a long time, untill it was rediscovered as a corollary of a stronger Pringsheim's criterion.

Pringsheim's criterion (cf. [6]). *If* $|a_n| + 1 \leq |b_n|, \quad n = 1, 2, \ldots,$ *then the continued fraction (1) converges.*

It is easy to see that $K_{n=1}^{\infty}\frac{a_n}{1}$ is equivalent to $K_{n=1}^{\infty}\frac{a_n^*}{b_n^*}$ with $a_1^* = 2a_1$, $a_n^* = 4a_n$, $n = 2, 3, \ldots$, $b_n^* = 2$, $n = 1, 2, \ldots$. It means that $|a_n| \leq 1/4, \quad n = 1, 2, \ldots \Longrightarrow |a_n^*| + 1 \leq |b_n^*|, \quad n = 1, 2, \ldots$ and Worpitsky's criterion is indeed a corollary of Pringsheim's.

The following statement consists of three different theorems that have the common assumption $A(5)$.

Theorem. *Consider the continued fraction (1) and suppose that the assumption* $A(5)$ *is fulfilled. Then*

1° (**Van Vleck** (cf. [7])) *the continued fraction (1) converges;*

2° (**Perron** (cf. [8])) $\lim_{m\to\infty} \lim_{n\to\infty} S_m \circ \ldots \circ S_n(0) = -\lambda^1$, *where* $S_n(W) = \frac{a_n}{W+b_n}$*;*

3° (**Thron-Waadeland** (cf. [9])) $\lim_{n\to\infty} \frac{S_1 \circ \ldots \circ S_n(-\lambda^1) - S}{S_1 \circ \ldots \circ S_n(0) - S} = 0$, *where* $S = \lim_{n\to\infty} S_1 \circ \ldots \circ S_n(0)$.

Van Vleck's theorem was generalized by Perron in the following way.

Perron's theorem *(cf. [8]). Suppose that the complex numbers* a *and* b *are such that the roots of polynomial* $\lambda^2 - b\lambda - a$ *are distinct in modulus (that is, the assumption* $A(1)$ *is fulfilled). Then there exist sufficiently small positive numbers* ϵ_1 *and* ϵ_2 *(depending on* a *and* b*) such that every continued* *fraction* $K_{n=1}^{\infty}\frac{a_n}{b_n}$ *with* $|a_n - a| \le \epsilon_1,\ |b_n - b| \le \epsilon_2, \quad n = 1, 2, \ldots$ *converges.*

It is easy to see that the assumptions of the last Perron's theorem are weaker than $A(5)$ because they don't require the existence of limits $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$: only a sufficient closeness of a_n and b_n to a and b, respectively, is assumed.

0.8 Refinements of theorems on convergence of continued fraction (1).

It is easy to see that the last Perron's theorem is a corollary of the following theorem.

Theorem 6 *(cf. [15]). Consider the continued fraction (1) and suppose that* $A(8)$ *is fulfilled. Then the continued fraction (1) converges.*

In addition to theorem 6, we can prove the following statements.

Theorem 7 *(cf. [15]). Consider the continued fraction (1) and suppose that* $A(9)$ *is fulfilled. Then* $S_1 \circ \ldots \circ S_n(-\lambda)$ *converges. If* $A(9)$ *holds with* $\theta > |\lambda|$*, then* $S_1 \circ \ldots \circ S_n(0)$ *(that is, the continued fraction (1) itself) converges, too.*

Theorem 8 *(cf. [15]). Consider the continued fraction (1) and suppose that* $A(7)$ *is fulfilled. Then*
1° $\lim_{m\to\infty} \lim_{n\to\infty} S_m \circ \ldots \circ S_n(-\lambda) = -\lambda$;
2° $\lim_{n\to\infty} \frac{S_1\circ\ldots\circ S_n(-\lambda)-S}{S_1\circ\ldots\circ S_n(0)-S} = 0$, *where* $S = \lim_{n\to\infty} S_1 \circ \ldots \circ S_n(-\lambda)$ *(the last limit exists by theorem 7).*

Theorems 6 and 7 generalize Van Vleck's theorem and are corollaries of the assertions concerning the existence of the exceptional solution of the recurrence equation (2) of minimal growth (see assertions 1° of theorems 3 and 4). Theorem 8 generalize Perron and Thron-Waadeland theorems, mentioned above in three author's theorem. Let us derive for example theorem 7 from the assertion 1° of theorem 4 and derive theorem 8 from the assertion 2° of theorem 4. Denote $\{v_n\}_{n=-1}^{\infty}$ the exceptional solution of minimal growth of the recurrence equation (2). Let us represent $\{v_n\}_{n=-1}^{\infty}$ as a linear combination of two linearly independent solutions $\{A_n\}_{n=-1}^{\infty}$ and $\{B_n\}_{n=-1}^{\infty}$ of (2)

$$\{v_n\}_{n=-1}^{\infty} = C_1\{A_n\}_{n=-1}^{\infty} + C_2\{B_n\}_{n=-1}^{\infty}.$$

Using the initial values A_{-1}, A_0, B_{-1}, B_0 we can find $C_1 = v_{-1}$ and $C_2 = v_0$. Then

$$\frac{A_n - \lambda A_{n-1}}{B_n - \lambda B_{n-1}} + \frac{v_0}{v_{-1}} = \frac{v_{-1}(A_n - \lambda A_{n-1}) + v_0(B_n - \lambda B_{n-1})}{v_{-1}(B_n - \lambda B_{n-1})} = \frac{v_n - \lambda v_{n-1}}{v_{-1}(B_n - \lambda B_{n-1})}.$$

The righthand side of this equality tends to 0 by the assertion 1° of theorem 4. Hence, using (7) for $W = -\lambda$, we have $\lim_{n\to\infty} S_1 \circ \ldots \circ S_n(-\lambda) = -v_0/v_{-1}$, that is we proved theorem 7. (Using assertion 1° of theorem 3 we can prove theorem 6 in the same manner). Furthermore, analogously, we have $\lim_{n\to\infty} S_m \circ \ldots \circ S_n(-\lambda) = -v_{m-1}/v_{m-2}$ because $\{v_n\}_{n=-1}^{\infty}$ is the unique solution of minimal growth. Using the assertion 2° of theorem 4 we get the claim 1° of theorem 8. And finally

$$\lim_{n\to\infty} \frac{S_1 \circ \ldots \circ S_n(-\lambda) - S}{S_1 \circ \ldots \circ S_n(0) - S} = \frac{\frac{v_n - \lambda v_{n-1}}{v_{-1}(B_n - \lambda B_{n-1})}}{\frac{v_n}{v_{-1} B_n}} = \frac{v_n - \lambda v_{n-1}}{v_n} \frac{B_n}{B_n - \lambda B_{n-1}}$$

The first multiplier on the righthand side of this equality tends to 0, by the assertion 2° of theorem 4, while it is not hard to see that the second one is bounded. Hence we arrive at the claim 2° of theorem 8. Thus, theorems 7 and 8 are a simple corollaries of theorem 4 (and theorem 6 is a simple corollary of theorem 3).

Let us also remark the following statement. It generalizes Pringsheim's criterion to the case of arbitrary linear fractional transformations

$$S_n(W) = \frac{\alpha_n W + \beta_n}{\gamma_n W + \delta_n}, \quad n = 1, 2, \ldots \tag{8}$$

and is a simple corollary of theorem 2.

Theorem 9 *(cf. [15]). Consider a non-degenerate linear fractional transformations (8) and suppose that the following assumption holds*

$$|\alpha_n| + |\gamma_n| \le q(|\delta_n| - |\beta_n|), \quad n = 1, 2, \ldots$$

with $q < 1$. *Then for each* $|w| < 1$ *there exists a limit* $\lim_{n\to\infty} S_1 \circ \ldots \circ S_n(W) = S$ *such that* S *is independent of* W, $|S| \le 1$ *and* $|S_1 \circ \ldots \circ S_n(W) - S| \le q^n \frac{1+|W|}{1-|W|}, \quad n = 1, 2, \ldots.$

If $\alpha_n = 0$, $\beta_n = a_n$, $\gamma_n = 1$, $\delta_n = b_n$, then theorem 9 practically coincides with Pringsheim's criterion. Theorem 9 is useful, for example, in the study of continued fractions with limit periodic coefficients (see theorems 10,11 and 14) and some other fractions.

0.9 Van Vleck's theorem for regular C-fractions with limit constant coefficients.

In Section 7, we formulated Van Vleck's theorem on convergence of the continued fraction (1) with limit constant coefficients. More precisely, Van Vleck's theorem is formulated as a statement on regular C-fractions with limit constant coefficients.

Van Vleck's theorem (cf. [7]). *Consider a regular C-fraction* $K_{n=1}^{\infty}\frac{a_n z}{1}$ *and suppose that its coefficients have limit* $\lim_{n\to\infty} a_n = a \neq 0$. *Then* $K_{n=1}^{\infty}\frac{a_n z}{1}$ *converges to a function* $f(z)$ *in the domain* $D = \mathcal{C} \setminus \Gamma$, *where* $\Gamma = \{z \in \mathcal{C} :\ z = -t/(4a), \quad t \geq 1\}$; $f(z)$ *is meromorphic (or identically equal to* ∞*) in* D *and the convergence is uniform on every compact* $K \subset D$ *that does not contain poles of* $f(z)$.

It is easy to see that the set Γ in VanVleck's theorem consists of all those $z \in \mathcal{C}$ for which the assumption $A(1)$ for the roots of the polynomial $\lambda^2 - \lambda - az$ fails to hold. In other words, the set Γ consists of all those $z \in \mathcal{C}$ for which the fixed points of linear fractional (by W) transformation $S(W,z) = \frac{az}{W+1}$ are neutral. Let us remind that a fixed point p of a linear fractional transformation $S(W)$ is called an *attractive*, *repulsive* or *neutral* fixed point, if $|S'(p)|$ is, repsectively, less than, bigger than, or equal to 1. If one of the fixed points is attractive, then the other will be necessarily repulsive, and vice versa.

Using theorem 9 we can generalize VanVleck's theorem to the case of arbitrary limit periodic linear fractional transformations.

Theorem 10 . *Consider a natural number* $m \in \mathcal{N}$, *a domain* Ω *of complex plane and linear fractional (by* W*) transformations* $S_n(W,z) = \frac{\alpha_n(z)W+\beta_n(z)}{\gamma_n(z)W+\delta_n(z)}$, $\quad n = 1, 2, \ldots$ *such that*

a) $\alpha_n(z), \beta_n(z), \gamma_n(z), \delta_n(z) \in H(\Omega)\,, n = 1, 2, \ldots$, *that is all coefficients of these linear fractional transformations are holomorphic functions of* z *in the domain* Ω;

b) $\lim_{n\to\infty} \alpha_{nm+l}(z) = \alpha^l(z)$, $\lim_{n\to\infty} \beta_{nm+l}(z) = \beta^l(z)$, $\lim_{n\to\infty} \gamma_{nm+l}(z) = \gamma^l(z)$, $\lim_{n\to\infty} \delta_{nm+l}(z) = \delta^l(z)$, $l = 1, \ldots, m$ *and these limits are uniform on compacts subsets of* Ω.

Let us introduce the following notation:

$$S^l(W,z) = \frac{\alpha^l(z)W + \beta^l(z)}{\gamma^l(z)W + \delta^l(z)}, \quad l = 1, \ldots, m\,, \qquad S(W,z) = S^1 \circ \ldots \circ S^m(W,z) = \frac{\alpha(z)W + \beta(z)}{\gamma(z)W + \delta(z)},$$

$$\Delta(z) = \alpha(z)\delta(z) - \beta(z)\gamma(z), \quad I(z) = \alpha(z) + \delta(z), \quad \Gamma = \{z \in \Omega :\ 0 \leq I(z)^2\Delta(z)^{-1} \leq 4\}\,,$$

$$p^1(z) = \frac{\alpha(z) - \delta(z) - I(z)\sqrt{1 - 4\Delta(z)I(z)^{-2}}}{2\gamma(z)}\,, \ p^l(z) = S^l \circ \ldots \circ S^m(p^1(z), z)\,, \ l = 2, \ldots, m\,, \ z \in \Omega \setminus \Gamma$$

(here, for a non-negative number t, *we select that value of* $\sqrt{t}$ *for which the modulus of* $1 + \sqrt{t}$ *is greater; note that* $1 - 4\Delta(z)I(z)^{-2}$ *is non-negative for all* $z \in \Omega \setminus \Gamma$*),*

$$G = \{(W,z) \in \mathcal{C}^2 :\ z \in \Omega \setminus \Gamma\,, \quad W \neq p^1(z), \ldots, W \neq p^m(z)\},$$

Suppose that $\Delta(z) \neq 0$, $z \in \Omega$. Then the sequence of compositions $S_1 \circ \ldots \circ S_n(W, z)$ converges to a function $f(z)$ in $\mathcal{C}^2$-domain G; $f(z)$ is independent of W and meromorphic in z, and the convergence is uniform in the spheric metric on compact subsets of G.

An interesting question arises in connection with VanVleck's theorem. Is the cut Γ indeed the set of singularities of the limit function $f(z)$? More precisely, is it true that $f(z)$ can not be meromorphically continued to a domain $\mathcal{C} \setminus (\Gamma \setminus \{|z - z_0| < \epsilon\})$, where $z_0 \in \Gamma$, ϵ is an arbitrary positive number? Gonchar proved that it is indeed true.

Gonchar's addition to VanVleck's theorem. *Under assumptions of VanVleck's theorem, the function $f(z)$ can not be meromorphically continued to a domain $\mathcal{C} \setminus (\Gamma \setminus \{|z - z_0| < \epsilon\})$, where $z_0 \in \Gamma$, $\epsilon > 0$.*

The analogous question can be formulated concerning the set Γ in theorem 10. We can answer in the positive in the cases of C- and T-fractions with limit periodic coefficients. The proof is based on an extremal property of the set Γ that can be formulated in terms of transfinite diameter in the case of regular C-fractions and in terms of two-point version of transfinite diameter in the case of T-fractions.

0.10 An extremal property of the set of singularities of a regular C-fraction with limit periodic coefficients.

By definition, the transfinite diameter of a compact set $F \subset \mathcal{C}$ is equal to $\lim_{n\to\infty} V_n^{\frac{2}{(n-1)n}}$, where $V_n = \max_{(z_1,\ldots,z_n)\in F} \prod_{i<j} |z_i - z_j|$. It is easy to prove that $V_n^{\frac{2}{(n-1)n}}$ is monotonically decreasing and consequently the corresponding limit exists. It is well known that $d(F) = \tau(F) = C(F)$, where $\tau(F)$ is Chebyshev's constant and $C(F)$ is the capacity of F (the definitions of $\tau(F)$ and $C(F)$ and proofs of these equalities can be found in [10]). As a simple corollary of the first of these equalities, we have the equality $d(F^*) = d(F)^{1/k}$, where $F^* = \{z \in \mathcal{C} : \, q(z) \in F\}$, $q(z) = z^k + \ldots$ is an arbitrary polynomial of degree k with leading coefficient 1 and the equality $d(E) = l(E)/4$, where E is an arbitrary segment with the length $l(E)$.

It is convenient to formulate the result for the fraction $K_{n=1}^{\infty} \frac{a_n z^{-1}}{1}$ (this fraction coincides, after substitution $z \to z^{-1}$, with the regular C-fraction).

The following theorem holds.

Theorem 11 . *Suppose that the coefficients of a continued fraction $K_{n=1}^{\infty} \frac{a_n z^{-1}}{1}$ have periodic limits $\lim_{n\to\infty} a_{nm+l} = a^l \neq 0, \quad l = 1, \ldots, m$. Then*

1° *the fraction* $K_{n=1}^{\infty}\frac{a_n z^{-1}}{1}$ *converges to a function* $f(z)$ *in the domain* $D = \mathcal{C} \setminus (\Gamma \bigcup E)$, *where* Γ *is defined like in theorem 10 (more precisely,* $\Gamma = \Gamma(a^1, \ldots, a^m) = \{z \in \mathcal{C} : z^m I(z)^2 = 4(-1)^m a^1 \ldots a^m t, \quad 0 \le t \le 1\}$, $I(z) = I(z; a^1, \ldots, a^m) = Tr \prod_{l=1}^{m} \begin{bmatrix} 0 & a^l z^{-1} \\ 1 & 1 \end{bmatrix}$*) and* E *is a finite subset of* $\mathcal{C}$ *(more precisely, in the notations of theorem 10* $E = E(a^1, \ldots, a^m)$ *is the set* $\{z \in \mathcal{C} \setminus \Gamma : 0 = p^1(z), \ldots, p^m(z)\}$*);* $f(z)$ *is meromorphic in* D *and the convergence is uniform on every compact subset* $K \subset D$ *that does not contain poles of* $f(z)$*. For* $m = 1$, *the finite set* $E(a^1)$ *is empty;*

2° $f(z)$ *can be meromorphically continued in the domain* $\mathcal{C} \setminus \Gamma$;

3° $f(z)$ *can not be meromorphically continued to* $\mathcal{C} \setminus (\Gamma \setminus \{|z - z_0| < \epsilon\})$, *where* $z_0 \in \Gamma$ *and* $\epsilon > 0$;

4° $d(\Gamma) = |a^1 \ldots a^m|^{1/m} = \min_F d(F)$, *where minimum is taken over all compact sets* F *such that* $f(z)$ *considered in a small neiborhood of the point* $z = \infty$ *can be meromorphically continued in the domain* $\mathcal{C} \setminus F$;

5° $d(\Gamma) = \min_G d(G)$, *where minimum is taken over all compact subsets* G *such that the algebraic function* $g(z) = \sqrt{I(z)^2 + 4(-1)^{m-1} a^1 \ldots a^m z^{-m}}$ *admits a selection of regular branch in the domain* $\mathcal{C} \setminus G$.

For $m = 1$ the assertion 1° of this theorem coincides with the classical Van Vleck's theorem on convergence of regular C-fractions with limit constant coefficients. For arbitrary $m \in \mathcal{N}$ the assertions 1° and 2° are corollaries of theorem 10. For $m = 1$ the assertion 3° coincides with Gonchar's addition to VanVleck's theorem. For each $m = 1, 2, \ldots$ the assertion 3° follows from the assertion 4° because of the evident strict inequality $d(\Gamma \setminus \{|z - z_0| < \epsilon\}) < d(\Gamma)$. The equality $d(\Gamma) = |a^1 \ldots a^m)|^{1/m}$ of 4° is a simple corollary of the definition of Γ and the properties of transfinite diameter mentioned above. The main contents of 4° is the inequality $d(\Gamma) \le d(F)$. The assertion 5° can be obtained from this main inequality if we consider the purely periodic case $a_{nm+l} = a^l$, $n = 1, 2, \ldots, l = 1, \ldots, m$ and find the explicit value of the limit function $f(z)$ (it is easy to see that algebraic functions $f(z)$ and $g(z)$ have the same branch points).

The proof of the main inequality $d(\Gamma) = |a^1 \ldots a^m)|^{1/m} \le d(F)$ is based on two well known results. One provides the expression for the coefficient a_n of regular C-fraction in terms of Hankel's determinants of the limit function $f(z)$ (cf. [11]), and as a consequence of this expression we have the equality $\lim_{n\to\infty} |H_n|^{1/n^2} = |a^1 \ldots a^m|^{1/m}$, where $H_n = \begin{vmatrix} f_1 & \ldots & f_n \\ \ldots & \ldots & \ldots \\ f_n & \ldots & f_{2n-1} \end{vmatrix}$ are Hankel's determinants of the expansion $\sum_{n=1}^{\infty} f_n z^{-n}$ of the limit function $f(z)$ in some neigborhood of the point $z = \infty$. The second result is the well known Polya's theorem on estimating of the upper limit of Hankel's determinants of holomorphic function in terms of the transfinite diameter of the set of singularities.

Polya's theorem (cf. [12]). *Let F be a compact set and let $f(z)$ be a holomorphic function in the component of $\mathcal{C} \setminus F$, containing the point $z = \infty$. If $f(z) = \sum_{n=1}^{\infty} f_n z^{-n}$ for sufficiently large z and* $H_n = \begin{vmatrix} f_1 & \dots & f_n \\ \dots & \dots & \dots \\ f_n & \dots & f_{2n-1} \end{vmatrix}$, *then* $\limsup_{n\to\infty} |H_n|^{1/n^2} \leq d(F)$.

One can choose the parameters $a^1, \dots, a^m$ in such a way that the polynomial $I(z) = I(z; a^1, \dots, a^m)$ of z^{-1} and the number $4(-1)^{m-1}a^1 \dots a^m$ will be equal to any given polynomial of the form $1+\alpha_1 z^{-1} + \dots + \alpha_{[m/2]} z^{-[m/2]}$ and for any given number c we can reformulate the assertion $5°$ of theorem 11 in the following way.

Corollary 1 . *Consider an algebraic function $g(z) = \sqrt{p(z)^2 + c}$ or $g(z) = \sqrt{z(zp(z)^2 + c)}$, where $p(z) = z^k + \dots$ is an algebraic polynomial of degree $k \geq 1$, $c \neq 0$. Then the transfinite diameter of the set $\Gamma = \{z \in \mathcal{C} : p(z)^2 + ct = 0, \quad o \leq t \leq 1\}$ in the first case and $\Gamma = \{z \in \mathcal{C} : zp(z)^2 + ct = 0, \quad o \leq t \leq 1\}$ in the second does not exceed the transfinite diameter of any compact set G such that $g(z)$ admits a selection of a regular branch in the domain $\mathcal{C} \setminus G$. Additionally, the transfinite diameter of the set Γ equals $|c|^{1/2k}$ in the first case and $|c|^{1/(2k+1)}$ in the second.*

This corollary enables in some cases to find the explicit form of the cuts Γ with minimal transfinite diameter in well known Stahl's theorem (cf. [13]) on convergence of Pade approximations of algebraic functions. For example, in the case of four branch points lying on the vertices $((z+a)^2+b)^2+c = 0$ of an arbitrary parallelogramm we have $\Gamma = \{z \in \mathcal{C} : ((z+a)^2+b)^2+ct = 0, \quad o \leq t \leq 1\}$. It is easy to see that Γ consists of two pieces of a hyperbola. Each of these pieces connects two vertices with the smaller mutual distance. In the case of a romboid, Γ consists of its diagonals. Another interesting example is constructed for branch points $0, 2, \pm i$. In this case $\Gamma = \{z \in \mathcal{C} : z(z-1)^2 - 2t = 0, \quad o \leq t \leq 1\}$ consists of the segment $[0, 2]$ and a curve passing through points $\pm i, 1/3$.

0.11 A theorem on circle convergence of T-fractions.

A continued fraction $K_{n=1}^{\infty} \frac{a_n z}{1+b_n z}$ is called T-fraction. It is well known that one can put in correspondence with each T-fraction two formal power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^{-n}$ such that at the point $z = 0$ $\quad A_n(z)/B_n(z) - f(z) = O(z^{n+1})$ and at the point $z = \infty$ $\quad A_n(z)/B_n(z) - g(z) = O(z^{-(n+1)})$. This means that the expansions of the n-th convergent $A_n(z)/B_n(z)$ of T-fraction have the same coefficients as $f(z)$ for $1, z, \dots, z^n$ and as $g(z)$ for $1, z^{-1}, \dots, z^{-n}$. But the convergence

$A_n(z)/B_n(z) \to f(z), \quad |z| < R_1$ and $A_n(z)/B_n(z) \to g(z), \quad |z| > R_2$ requires a special proof. This proof is simple, for example, in the case $a_n = -b_n$, $n = 1, 2, \ldots$, because in this case one can easily find the explicit formulas for $A_n(z)/B_n(z)$, using the recurrence equation (2).

In the general case we can prove, using theorems 6 and 7, the following statement.

Theorem 12 . *Consider a T-fraction* $K_{n=1}^{\infty}\frac{a_n z}{1+b_n z}$. *The following assertions hold true.*

1° *T-fraction* $K_{n=1}^{\infty}\frac{a_n z}{1+b_n z}$ *converges to a function* $f(z)$ *in the circle* $|z| < R_1 = (\sqrt{B+H}+\sqrt{H})^{-2}$, *where* $B = \limsup_{n\to\infty}|b_n|$, $H = \limsup_{n\to\infty}|a_n + b_n|$; $f(z)$ *is meromorphic in* $|z| < R_1$ *and the convergence is uniform on every compact set* $K \subset \{|z| < R_1\}$ *that does not contain poles of* $f(z)$;

2° *the sequence of compositions* $S_1 \circ \ldots \circ S_n(-1, z)$, $n = 1, 2, \ldots$, *where* $S_n(W, z) = \frac{a_n z}{W+1+b_n z}$, *converges to a function* $g(z)$ *in the set* $|z| > R_2 = b(\sqrt{1+h}+\sqrt{h})^2$, *where* $b = \limsup_{n\to\infty}|b_n^{-1}|$, $h = \limsup_{n\to\infty}|1 + a_n/b_n|$; $g(z)$ *is meromorphic in* $|z| > R_2$ *and the convergence is uniform on every compact set* $K \subset \{|z| > R_2\}$ *that does not contain poles of* $g(z)$;

3° *T-fraction* $K_{n=1}^{\infty}\frac{a_n z}{1+b_n z}$ *converges to a function* $g(z)$ *in the set* $|z| > R_2^*$, *where* $R_2^* = R_2$, *if* $h \geq 1/3$, *and* $R_2^* = 2b(1-h)^{-1} \geq R_2$, *if* $h \leq 1/3$, *and the convergence is uniform on every compact set* $K \subset \{|z| > R_3\}$ *that does not contain poles of* $g(z)$.

The formulas for R_1 and R_2 are sharp in the sense that we can construct examples of T-fractions realizing values of R_1 and R_2. This will be shown in the next section.

0.12 An extremal property of the singularities set of T-fractions with limit periodic coefficients.

Now we need to define the notion of two-point version of transfinite diameter of a compact sets not containing the point $z = 0$.

Consider a compact set F such that $0 \notin F$. The two-point version of transfinite diameter $e(F)$ is defined as $\lim_{n\to\infty} V_n^{\frac{2}{(n-1)n}}$, where $V_n = \max_{(z_1,\ldots,z_n)\in F}\prod_{i<j}|z_i - z_j||z_i^{-1} - z_j^{-1}| = \max_{(z_1,\ldots,z_n)\in F}\prod_{i<j}|z_i - z_j|^2|z_i z_j|^{-1}$.

In analogy with the usual transfinite diameter, it is easy to prove that the sequence $V_n^{\frac{2}{(n-1)n}}$ is monotonically decreasing and consequently the limit exists. It is evident that two-point version of transfinite diameter is invariant under transformations $z \to \alpha z^{\pm 1}$, $\alpha \in \mathcal{C}$. It is possible to prove the properties of two-point version of transfinite diameter analogous to the corresponding properties of the usual transfinite diameter. In particular, $e(F) = \tau^*(F) = C^*(F)$, where the two-point version of Chebyshev's constant is defined as $\tau^*(F) = \lim_{n\to\infty}(\min_{p_n(z)\in\mathcal{P}_n(F)}\max_{z\in F}|\frac{p_n(z)^2}{z^n p_n(0)}|)^{1/n}$, $\mathcal{P}_n(F)$

denotes the set of all polynomials $z^n + \ldots$ of degree n with leading coefficient $= 1$ and whose roots belong to F. Two-point version of capacity of F is defined as $C^*(F) = e^{-(\gamma_0+\gamma_\infty+g)/2}$, where $\gamma_0 = \lim_{z\to 0}(g_0(z,0) + \ln|z|)$, $\gamma_\infty = \lim_{z\to\infty}(g_\infty(z,\infty) - \ln|z|)$, $g_0(z,0)$ and $g_\infty(z,\infty)$ are Green's functions of components of $\mathcal{C} \setminus F$, containing the points $z = 0$ and $z = \infty$, respectively, $g = 0$, if these components are different and $g = g_0(\infty, 0) = g_\infty(0,\infty)$, if these components coincide.

The following analog of Polya's theorem can be also proved.

Theorem 13 . *Let F be a compact set, not containing the point $z = 0$, and let $f(z)$ be a function meromorphic in the components of $\mathcal{C} \setminus F$ that contains the points $z = 0$ and $z = \infty$. If $f(z) = \sum_{n=k}^{\infty} a_n z^n$ for sufficiently small $|z|$, $f(z) = \sum_{n=-\infty}^{l} b_n z^n$ for sufficiently large $|z|$, $f_n = a_n - b_n$, $n = 0, \pm 1, \ldots$, where $a_n = 0$, if $n < k$, and $b_n = 0$, if $n > l$, and $H_n^* = \begin{vmatrix} f_{-n} & \ldots & f_0 \\ \ldots & \ldots & \ldots \\ f_0 & \ldots & f_n \end{vmatrix}$, then $\limsup_{n\to\infty} |H_n^*|^{2/n^2} \le e(F)$.*

Finally, let us formulate an analog of Van Vleck's theorem for T-fractions with limit periodic coefficients.

Theorem 14 . *Consider a T-fraction $K_{n=1}^{\infty} \frac{a_n z}{1+b_n z}$ and suppose that its coefficients have periodic limits $\lim_{n\to\infty} a_{nm+l} = a^l \neq 0$, $\lim_{n\to\infty} b_{nm+l} = b^l \neq 0$, $l = 1, \ldots, m$. Then*

1° *T-fraction $K_{n=1}^{\infty} \frac{a_n z}{1+b_n z}$ converges to a function $f(z)$ in $D = \mathcal{C} \setminus (\Gamma \bigcup E)$, where Γ is defined like in theorem 10 (more precisely, $\Gamma = \Gamma(a^1, \ldots, a^m; b^1, \ldots, b^m) = \{z \in \mathcal{C} : I(z)^2 = 4(-1)^m z^m a^1 \ldots a^m t, \ 0 \le t \le 1\}$, $I(z) = I(z; a^1, \ldots, a^m; b^1, \ldots, b^m) = Tr \prod_{l=1}^{m} \begin{bmatrix} 0 & a^l z \\ 1 & 1 + b^l z \end{bmatrix}$), E is a finite set of $\mathcal{C}$ (more precisely, in notation of theorem 10 $E = E(a^1, \ldots, a^m; b^1, \ldots, b^m)$ is the set $\{z \in \mathcal{C} \setminus \Gamma : 0 = p^1(z), \ldots, p^m(z)\}$); $f(z)$ is meromorphic in D and the convergence is uniform on every compact set $K \subset D$ which does not contain poles of $f(z)$; if $m = 1$, then $E = E(a^1; b^1)$ is empty;*

2° *$f(z)$ can be meromorphically continued in the domain $\mathcal{C} \setminus \Gamma$;*

3° *$f(z)$ cannot be meromorphically continued to $\mathcal{C} \setminus (\Gamma \setminus \{|z - z_0| < \epsilon\})$, where $z_0 \in \Gamma$ and $\epsilon > 0$;*

4° *$e(\Gamma) = |\frac{a^1 \ldots a^m}{b^1 \ldots b^m}|^{1/m} = \min_F e(F)$, where minimum is taken over all compact sets F such that $0 \notin F$ and $f(z)$ considered at some small neighborhoods of the points $z = 0$ and $z = \infty$ can be meromorphically continued in the components of $\mathcal{C} \setminus F$ containing points $z = 0$ and $z = \infty$ (if these components coincide then the meromorphic continuations of $f(z)$ must be the same).*

The assertions 1° and 2° of this theorem are corollaries of theorem 10. The assertion 3° follows from the assertion 4°, because of the evident strict inequality $e(\Gamma \setminus \{|z - z_0| < \epsilon\}) < e(\Gamma)$. The equality $e(\Gamma) = |\frac{a^1 \ldots a^m}{b^1 \ldots b^m}|^{1/m}$ is a simple corollary of the definition of Γ and the properties of two-point version of transfinite diameter mentioned above.

The proof of the main inequality $e(\Gamma) = |\frac{a^1 \ldots a^m}{b^1 \ldots b^m}|^{1/m} \leq e(F)$ is based on the analog Polya's theorem and the well known expressions for the coefficients a_n and b_n of T-fraction by means of Hankel's determinants of the expansions of the limit function at the points $z = 0$ and $z = \infty$ (cf. [11]). As a consequence of these expressions we have the equality $\lim_{n\to\infty} |H_n^*|^{2/n^2} = |\frac{a^1 \ldots a^m}{b^1 \ldots b^m}|^{1/m}$, where the Hankel's determinants H_n^* are the same as in the analog of Polya's theorem.

Let us note that $e(F) = e(F^{-1})$, where $F^{-1} = \{z \in \mathcal{C} : z^{-1} \in F\}$, and consequently theorem 14 admits a symmetric formulation for fractions $K_{n=1}^{\infty} \frac{a_n z^{-1}}{1 + b_n z^{-1}}$.

Let us pinpoint for the case $m = 1$ the geometry of the set $\Gamma = \Gamma(a;b) = \{z \in \mathcal{C} : (1 + bz)^2 + 4atz = 0, \quad 0 \leq t \leq 1\}$, for various values of $k = a/b$. For simplicity, we assume that $b = 1$. Evidently, Γ contains the point $z = -1$ and is invariant under transformation $z \to z^{-1}$.

If $k > 0$, then Γ is the segment $[-R, -1/R]$, where $R = 1 + 2k + 2\sqrt{k^2 + k} > 1$. If $k = 0$, then Γ is the point $z = -1$. If $-1 < k < 0$, then Γ is the piece of the circle $|z| = 1$. If $k = -1$, then Γ is the circle $|z| = 1$. If $k < -1$, then Γ is the circle $|z| = 1$ united with the segment $[1/R, R]$, where $R = 2|k| - 1 + 2\sqrt{k^2 - |k|} > 1$. If $Imk \neq 0$, then $\mathcal{C} \setminus \Gamma$ is connected. Thus, the domain $\mathcal{C} \setminus \Gamma$ consists of two components only in the case $k \leq -1$, in other cases $\mathcal{C} \setminus \Gamma$ is connected.

It is easy to see that in the case $k \leq -1$ the numbers $R^{\pm 1}$ coincide with numbers $R_{1,2}$ defined in theorem 12. Consequently, the formulas for $R_{1,2}$ in theorem 12 are sharp.

Having in mind Stahl's theorem, it is natural to formulate the following question.

Question. *Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^{-n}$ be expansions of algebraic functions $f(z)$ and $g(z)$ at the points $z = 0$ and $z = \infty$, respectively, and let Γ be a compact set such that*
a) $0 \notin \Gamma$,
b) $f(z)$ and $g(z)$ can be holomorphically continued to the components of $\mathcal{C} \setminus \Gamma$ that contain the points $z = 0$ and $z = \infty$, respectively, (if these components coincide the holomorphic continuations of $f(z)$ and $g(z)$ must be the same),
c) two-point version of transfinite diameter $e(\Gamma)$ is minimal among all compact sets for which the properties a) and b) are valid.

Is it true that two-point Pade approximations of these expansions converge in the components of $\mathcal{C} \setminus \Gamma$ that contain the points $z = 0$ and $z = \infty$ to the holomorphic continuations of $f(z)$ and $g(z)$, respectively ?

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