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Indefinitely oscillating function

T. Elbouayachi

Department of Mathematics
University of South Carolina

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T.Elbouayachi[*]

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Abstract

This article is a detailled study of a class of indefinitely oscillating function in $H^s(\mathbb{R})$. It's a class of functions of Sobolev space $H^s(\mathbb{R})$ which have for all m integer one primitive of the order m in the same space.

We will try to characterize at the same time the oscillations as well as the Holder exponents. Our contribution consists of measuring these two parameters by using the L^2 instead of L^∞ norm. This work is an extension of a previous study that has been done by Y.Meyer and S.Jaffard. We introduced chirp's functional space by using Sobolev spaces. We observe that a chirp is an asymptotic signal which is of the form $s(t) = A(t)e^{i\lambda\phi(t)}$, where $A(t)$ and $\phi(t)$ are two regular functions and $\lambda >> 1$. (Actually $\phi'(t) \longrightarrow \infty$ when $t \longrightarrow t_0$). The function e^{ix} is fundamental in the last definition. It's an indefinitely oscillating function in L^∞ sense. It will be replaced by what we call an indefinitely oscillating function. Hong Xu has worked on chirps by using L^p indefinitely oscillating function defined on $\mathbb{R}^n$. Our contribution consists of studying the behavior of the Fourier transform of indefinitely oscillating function in H^s on $\mathbb{R}$ around 0. The motivation for studying the indefinitely oscillating function is given by chirps. The first example considered herein is the cry of a bat. The signal is given by the formula: $F(x) = e^{\frac{-i}{x}} - 1$ which is a function of real variable. Its Fourier Transform on the real axe is given by: $\hat{F}(\xi) = \frac{J_1(\xi^{\frac{1}{2}})}{\xi}^{\frac{1}{2}}$ if $\xi > 0$ $\hat{F}(\xi) = 0$ elsewhere. J_1 is the Bessel function of indicia one. We have then a discontinuity at the origin, this is obviously shown by the fact that $e^{\frac{-i}{x}} - 1 \sim \frac{-i}{x}$ at the infinity. A second example is the emission of chirps by vibrating lorries to localize petroleum fields. It concerns signals with large band of frequency with short-lived. So that the detection possibility of a large range of objects by avoiding the interference thanks to short duration of those signals. The last example is given by gravitational waves. The existence of these waves follows from the theory of general relativity. The scientist world has already indirect evidences of their existence. But the gravitational waves have never been measured by experiences. Several sources are susceptible to product these gravitational waves:

[*]Faculte Scientifique et Technique de Tanger

1. Coalescence of a binary star gives birth to a chirp.

2. Collapse of neutrons star.

3. Collapse of black holes.

Let E a functional Banach space, which means $\mathcal{S}(I\!R) \subset E \subset \mathcal{S}'(I\!R)$ where the two injections are continuous. We say that $f \in E$ is indefinitely oscillating (in E sense) if, for all m, there exist $f_m \in E$ such that $f(x) = (\frac{d}{dx})^m f_m(x)$. A particular case is extremely simple and let us to understood the nature of the difficulties which occurs. If $E = L^2(I\!R)$, the problem can be solved by Fourier transform and it becomes $\hat{f}(\xi) = (i\xi)^m \hat{f}_m(\xi)$. Then $f \in L^2(I\!R)$ is indefinitely oscillating in $L^2(I\!R)$ sense if and only if $\int_{-\epsilon}^{\epsilon} \mid \hat{f}(\xi) \mid^2 \, d\xi = \mathcal{O}(\epsilon^q)$ for all integer q.
We will systematically study the case of Sobolev spaces $H^s(I\!R)$. The characterization of functions indefinitely oscillating is then similar.
In the second parte of this chapter we will interested on a complex problem, those of functions which are only defined on a half axe, $[0, \infty)$ for example. The difficulty results from that we can not use the notion of Fourier Transform. We can get round this difficulty by the following way: Either one defines the space of functions indefinitely oscillating on $[0, \infty)$ as the space of restrictions to $[0, \infty)$ of functions indefinitely oscillating on $I\!R$, or one defines function f indefinitely oscillating on $[0, \infty)$ using the scalar product $< f, \varphi_{(a,b)} >$ where $\varphi \in C_0^\infty([-1, 1])$, which have a sense if $\varphi_{(a,b)}(x) = \frac{1}{a}\varphi(\frac{x-b}{a})$ and $b \geq a + 1$. One shows than that the two definitions are identical.

1 Indefinitely oscillating function theory on real axe

1.1 Fourier transform characterization

1.1.1 $L^2(I\!R)$ or $H^s(I\!R)$ case

One considers a function $f(x)$ of real or complex value, defined on $I\!R$.

Definition 1 *f is an indefinitely oscillating function in $L^2(I\!R)$ (resp $H^s(I\!R)$) sense if, for all integer $n \geq 0$, an n-th primitive function $f_n(x)$ de $f(x)$, defined by $(\frac{d}{dx})^n f_n(x) = f(x)$, belongs to $L^2(I\!R)$ (resp $H^s(I\!R)$).*

There is a general definition in the general arbitrary Banach space. This definition is a particular case of the previously mentioned general definition.
Studying the behavior of the Fourier transform of the function around zero is another way of characterizing the indefinitely oscillating functions in $H^s(I\!R)$.

Lemma 1 *f is an indefinitely oscillating function in $H^s(I\!R)$ if and only if f belongs to $H^s(I\!R)$ and for all N one has:*

$$\int_{-1}^{1} \frac{\mid \hat{f}(\xi) \mid^2}{\mid \xi \mid^{2N}} \, d\xi < \infty$$

Proof of Lemma

Using the classical definition of indefinitely oscillating function:

$$\forall N,\ \exists f_N \in H^s(\mathbb{R}) \text{ such that } f = \frac{d^N f_N}{dx^N}$$

and by application of Fourier transform one has: $\hat{f}(\xi) = \hat{f}_N(\xi)(i\xi)^N$. A necessary and sufficient condition that f is an indefinitely oscillating in H^s on all IR is that for all integer N one has:

$$\int_{-\infty}^{+\infty} \frac{|\ \hat{f}(\xi)\ |^2}{|\ \xi\ |^{2N}}\, d\xi < \infty$$

$\hat{f}(\xi)$ belongs to H^s, then $\hat{f}$ is in $L^2([-1,1])$ and

$$\int_1^{+\infty} |\ \xi\ |^{2s} |\ \hat{f}(\pm\xi)\ |^2\ d\xi < \infty.$$

IRemark:
The condition1 does not depend on s. Hence, the characterization of indefinitely oscillating functions' spaces either in $L^2(\mathbb{R})$ or in $H^s(\mathbb{R})$ is the same.

Another equivalent definition is:

Lemma 2 *The two following properties are equivalent:*

1. *f is an indefinitely oscillating function in $H^s(\mathbb{R})$*
2. *f belongs to $H^s(\mathbb{R})$ and for all integer q and all $|\ \xi\ |\leq 1$ one has:*

$$\int_{-\xi}^{\xi} |\ \hat{f}(t)\ |^2\ dt = \mathcal{O}(|\ \xi\ |^q)$$

Proof:

- **(1) $\longrightarrow$ (2)**

$$\begin{aligned}\int_{-\xi}^{\xi} |\ \hat{f}(t)\ |^2\ dt &= \int_{-\xi}^{\xi} \frac{|\ \hat{f}(t)\ |^2}{|\ t\ |^{2N}} |\ t\ |^{2N}\ dt \\ &\leq |\ \xi\ |^{2N} \int_{-\xi}^{\xi} \frac{|\ \hat{f}(t)\ |^2}{|\ t\ |^{2N}}\, dt\end{aligned}$$

- **(2) $\longrightarrow$ (1)**

A dyadic decomposition gives:

$$\begin{aligned}\int_{2^{-j}}^{2^{-j+1}} \frac{|\ \hat{f}(\xi)\ |^2}{|\ \xi\ |^{2N}}\, d\xi &\leq 2^{2jN} \int_{2^{-j}}^{2^{-j+1}} |\ \hat{f}(\xi)\ |^2\ d\xi \\ &\leq C 2^{j(2N-q)}\end{aligned}$$

Normally convergent series for $q > 2N$.

1.2 Characterization by Littlewood-Paley analysis

1.2.1 $L^2(I\!R)$ CASE

Theorem 3 *The three following proprieties are equivalents*

1. f *belongs to* $L^2(I\!R)$ *and* $|| \triangle_j (f)||_2 \leq C_N 2^{jN}$ *for* $j \leq -1$
2. f *belongs to* $L^2(I\!R)$ *and* $\forall n \geq 0$ $f = \dfrac{d^n f_n(x)}{dx^n}$ *where* f_n *is in* $L^2(I\!R)$.
3. $\forall\, n$ $f = \dfrac{d^n f_n(x)}{dx^n}$ *where* f_n *is in* $H^n(I\!R)$.

Proof of theorem (3)

- **(1)** $\Longrightarrow$ **(2)** Littlewood-Paley decomposition gives $f = \sum_j \triangle_j(f)$.

  Let f_n a n-th primitive defined by $f(x) = \dfrac{d^n f_n(x)}{dx^n}$. If f and f_n belong to $L^2(I\!R)$, and $f = (\dfrac{d}{dx})^n f_n$ then one has

$$\hat{f}(\xi) = (i\xi)^n \hat{f}_n(\xi)$$

with

$$\hat{f}(\xi) = \sum_j \widehat{\triangle_j(f)}(\xi)$$

$\hat{f}$ and $\hat{f}_n$ belong to $L^2(I\!R)$. Then $\hat{f}_n$ is defined almost everywhere by

$$\hat{f}_n(\xi) = \frac{\hat{f}(\xi)}{(i\xi)^n}$$

$$\begin{aligned} \widehat{\triangle_j(f)}(\xi) &= \widehat{f * \psi_j}(\xi) \\ &= \hat{f}(\xi)\hat{\psi}(\frac{\xi}{2^j}) \end{aligned}$$

$\widehat{\triangle_j f}$ is supported by $\alpha 2^j \leq |\xi| \leq \beta 2^j$ with $0 < \alpha < \beta$. One has then

$$\hat{f}_n(\xi) = \sum \frac{\widehat{\triangle_j(f)}(\xi)}{(i\xi)^n}$$

and

$$||\frac{\widehat{\triangle_j(f)}(\xi)}{(i\xi)^n}||_2 \leq C_n 2^{-jn} || \triangle_j f||_2$$

- **(2) $\Longrightarrow$ (1)** Let n a positive integer, f is in $L^2(I\!R)$ and $\forall n \geq 0$ one has $f(x) = \dfrac{d^n f_n(x)}{dx^n}$ where f_n and f belong to $L^2(I\!R)$. Then one has :

$$\hat{f}(\xi)\hat{\psi}(2^{-j}\xi) = (i\xi)^n \hat{f}_n(\xi)\hat{\psi}(2^{-j}\xi)$$

We deduct that:

$$\begin{aligned} || \triangle_j f||_2 &= ||\widehat{\triangle_j f}||_2 \\ &= ||(i\xi)^n \hat{f}_n(\xi)\hat{\psi}(2^{-j}\xi)||_2 \\ &\leq C_n 2^{jn} ||\hat{f}_n(\xi)\hat{\psi}(2^{-j}\xi)||_2 \end{aligned}$$

For $j \leq -1$ one has

$$|| \triangle_j f||_2 \leq C_n 2^{jn}$$

And for $j \geq 0$

$$|| \triangle_j f||_2 \leq C_n 2^{jn} ||\hat{f}_n(\xi)\hat{\psi}(2^{-j}\xi)||_2 \leq \epsilon_{j,n}$$

with $\sum_0^{\infty} \epsilon_{j,n}^2 < \infty$, since f belongs to $L^2(I\!R)$.

1.2.2 $H^s(I\!R)$ CASE

Theorem 4 *The two following proprieties are equivalents:*

1. *$\forall n$, there exist f_n in $H^s(I\!R)$ such that $f = \dfrac{d^n f_n}{dx^n}$ in a distributions sense.*

2. *$f \in H^s(I\!R)$ and $|| \triangle_j (f)||_2 \leq C_N 2^{jN}$ $\forall N$ and $\forall j \leq -1$.*

We say that f is indefinitely oscillating in $H^s(I\!R)$. We remark then, that f_n is in $H^{s+n}(I\!R)$.

We start by giving the following lemma:

Lemma 5 *If $\hat{f}$ is supported by par $[\alpha, \beta]$ where $0 < \alpha < \beta$ then $||f||_2 \simeq ||f||_{H^s}$ for all s in IR.*

- **Proof of (1) $\Longrightarrow$ (2)**

f is in $H^s(I\!R)$ and $\forall n$; $\exists f_n \in H^s(I\!R)$ such that $f = \dfrac{d^n f_n}{dx^n}$.

Using Fourier transform one has: $\hat{f}(\xi) = (i\xi)^n \hat{f}_n(\xi)$, and hence f_n is in $H^s(I\!R)$ which is giving either by:

$$\int (1+\xi^2)^s \frac{|\hat{f}(\xi)|^2}{\xi^{2n}} \, d\xi < \infty$$

So we have

$$\int_{-1}^{1} \frac{|\hat{f}(\xi)|^2}{\xi^{2n}}\, d\xi < \infty$$

.

Using Littlewood-Paley decomposition we can write: $f = \sum_{-\infty}^{-1} \triangle_j f + S_0(f)$. Applying Fourier we have: $\hat{f}(\xi) = \sum_{-\infty}^{+\infty} \hat{\triangle}_j f(\xi)$. As $\widehat{\triangle_j f}$ is supported by dyadic corona $\alpha 2^j \leq |\xi| \leq \beta 2^j$ with $0 < \alpha < \beta$. Hence the inequality

$$\int_{-1}^{1} \frac{|\hat{f}(\xi)|^2}{\xi^{2n}}\, d\xi < \infty$$

become then :

$$\int_{-1}^{1} \frac{|\sum_{-\infty}^{+\infty} \triangle_j \widehat{f(\xi)}|^2}{\xi^{2n}}\, d\xi < \infty$$

Thanks to the quasiorthogonality of the terms, we have then:

$$\sum_{-\infty}^{+\infty} \int_{-1}^{1} \frac{|\widehat{\triangle_j} f(\xi)|^2}{\xi^{2n}}\, d\xi < \infty$$

In the same way we have:

$$\int_{\alpha 2^j}^{\beta 2^j} \frac{|\triangle_j \widehat{f(\xi)}|^2}{\xi^{2n}}\, d\xi < \infty$$

$$\Longrightarrow ||\widehat{\triangle_j} f||_2 \leq C_n 2^{jn}$$

for all n and for all $j \leq -1$

- **Proof of (2) $\Longrightarrow$ (1):**

f is in $H^s(I\!R)$ and $|| \triangle_j f||_2 \leq C_n 2^{jn}$ for all n and for all $j \leq -1$.
Let f_n such that $f = \frac{d^n f_n}{dx^n}$. By using Fourier transform we will have: $\hat{f}(\xi) = (i\xi)^n \hat{f}_n(\xi)$.

As f is in $H^s(I\!R)$, we have then:

$$\int_{(I\!R)} (1+\xi^2)^s |\hat{f}(\xi)|^2\, d\xi \quad < \quad \infty \quad \text{or again}$$

$$\int_{(I\!R)} (1+\xi^2)^s |\xi|^{2n} |\hat{f}_n(\xi)|^2\, d\xi \quad < \quad \infty$$

which gives $\int_{V(\infty)} (\xi^2)^{n+s} |\hat{f}_n(\xi)|^2\, d\xi$. Now there remain to verify that we have $\int_{-1}^{1} |\hat{f}_n(\xi)|^2\, d\xi < \infty$, and thus we will prove that f_n is in $H^{s+n}(I\!R)$. Littlewood-Paley decomposition of f_n is given by $f_n = \sum_{-\infty}^{+\infty} \triangle_j f_n$, using Fourier one has:

$$\hat{f}_n(\xi) = \sum_{-\infty}^{-1} \hat{\triangle}_j f_n(\xi) + \sum_{0}^{+\infty} \hat{\triangle}_j f_n(\xi)$$

Show that each of the two terms is in $L^2([-1,1])$. Hence, since supports of the different terms are 2 by 2 disjoints

$$\int_{-1}^{1} |\sum_{-\infty}^{-1} \hat{\triangle}_j f_n(\xi)|^2 \, d\xi \leq 2 \int_{-1}^{1} \sum_{-\infty}^{-1} |\hat{\triangle}_j f_n(\xi)|^2 \, d\xi$$

Then one has:

$$\begin{aligned} \int_{-1}^{1} |\sum_{-\infty}^{-1} \hat{\triangle}_j f_n(\xi)|^2 \, d\xi &\leq 2 \sum_{-\infty}^{-1} \int_{\alpha 2^j}^{\beta 2^j} \frac{|\hat{\triangle}_j f(\xi)|^2}{\xi^{2n}} \, d\xi \\ &\leq C_{n,N} \sum_{-\infty}^{-1} 2^{2j(N-n)} \end{aligned}$$

And then for a good choice of N this series is normally convergent. Concerning the other term we have, by again using that supports are disjoints:

$$\begin{aligned} \int_{-1}^{1} |\sum_{0}^{+\infty} \hat{\triangle}_j f_n(\xi)|^2 \, d\xi &\leq 2 \int_{-1}^{1} \sum_{0}^{+\infty} |\hat{\triangle}_j f_n(\xi)|^2 \, d\xi \\ &\leq 2 \int_{-1}^{1} \sum_{0}^{+\infty} \frac{|\hat{\triangle}_j f(\xi)|^2}{\xi^{2n}} \, d\xi \end{aligned}$$

Or $||\hat{\triangle}_j(f)(\xi)||_2 \leq \epsilon_j$ for $j \geq 0$ with $\sum_{0}^{+\infty} \epsilon_j^2 < \infty$. which indicates:

$$\begin{aligned} \int_{-1}^{1} |\sum_{0}^{+\infty} \hat{\triangle}_j f_n(\xi)|^2 \, d\xi &\leq 2 \sum_{0}^{+\infty} \int_{\alpha 2^j}^{\beta 2^j} \frac{|\hat{\triangle}_j f(\xi)|^2}{\xi^{2n}} \, d\xi \\ &\leq 2 \sum_{0}^{+\infty} C_n \epsilon_j^2 2^{-2jn} \end{aligned}$$

1.3 Generalization in an arbitrary Banach space

Let E a functional Banach space : $\mathcal{S}(I\!R) \subset E \subset \mathcal{S}'(I\!R)$. Assuming that the norm in E is invariant by translation and that E satisfy the following propriety

(1)
$$\begin{aligned} f_j \in E,\ \| f_j \|_E &\leq C \text{ and } f_j \longrightarrow f \text{ in } \sigma(\mathcal{S}', \mathcal{S}) \text{ sens} \\ \Longrightarrow f \in E \text{ and } \| f \|_E &\leq C \end{aligned}$$

In other hand, one has $\| f \|_E \leq limsup_{j \longrightarrow +\infty} \| f_j \|_E$ each time that f_j converge in distributions sense.

Definition 2 *A function $f \in E$ is indefinitely oscillating relatively to E if, for all $m \geq 1$, there exist $f_m \in E$ such that $f(x) = (\frac{d}{dx})^m f_m(x)$.*

$I = \sum_{-\infty}^{+\infty} \triangle_j$ means the Littlewood-Paley decomposition. One has then

Theorem 6 *The three following proprieties are equivalents*

1. *f is indefinitely oscillating in E sense*
2. *$\| \triangle_j(f) \|_E \leq C_m 2^{jm}$ for all $m \geq 0$ and all $j \leq 0$*
   and $f = \sum_{-\infty}^{+\infty} \triangle_j(f)$ in $\sigma(\mathcal{S}'(IR), \mathcal{S}(IR))$ sense
3. *$\| S_j(f) \|_E \leq C'_m 2^{jm}$ for all $m \geq 0$ and all $j \leq 0$.*

We start by the following lemma

Lemma 7 *If $\omega \in L^1(I\!\!R)$, $f \in E$, then $f * \omega \in E$ and*
$\| f * \omega \|_E \leq \| \omega \|_1 \| f \|_E$.

- **Starting by (2) $\Longrightarrow$ (1)**

One calls $\tilde{\psi}$ a function of Schwartz class $\mathcal{S}(I\!\!R)$ for which Fourier transform is equal to 1 over $\frac{1}{4} \leq | \xi | \leq 4$ and 0 if $| \xi | \geq \frac{1}{10}$ and $| \xi | \geq 10$. The Fourier transform of ψ is taken by $\frac{1}{3} \leq | \xi | \leq 3$.One writes then

$$\triangle_j(f) = \tilde{\triangle}_j(\triangle_j f) = (\frac{d}{dx})^m 2^{-jm} \tilde{\triangle}_{j_{(m)}}, (\triangle_j f)$$

or by Fourier transform,

$$\tilde{\psi}(2^{-j}\xi) = (i\xi 2^{-j})^m \tilde{\psi}_{(m)}(2^{-j}\xi)$$

which means

$$\tilde{\psi}(\xi) = (i\xi)^m \hat{\tilde{\psi}}_m(\xi).$$

It is then evident that $\tilde{\psi}_m \in \mathcal{S}(I\!\!R)$.
One applies then the Lemma 7 and will have

$$\triangle_j(f) = 2^{-jm} (\frac{d}{dx})^m f_{j,m}$$

where

$$\| f_{j,m} \|_E \leq C_N 2^{jN}$$

for all integer N.
Then

$$\sum_{-\infty}^{0} 2^{-jm} f_{j,m}$$

converge in E norm sense.
Let

$$\sigma_q(f) = \sum_{j \geq -q} \triangle_j(f)$$

one has $\sigma_q(f) \longrightarrow f$ (in distributions sense) when $q \longrightarrow +\infty$.

$$\sigma_q(f) = (\frac{d}{dx})^m \sigma_{q,m}(f)$$

and $\sigma_{q,m}(f) \longrightarrow I_m(f)$ where $q \longrightarrow +\infty$.
Then

$$(\frac{d}{dx})^m I_m(f) = f$$

- **(1) $\Longrightarrow$ (3)**
  Hence $f = (\frac{d}{dx})^m f_m$ and $f_m \in E$.

  Then

$$S_j(f) = (\frac{d}{dx})^m S_j(f_m)$$

and

$$(\frac{d}{dx})^m S_j = 2^{jm} S_j^{(m)}$$

$S_j^{(m)}$ is the convolution with $2^j \varphi^{(m)}(2^j x)$ where $\varphi^{(m)}(x) = (\frac{d}{dx})^m \varphi(x)$. It is sufficient then to apply the lemma (7).

- **(3) $\Longrightarrow$ (2)**
  This implication is evident since $\triangle_j = S_{j+1} - S_j$.

2 Theory of indefinitely oscillating functions on the half real axe

We can't then use the Fourier transform. We have two ways to do it:

- Restrictions to a half real axe of functions indefinitely oscillating on all IR
- Direct definition

Before studying the case of functions indefinitely oscillating on $[0, \infty)$ relatively to space $H^s[0, \infty)$, we will recall the results now classics of functions indefinitely oscillating on $[0, \infty)$ relatively to space $L^\infty[0, \infty)$.

2.1 $L^\infty([0,\infty))$ case

Definition 3 *Considering a function f defined on the half real axe $[0,\infty)$. One says that f is indefinitely oscillating in $L^\infty([0,\infty))$ sense if $f \in L^\infty([0,\infty))$ and if for all integer m there exist $f_m \in L^\infty([0,\infty))$ such that $f(x) = (\frac{d}{dx})^m f_m(x)$ (in distributions sense on $[0,\infty)$).*

Theorem 8 *Let f a function indefinitely oscillating in $L^\infty([0,\infty))$ sense.*
Then f is the restriction to $[0,\infty)$ of a function g indefinitely oscillating on IR for all integer.

Proof of theorem:

To prove this remarkable result, we will start by defining the generalized moments $\mu_k$of f by:

$$\mu_k = \lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x}x^k f(x)\,dx$$

by showing the existing of this limit.

$$\int_0^\infty e^{-\epsilon x}x^k \frac{df_1}{dx}(x)\,dx = [e^{-\epsilon x}x^k f_1(x)]_0^\infty - \int_0^\infty \frac{d}{dx}(e^{-\epsilon x}x^k)f_1(x)\,dx$$

The first parte in the right hand side is nul. Then we have:

$$\begin{aligned}
\int_0^\infty e^{-\epsilon x}x^k f(x)\,dx &= (-1)^k\int_0^\infty (\frac{d}{dx})^k(e^{-\epsilon x}x^k)f_k(x)\,dx \\
&= (-1)^k[k!\int_0^\infty e^{-\epsilon x}f_k(x)\,dx + ... \\
&+ \epsilon^{k-2}\frac{k^2(k-1)^2}{2}\int_0^\infty e^{-\epsilon x}x^{k-2}f_k(x)\,dx \\
&- k^2\epsilon^{k-1}\int_0^\infty e^{-\epsilon x}f_k(x)x^{k-1}\,dx \\
&+ \epsilon^k\int_0^\infty e^{-\epsilon x}x^k f_k(x)\,dx]
\end{aligned}$$

We reason by recurrence on the integer k. One assumes
$\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x}x^q f(x)\,dx$ exists for $q = 0, 1, .., k-1$.
One remarks, that if f is an indefinitely oscillating function on $[0,\infty)$ its primitives functions $f_1, .., f_k, ...$ giving by the definition of indefinitely oscillating function are so.
And because of the recurrent hypotheses, the quantity $\int_0^\infty e^{-\epsilon x}x^q f_n(x)\,dx$ has a limit when ϵ goes to 0, for all integer n and for all integer q satisfies $0 \le q \le k-1$. However precisely we have:

$$\int_0^\infty e^{-\epsilon x}x^k f(x)\,dx = (-1)^k[k!\int_0^\infty e^{-\epsilon x}f_k(x)\,dx + ...$$

$$+\epsilon^{k-2}\frac{k^2(k-1)^2}{2}\int_0^\infty e^{-\epsilon x}x^{k-2}f_k(x)\,dx$$
$$-k^2\epsilon^{k-1}\int_0^\infty e^{-\epsilon x}f_k(x)x^{k-1}\,dx$$
$$+\epsilon^k\int_0^\infty e^{-\epsilon x}x^k f_k(x)\,dx]$$

To have the wanted result, it is sufficient to show the existence of

$$\lim_{\epsilon\to 0}\epsilon^k\int_0^\infty e^{-\epsilon x}x^k f_k(x)\,dx$$

By another integration and using the recurrent hypotheses we have :

$$\epsilon^{k+1}\int_0^\infty e^{-\epsilon x}x^k f_{k+1}(x)\,dx$$

And another last integration gives us:

$$\epsilon^{k+2}\int_0^\infty e^{-\epsilon x}x^k f_{k+2}(x)\,dx$$

that can be majored by $||f_{k+2}||_\infty\epsilon\int_0^\infty x^k e^{-x}\,dx$, which goes to 0 when ϵ goes to 0. The demonstration will be complete if the propriety is well satisfy to zero rank:

$$\begin{aligned}\int_0^\infty e^{-\epsilon x}f(x)\,dx &= [f_1(x)e^{-\epsilon x}]_0^\infty+\epsilon\int_0^\infty f_1(x)e^{-\epsilon x}\,dx\\ &= -f_1(0)+\epsilon\int_0^\infty f_1(x)e^{-\epsilon x}\,dx\end{aligned}$$

a new integration gives:

$$\begin{aligned}\epsilon\int_0^\infty f_1(x)e^{-\epsilon x} &= \epsilon[f_2(x)e^{-\epsilon x}]_0^\infty+\epsilon^2\int_0^\infty f_2(x)e^{-\epsilon x}\,dx\\ &= -\epsilon f_2(0)+\epsilon^2\int_0^\infty f_2(x)e^{-\epsilon x}\,dx\end{aligned}$$

Precisely we have:

$$0\le\epsilon^2|\int_0^\infty f_2(x)e^{-\epsilon x}\,dx|\le\epsilon||f_2||_\infty\int_0^\infty e^{-x}\,dx$$

these last quantity goes to 0 when ϵ goes to 0.

Corollary 9

$$\mu_k=\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x}x^k f(x)\,dx=(-1)^{k+1}k!f_{k+1}(0).$$

or again

$$\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x}f_k(x)\,dx=-f_{k+1}(0).$$

Return back to the proof of the theorem. Applying Borel's theorem : there exist then a function h belonging to a Schwartz class taken by $(-\infty, 0]$ such that:

$$-\mu_k = \int_{-\infty}^{0} x^k h(x)\, dx$$

We will show that $h(x) + f\chi_{[0,\infty)}$ is the function $g(x)$ that we seek. For this, it is sufficient to show that g is indefinitely oscillating on all $I\!R$. Hence $g(x) = h(x) + f\chi_{[0,\infty)}$ is in $L^\infty(\text{IR})$. by definition $g_1(x) = \int_{-\infty}^{x} g(t)\, dt$.

$$\begin{array}{lllcl}
\text{Si} & x < 0 & \text{then} & g_1(x) & = \int_{-\infty}^{x} h(t)\, dt \\
 & & & g_1 & \in L^\infty((-\infty, 0[) \\
\text{Si} & x = 0 & \text{then} & g_1(0) & = \int_{-\infty}^{0} h(t)\, dt = -\mu_0 \\
 & & & & = f_1(0) \\
\text{et si} & x \geq 0 & \text{then} & g_1(x) & = \int_{-\infty}^{0} h(t)\, dt + \int_{0}^{x} f(t)\, dt \\
 & & & & = f_1(0) + f_1(x) - f_1(0)
\end{array}$$

Which indicates $||g_1||_\infty \leq C$.
Also $g_2(x) = \int_{-\infty}^{x} g_1(t)\, dt$. And for $x \geq 0$ one has:

$$\begin{aligned}
g_2(x) &= \int_{-\infty}^{0} g_1(t)\, dt + \int_{0}^{x} f_1(t)\, dt \\
&= \int_{-\infty}^{0} g_1(t)\, dt + f_2(x) - f_2(0) \\
&= -\int_{-\infty}^{0} th(t)\, dt + f_2(x) - f_2(0)
\end{aligned}$$

Or $\int_{-\infty}^{0} th(t)\, dt = -\mu_1 = -f_2(0)$ by the construction h .

IRemark :

$$-\mu_k = \int_{-\infty}^{0} x^k h(x)\, dx = \int_{-\infty}^{0} h_k(x)\, dx = h_{k+1}(0)$$

Showing this by recurrence, assuming that $||g_k||_\infty \leq C_k < \infty$ for $k < n$ and showing that g_n follows that same inequality :
One has

$$\begin{aligned}
g_n(0) &= \int_{-\infty}^{0} g_{n-1}(t)\, dt \\
&= f_n(0) \\
&= \frac{(-1)^{n-1}}{(n-1)!} \int_{-\infty}^{0} x^{n-1} h(x)\, dx
\end{aligned}$$

For $x < 0$ one has $g_n(x) = h_n(x)$ which belongs to Schwartz class.
And for $x \geq 0$ one has:

$$\begin{aligned} g_n(x) &= \int_{-\infty}^{x} g_{n-1}(t)\,dt \\ &= \int_{-\infty}^{0} h_{n-1}(t)\,dt + \int_{o}^{x} f_{n-1}(t)\,dt \\ &= f_n(0) + f_n(x) - f_n(0) \end{aligned}$$

Or $f_n(x)$ is bounded by hypotheses. It is also the same for $g_n(x)$.

2.2 $L^2([0,\infty))$ case

Definition 4 *Let f a function in $L^2([0,\infty))$. One says that f is indefinitely oscillating in $L^2([0,\infty))$ sense, if there exist functions in $L^2[0,\infty)$ such that $f(x) = \dfrac{d^n f_n(x)}{dx^n}$ in distributions sense.*

Let f a function indefinitely oscillating in L^2 sense, and using the same notations, showing that $f_n \in H^n([0,\infty))$.
If $0 < x < y$ on a $f_1(y) - f_1(x) = \int_x^y f(t)\,dt$.
By consequence $|f_1(y) - f_1(x)| \leq \sqrt{y-x}||f||_2$. Also f_1 is uniformly continuous on $[0,\infty)$ and can be extended at 0, it is also the same for all $f_n, n \geq 1$.
Showing now that f_n goes to 0 at infinity for all $n \geq 1$. For this writing the following lemma:

Lemma 10 *Let u an uniformly continuous function on $[0,\infty)$.
If u is in $L^2[0,\infty)$ then $\lim_{x\to\infty} u(x) = 0$.*

The proof of this lemma is known. We will give it herein just for reader convenient.
To prove this lemma, we reason by absurd. We assume that u does not go to 0 when x goes to infinity. By consequent there exist a net x_k which goes to infinity such that $|u(x_k)| \geq \delta > 0$. u is uniformly continuous, there exist $\epsilon > 0$ such that $|u(x_k + t)| \geq \frac{\delta}{2}$ if $|t| \leq \epsilon$. Using sub net, we can always assume that intervals $[x_k - \epsilon, x_k + \epsilon]$ are two by two disjoints. Then

$$||u||_2 \geq \sum_k \int_{x_k-\epsilon}^{x_k+\epsilon} |u(s)|^2\,ds = \infty$$

Remark: If f is in $L^2[0,\infty)$ and is indefinitely oscillating, then primitives given by the definition $f_1, f_2, ...f_n, ...$ are indefinitely oscillating in $L^\infty([0,\infty))$ sense. By consequent the f generalized moments μ_k exist for all integer $k \geq 1$.

Lemma 11 *If $f \in L^2([0,\infty))$ is indefinitely oscillating in $L^2([0,\infty))$ sense, then*

$$lim_{\epsilon\longrightarrow 0} \int_0^{+\infty} e^{-\epsilon x} f(x)\,dx$$

exists and equal to $-f_1(0)$.

Proof of lemma

integrating by party

$$\int_0^{+\infty} e^{-\epsilon x} f(x)\, dx = [e^{-\epsilon x} f_1(x)]_0^{+\infty} + \epsilon \int_0^{+\infty} e^{-\epsilon x} f_1(x)\, dx$$

Or $f_1(+\infty) = 0$, the first term of the right hand side is equal $-f_1(0)$ and the second can be bounded using Cauchy-Schwarz by $\dfrac{\| f_1 \|_2}{\sqrt{\epsilon}}$

Theorem 12 *If $f \in L^2[0,\infty)$ is indefinitely oscillating, then there exist a function g in $L^2(\mathbb{R})$, indefinitely oscillating, for which its restriction to $[0,\infty)$ is f.*

To use the Borel theorem, it is necessary to begin by defining the generalized moments of f. Let then f in $L^2[0,\infty)$, indefinitely oscillating.

$$\forall n \;;\; f(x) = \frac{d^n f_n(x)}{dx^n}$$

where f_n is in $L^2[0,\infty)$. We have already proved that $\forall n \geq 1$ f_n is uniformly continuous on $[0,\infty)$ and that $\lim_{x\to\infty} f_n(x) = 0$.
One has then the following corollary:

Corollary 13 *Let $f \in L^2([0,\infty))$ and f indefinitely oscillating, then for all integer k:*

$$\begin{aligned} \lim_{\epsilon\to 0} \int_0^\infty e^{-\epsilon x} x^k f(x)\, dx &= (-1)^{k+1} k!\, f_{k+1}(0) \\ \textit{or again } \lim_{\epsilon\to 0} \int_0^\infty e^{-\epsilon x} f_k(x)\, dx &= -f_{k+1}(0) \end{aligned}$$

We can then apply the Borel theorem:
There exist a Schwartz class function h taken by $(-\infty,0]$, such that $h(x) + f(x)\chi_{[0,\infty)}$ is a function in $L^2(\mathbb{R})$, indefinitely oscillating on any real axe. Finally one shows, by an analogue reasoning to that in the case L^∞, that $h(x) + f(x)\chi_{[0,\infty)}$ is the wanted function g.

2.3 $H^s([0,\infty))$ case

Definition 5 *Let f in $H^s([0,\infty))$. One says that f is indefinitely oscillating if, for all integer n, there exist a function (or a distribution) f_n in $H^s([0,\infty))$ such that $f = \dfrac{d^n f_n}{dx^n}$ (in distributions sense).*

Theorem 14 *Every function (resp. distribution) f in $H^s([0,\infty))$, indefinitely oscillating, is the restriction to $[0,\infty)$ of a function (resp. distribution) g belonging to $H^s(\mathbb{R})$ and indefinitely oscillating.*

One starts by $\mathbf{s \geq 0}$ case.
Then $f_1, f_2, ..., f_n, ...$ are uniformly continuous and can be extended to 0. We can then apply the Borel theorem, and constructs a h Schwartz class function taken by $(-\infty, 0]$ such that:

$$\begin{aligned} f_1(0) &= \int_{-\infty}^{0} h(x)\,dx \\ f_2(0) &= \int_{-\infty}^{0} h_1(x)\,dx \quad \text{where} \quad h_1(x) = \int_{-\infty}^{x} h(t)\,dt \\ &. \\ &. \\ &. \\ f_m(0) &= \int_{-\infty}^{0} h_{m-1}(x)\,dx \quad \text{where} \quad h_{m-1}(x) = \int_{-\infty}^{x} h_{m-2}(t)\,dt \end{aligned}$$

Then one lets $g(x) = h(x) + f(x)\chi_{[0,\infty)}$. This approach is possible only in the case of $\mathbf{0 \leq s < \frac{1}{2}}$. The reason is that if f is in $H^s([0,\infty))$, it is true that f is the restriction to $[0,\infty)$ of a function in $H^s(I\!R)$, but it is not necessary that this function is obtained by multiplying it by the indicating function $\chi_{[0,\infty)}$. Then we are arrived to proof the following result:

Theorem 15 *If $-\frac{1}{2} < s < \frac{1}{2}$ and f is in $H^s(I\!R)$, then $f.\chi_I$ is in $H^s(I\!R)$ for all interval I (either for a finite or infinite length).*

One has first to return to the following particular case: The multiplication by $sign(x) = \frac{x}{|x|}$ and this case will be find out its importance since it can let us to prove the general case. Let then f in $H^s(I\!R)$ with $0 \leq s < \frac{1}{2}$, observing the Fourier transform of $sign(x)f(x)$. We will have

$$\mathcal{F}(sign(x)f(x)) = \frac{1}{i\pi}\hat{f} * V_p(\frac{1}{\xi})$$

As one remarks, the Borel theorem can be applied only in the case $s \geq 0$, and the theorem (15) in the case $0 \leq s < \frac{1}{2}$. For proving the theorem (14), for the case where s is an arbitrary real, it is sufficient to use the following lemma.

Starting by making the following remark:
f is indefinitely oscillating in H^s sense (either on half axe or on the whole axe) if and only if $D^m f$ is indefinitely oscillating in H^{s-m} sense for all m relative integer.

Lemma 16 *f is indefinitely oscillating in H^s sense(either on some subset of IR or on IR) if and only if $D^\alpha f$ is indefinitely oscillating in $H^{s-\alpha}$ sense for all real α.*

Proof of lemma:

- **Let** $0 < \alpha < 1$ we define the fractional derivative by:

$$D^{\alpha} f(x) = c_{\alpha} \int_x^{\infty} \frac{f(y)}{(x-y)^{1-\alpha}} \, dy$$

The equality (2.3) has a sense as f is defined on $[T, \infty)$.Let f a function indefinitely oscillating. If f is in H^s, then $D^{\alpha} f$ is in $H^{s-\alpha}$ (it is a fractional derivative). By hypothesis $\forall m \quad \exists f_m \in H^{s+m}$ such that $f = \dfrac{d^m f_m}{dx^m}$. We have $D^{\alpha} f_m \in H^{s+m-\alpha}$. So $D^{\alpha} f$ is indefinitely oscillating in $H^{s-\alpha}$ sense.
Inversely, if $D^{\alpha} f$ is indefinitely oscillating in $H^{s-\alpha}$ sense, posing also $g = D^{\alpha} f$ then g_m is in $H^{s-\alpha+m}$. As $0 < 1-\alpha < 1$ with $D^{1-\alpha} g_1 \in H^s$ and $D^{1-\alpha} g_1 = f$. One deducts $D^{1-\alpha} g_m \in H^{s+m-1}$ and $D^{1-\alpha} g_m = f_{m-1}$.

- **cas** $-1 < \alpha < 0$
Taking f indefinitely oscillating. Then $\forall m \quad \exists f_m \in H^{s+m}$ such that $f = \dfrac{d^m f_m}{dx^m}$. As f_1 is in H^{s+1}. One deducts that $D^{1+\alpha} f_1$ is in $H^{s-\alpha}$. (The operator $D^{1+\alpha}$ is a derivative operator since $0 < 1+\alpha < 1$). As f_1 is indefinitely oscillating in H^{s+1} sense which is equivalent to say that $D^{1+\alpha} f_1$ is indefinitely oscillating in $H^{s-\alpha}$ sense.

2.4 Generalization to an arbitrary Banach space

We take a Banach space E satisfies the following conditions:

1. $E \subset \mathcal{D}'([0, \infty))$ and the injection is continuous
2. If $f_j \in E, \ \| f_j \|_E \leq C$ and if $f_j \longrightarrow f$ (in $\mathcal{D}'([0, \infty))$ sense), then $f \in E$
3. if $f \in E, \ \tau \geq 0$, then $T^{\tau} f$, defined by $T^{\tau} f(x) = f(x+\tau)$, belongs to E with

$\| T^{\tau} f \|_E \leq \| f \|_E$

Definition 6 *A function (or distribution) $f \in E$ is indefinitely oscillating, relatively to E, if and only if, for all integer $n \geq 1$, there exists $f_n \in E$ such that one has $f(x) = (\dfrac{d}{dx})^n f_n(x)$*

(the derivatives are considered in distributions sense).

Definition 7 *A function $\psi \in C_0^{\infty}(0,1)$ is admissible if*
$\int_0^{\infty} u^{-q} \psi(u) \, du = \gamma_q \neq 0$ for all integer $q \geq 1$.

This is compatible with the fact that ψ has a nul integral.

Theorem 17 *With the previous notations, the two following proprieties of the function $f \in E$ are equivalents.*

1. *f is indefinitely oscillating relatively to E*

2.

$$W_{(a,b)} \;\; = \int_0^\infty f(x)\frac{1}{a}\psi(\frac{x-b}{a})\,dx \;\;,\; b>0,\;\; a>0 \textit{ satisfy, for all } a \geq 1,$$

$$\parallel W_{(a,.)} \parallel_E \leq C_N a^{-N}$$

Proof of theorem

- $1 \Longrightarrow 2$ is evident.

  We can show the following lemma.

  Lemma 18 *if* $\omega \in L^1(0,\infty)$, $f \in E$, *then* $\int_0^\infty (T^\tau f)\omega(\tau)\,d\tau \in E$

  Using this result, we write

$$\begin{aligned} W_{(a,b)} &= \int_0^\infty f(x)\frac{1}{a}\psi(\frac{x-b}{a})\,dx \\ &= \int_0^\infty (\frac{d}{dx})^n f_n(x)\frac{1}{a}\psi(\frac{x-b}{a})\,dx \\ &= (-1)^n a^{-n}\int_0^\infty f_n(x)\frac{1}{a}\psi^{(n)}(\frac{x-b}{a})\,dx \\ &= (-1)^n a^{-n}\int_0^\infty (T^{(x)}f_n)(b)\frac{1}{a}\psi^{(n)}(\frac{x}{a})\,dx \\ &= (-1)^n a^{-n} W_{(n)_{(a,b)}} \end{aligned}$$

  Owing to lemma (18), $W_{(n)_{(a,b)}} \in E$ with a norm bound uniformly by a. The estimation (2) by result.

- **The implication** $2 \Longrightarrow 1$ is more subtle.

  We use the identity

$$\int_0^\infty a^{q-1}\psi(\frac{t-x}{a})\,\frac{da}{a} = \gamma_q(t-x)_+^{q-1}$$

  We conclude that we have

$$(q-1)!\,\gamma_q f_q(x) = \int_0^\infty a^{q-1}W_{(a,x)}\,da$$

  where $f_q(x)$ is the q-th primitive of $f(x)$.
  If $q \geq 1$, $\int_0^1 a^{q-1}W_{(a,x)}\,da \in E$ since $\parallel W_{(a,.)} \parallel_E \leq C \parallel f \parallel_E$ (lemma (18)). Now $\int_1^\infty a^{q-1}W_{(a,x)}\,da \in E$ since (2) let to satisfy that the integral is the Bochner integral.

2.5 Some interesting examples of the indefinitely oscillating functions

Some links between functions indefinitely oscillating in L^∞ and H^s sense can be more interesting. It is evident to constat that if f is indefinitely oscillating in H^s and if $s > \frac{1}{2}$, then f is indefinitely oscillating in L^∞ sense (because of the Sobolev injections).
One has also the following assertion:

Lemma 19 *If f is in $H^s(I\!R) \bigcap L^\infty(I\!R)$ for $s < \frac{1}{2}$ and f is indefinitely oscillating in $H^s(I\!R)$ sense then f is a functions indefinitely oscillating in $L^\infty(I\!R)$ sense.*

Proof of lemma

By hypothesis one has $\| \triangle_j f \|_2 \leq C_N 2^{jN} \;\; \forall N, \; \forall j \leq -1$. As $\widehat{\triangle_j} f(\xi)$ is taken by the dyadic corona $\alpha 2^j \leq \mid \xi \mid \leq \beta 2^j$ with $0 < \alpha < \beta$. It becomes then:

$$\| \widehat{\triangle_j} f \|_{L^1} = \int_{\alpha 2^j}^{\beta 2^j} \mid \widehat{\triangle_j} f(\xi) \mid \, d\xi$$

Applying Cauchy-Schwarz one has:

$$\| \widehat{\triangle_j} f \|_{L^1} \leq C_N 2^{jN} 2^{\frac{j}{2}} \;\; \forall N, \; \forall j \leq -1$$

or

$$\triangle_j f(x) = \int \widehat{\triangle_j} f(\xi) e^{i\xi x} \, d\xi$$

By consequent we have:

$$\begin{aligned} \| \triangle_j f \|_\infty &\leq \| \widehat{\triangle_j} f \|_{L^1} \\ &\leq C_N 2^{jN} 2^{\frac{j}{2}} \end{aligned}$$

Another interesting example, is to show that if f is in $H^s(I\!R)$ and is indefinitely oscillating in $L^\infty(I\!R)$ sense, then we have not necessarily f indefinitely oscillating in $H^s(I\!R)$ sense. We will construct such function in the following way:
Let $\hat{\varphi}$ a function of compact support and indefinitely derivative (It is then in Schwartz class) such that $\hat{\varphi}(\xi) \geq 0$. Writing

$$f(x) = \sum_0^{+\infty} 2^{-k^2} \varphi(\frac{x}{2^{k^3}}) e^{i 2^{-k} x}$$

By using Fourier transforms it becomes:

$$\hat{f}(\xi) = \sum_0^{+\infty} \hat{\varphi}(2^{k^3}(\xi - 2^{-k})) 2^{k^3 - k^2}$$

It clair that one has

$$\int_{2^{-j} \leq |\xi| \leq 2\, 2^{-j}} \mid \hat{f}(\xi) \mid \, d\xi = \mathcal{O}(2^{-Nj})$$

It can be remarked by the proof of the last lemma that f is indefinitely oscillating in L^∞ sense, but

$$\lim_{j\to+\infty}\int_{2^{-j}\leq|\xi|\leq 2\,2^{-j}} \mid \hat{f}(\xi)\mid^2\ d\xi = +\infty$$

By consequent f is not indefinitely oscillating in L^2 sense.

IRemark:

One knows that an indefinitely oscillating function in H^s sense can be writing under the forme $f = f_0 + f_1$, where f_0 is the basic component frequency which defines the oscillating characteristic of the function f, f_1 is a function of H^s indefinitely oscillating since the support of $\hat{f}_1$ does not content 0. Finally f_0 is a function indefinitely oscillating in L^2 sense.
An interesting example of indefinitely oscillating function in L^2 sense is the A.Grossmann wavelet defined by:

$$\begin{cases} f(x) &= \quad e^{-(logx)^2} \quad \text{si} \quad x > 0 \\ f(x) &= e^{-(log|x|+i\pi)^2} \quad \text{si} \quad x < 0 \end{cases}$$

We can show easily that f is in the Schwartz class. By consequent $\hat{f}$ is in the Schwartz class. By Paley-Wiener's theorem, $\hat{f}(\xi) = 0$ for $\xi < 0$ and $\frac{\hat{f}(\xi)}{\xi^N}$ is Schwartz class, for all N. One deducts that 0 is a zero of infinity order for $\hat{f}$. By consequent f is indefinitely oscillating L^2 sense.
We can observe that the same function is also indefinitely oscillating in L^∞ sense.

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