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A criterion for convergence of weak greedy algorithms

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1. Introduction

This paper completes the investigation of necessary and sufficient conditions on the "weakness" sequence $\tau := \{t_k\}_{k=1}^{\infty}$ for convergence of Weak Greedy Algorithm for all dictionaries $\mathcal{D}$ and each function (vector) f in Hilbert space H. This paper is a follow up to the papers [T] and [LT]. The Weak Greedy Algorithms (WGA) were introduced in [T]. The paper [T] contains also historical remarks and some motivation of studying greedy and weak greedy algorithms. We will not repeat historical remarks from [T] here and refer the reader to [T] for prehistory of WGA. We discuss here results on WGA in detail.

We remind first some notations and definitions from the theory of greedy algorithms. Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the norm $\|x\| := \langle x, x\rangle^{1/2}$. We say a set $\mathcal{D}$ of functions (elements) from H is a dictionary if each $g \in \mathcal{D}$ has norm one ($\|g\| = 1$) and $\overline{\operatorname{span}}\mathcal{D} = H$. We give now the definition of WGA (see [T]). Let a weakness sequence $\tau = \{t_k\}_{k=1}^{\infty}$, $0 \le t_k \le 1$, be given.

Weak Greedy Algorithm. *We define $f_0^\tau := f$. Then for each $m \ge 1$, we inductively define:*

1). $\varphi_m^\tau \in \mathcal{D}$ is any satisfying

$$|\langle f_{m-1}^\tau, \varphi_m^\tau \rangle| \ge t_m \sup_{g \in \mathcal{D}} |\langle f_{m-1}^\tau, g\rangle|;$$

2).

$$f_m^\tau := f_{m-1}^\tau - \langle f_{m-1}^\tau, \varphi_m^\tau \rangle \varphi_m^\tau;$$

3).

$$G_m^\tau(f, \mathcal{D}) := \sum_{j=1}^{m} \langle f_{j-1}^\tau, \varphi_j^\tau \rangle \varphi_j^\tau.$$

In the case $t_k = 1$, $k = 1, 2, \dots$, we call WGA by Pure Greedy Algorithm (PGA). The convergence of PGA and WGA with $t_k = t$, $0 < t < 1$, was established in [J] and [RW]. The first sufficient condition on τ which includes sequences with $\lim_{k\to\infty} t_k = 0$ was obtained in [T].

Theorem A. *Assume*

$$\sum_{k=1}^{\infty} \frac{t_k}{k} = \infty. \tag{1.1}$$

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Then for any dictionary $\mathcal{D}$ and any $f \in H$ we have

$$\lim_{m\to\infty} \|f - G_m^\tau(f,\mathcal{D})\| = 0.$$

In [T] we reduced the proof of convergence of WGA with weakness sequence τ to some properties of l_2-sequences with regard to τ. Theorem A was derived from the following two statements proved in [T].

Proposition 1.1. *Let τ be such that for any $\{a_j\}_{j=1}^\infty \in l_2$, $a_j \ge 0$, $j = 1,2,\dots$ we have*

$$\liminf_{n\to\infty} a_n \sum_{j=1}^n a_j/t_n = 0.$$

Then for any H, $\mathcal{D}$, and $f \in H$ we have

$$\lim_{m\to\infty} \|f_m^\tau\| = 0.$$

Proposition 1.2 (Lemma 2.3,[T]). *If τ satisfies the condition (1.1) then τ satisfies the assumption of Proposition 1.1.*

The following simple necessary condition

$$\sum_{k=1}^\infty t_k^2 = \infty$$

was mentioned in [T]. The first nontrivial necessary conditions were obtained in [LT]. We proved in [LT] the following theorem.

Theorem B. *In the class of monotone sequences $\tau = \{t_k\}_{k=1}^\infty$, $1 \ge t_1 \ge t_2 \ge \dots \ge 0$, the condition (1.1) is necessary and sufficient for convergence of Weak Greedy Algorithm for each f and all Hilbert spaces H and dictionaries $\mathcal{D}$.*

The proof of this theorem is based on a special procedure which we called Equalizer. The generalization of that procedure plays an important role in this paper also (see S.3). In [LT] we gave an example of a class of sequences τ for which the condition (1.1) is not a necessary condition for convergence. We also proved in [LT] a theorem which covers Theorem A.

Theorem C. *Assume*

$$\sum_{s=0}^\infty \Big(2^{-s} \sum_{k=2^s}^{2^{s+1}-1} t_k^2\Big)^{1/2} = \infty.$$

Then for any dictionary $\mathcal{D}$ and any $f \in H$ we have

$$\lim_{m\to\infty} \|f - G_m^\tau(f,\mathcal{D})\| = 0.$$

We prove in this paper a criterion on τ for convergence of WGA. Let us introduce some notation.

We define by $\mathcal{V}$ the class of sequences $x = \{x_k\}_{k=1}^\infty$, $x_k \ge 0$, $k = 1,2,\dots,$ with the following property: there exists a sequence $0 = q_0 < q_1 < \dots$ such that

$$\sum_{s=1}^\infty \frac{2^s}{\Delta q_s} < \infty; \tag{1.2}$$

and

$$\sum_{s=1}^{\infty} 2^{-s} \sum_{k=1}^{q_s} x_k^2 < \infty, \tag{1.3}$$

where $\Delta q_s := q_s - q_{s-1}$.

Remark 1.1. It is clear from this definition that if $x \in \mathcal{V}$ and for some N and c we have $0 \le y_k \le c x_k$, $k \ge N$, then $y \in \mathcal{V}$.

Theorem 1.1. *The condition $\tau \notin \mathcal{V}$ is necessary and sufficient for convergence of Weak Greedy Algorithm with weakness sequence τ for each f and all Hilbert spaces H and dictionaries $\mathcal{D}$.*

Sufficient part is proved in Section 2 and necessary part is proved in Section 3.

2. Proof of convergence

We begin this section with the following lemma.

Lemma 2.1. *Let $\{a_j\}_{j=1}^{\infty} \in l_2$, $a_j \ge 0$, $j = 1, 2, \dots$. Then $\{a_n \sum_{j=1}^{n} a_j\}_{n=1}^{\infty} \in \mathcal{V}$.*

Proof. Assume $\{a_j\}_{j=1}^{\infty}$ contains infinitely many nonzero terms (if not the statement is trivial). Denote $y_n := a_n \sum_{j=1}^{n} a_j$ and define $q_s := q_s(y)$ inductively: $q_0 := 0$ and for $q_0, \dots, q_{s-1}$ defined we choose q_s as the smallest q such that

$$(q - q_{s-1}) \sum_{n=q_{s-1}+1}^{q} y_n^2 \ge 2^{2s}. \tag{2.1}$$

Denote $Q_s := (q_{s-1}, q_s]$. Then (2.1) implies

$$\frac{2^s}{\Delta q_s} \le 2^{-s} \sum_{n \in Q_s} y_n^2 \le 2^{-s} \sum_{n=1}^{q_s} y_n^2.$$

Thus it is sufficient to check only (1.3)

$$\sum_s 2^{-s} \sum_{n=1}^{q_s} y_n^2 < \infty.$$

From the definition of q_s we have

$$\sum_{n=q_{s-1}+1}^{q_s - 1} y_n \le (\Delta q_s - 1)^{1/2} \Big(\sum_{n=q_{s-1}+1}^{q_s - 1} y_n^2 \Big)^{1/2} < 2^s. \tag{2.2}$$

Next for any $N \le M$ we have

$$\sum_{n=N}^{M} a_n \sum_{j=1}^{n} a_j \ge \sum_{N \le j \le n \le M} a_n a_j =$$

$$= 1/2(\sum_{j=N}^{M} a_j^2 + (\sum_{j=N}^{M} a_j)^2) \ge (\sum_{j=N}^{M} a_j)^2/2. \tag{2.3}$$

Combining (2.2) and (2.3) we get

$$\sum_{j\in Q_s} a_j = \sum_{j=q_{s-1}+1}^{q_s-1} a_j + a_{q_s} \le 2^{(s+1)/2} + \|a\|_\infty .$$

This implies

$$\sum_{j=1}^{q_s} a_j \le C(a)2^{s/2}. \tag{2.4}$$

We have now

$$\sum_s 2^{-s}\sum_{n=1}^{q_s} y_n^2 = \sum_s 2^{-s}\sum_{v=1}^{s}\sum_{n\in Q_v} y_n^2 \le 2\sum_s 2^{-s}\sum_{n\in Q_s} y_n^2 \le$$

$$2\sum_s 2^{-s}(\sum_{j=1}^{q_s} a_j)^2 \sum_{n\in Q_s} a_n^2 \le C(a)\sum_n a_n^2 < \infty .$$

Lemma 2.1 is proved now.

Theorem 2.1. *The following two conditions are equivalent*

$$\tau \notin \mathcal{V}, \tag{C.1}$$

$$\forall \{a_j\}_{j=1}^\infty \in l_2, \quad a_j \ge 0, \quad \liminf_{n\to\infty} a_n \sum_{j=1}^{n} a_j/t_n = 0. \tag{C.2}$$

Proof. We prove first that (C.1) $\Rightarrow$ (C.2). Assume (C.2) is not satisfied: $\exists\{a_j\}_{j=1}^\infty \in l_2$, $a_j \ge 0$, such that

$$\liminf_{n\to\infty} a_n \sum_{j=1}^{n} a_j/t_n > 0. \tag{2.5}$$

Relation (2.5) implies that for some N and $c > 0$ we have for $n \ge N$ that

$$a_n \sum_{j=1}^{n} a_j/t_n \ge c$$

or

$$t_n \le Ca_n \sum_{j=1}^{n} a_j .$$

This inequality, Lemma 2.1, and Remark 1.1 imply that $\tau \in \mathcal{V}$. The first implication is proved now.

We proceed to the second implication (C.2) $\Rightarrow$ (C.1). Let $\tau \in \mathcal{V}$. We construct a sequence $\{a_j\}_{j=1}^{\infty} \in l_2$ such that for all n

$$t_n \leq C a_n \sum_{j=1}^{n} a_j$$

with some C. This will imply that (C.2) is not satisfied. Let $\{q_s\} := \{q_s(\tau)\}$ be a sequence from the definition of $\mathcal{V}$. We define a sequence $\{a_j\}_{j=1}^{\infty}$ as follows. For $n \in Q_s$ we set

$$a_n := t_n 2^{-s/2} + 2^{s/2}/\Delta q_s.$$

Then

$$a_n^2 \leq 2(t_n^2 2^{-s} + 2^s (\Delta q_s)^{-2})$$

and

$$\sum_n a_n^2 \leq 2 \sum_s 2^{-s} \sum_{n \in Q_s} t_n^2 + 2 \sum_s \frac{2^s}{\Delta q_s} < \infty.$$

Next,

$$\sum_{n \in Q_s} a_n \geq 2^{s/2}.$$

Thus for $n \in Q_s$ we have

$$a_n \sum_{j=1}^{n} a_j \geq a_n \sum_{j \in Q_{s-1}} a_j \geq t_n 2^{-1/2}$$

and

$$t_n \leq \sqrt{2} a_n \sum_{j=1}^{n} a_j$$

for all n.

Theorem 2.1 is proved now.

The sufficient part of Theorem 1.1 follows from Theorem 2.1 and Proposition 1.1.

3. Construction of a counterexample

The following procedure which is the generalization of Equalizer from [LT] plays an important role in the construction. Let H be a Hilbert space with an orthonormal basis $\{e_j\}_{j=1}^{\infty}$. We take two elements e_i, e_j, $i \neq j$, and define the following procedure.

Equalizer with schedule $\gamma := \{\gamma_k\}$**.** Let $\gamma_k \leq 1/5$, $f_0 := e_i$. Define:

(3.1) $$g_1 := \alpha_1 e_i - (1 - \alpha_1^2)^{1/2} e_j; \quad \alpha_1 = \gamma_1; \quad \langle f_0, g_1 \rangle = \gamma_1;$$

(3.2) $$f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n; \quad g_n := \alpha_n e_i - (1 - \alpha_n^2)^{1/2} e_j;$$

$$\langle f_n, g_{n+1}\rangle = \gamma_{n+1}; \quad f_n = a_n e_i + b_n e_j.$$

We check

$$a_{n-1} - b_{n-1} \ge 3\sqrt{2}\gamma_n$$

to continue. If

$$a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n$$

then we take $g_n := 2^{-1/2}(e_i - e_j)$ and

$$f_n := f_{n-1} - \langle f_{n-1}, g_n\rangle g_n,$$

and stop after this step. We call this step "the final step" and all other steps "regular steps". At each regular step l we have

$$a_l - b_l = a_{l-1} - b_{l-1} - \gamma_l(\alpha_l + (1-\alpha_l^2)^{1/2}) \ge a_{l-1} - b_{l-1} - 2^{1/2}\gamma_l > 0.$$

After the final step we have

$$a_n = b_n.$$

At each regular step we have by definition that

$$\langle f_{l-1}, g_l\rangle = \gamma_l.$$

At the final step we have

$$\langle f_{n-1}, g_n\rangle = 2^{-1/2}(a_{n-1} - b_{n-1}) \ge 2^{-1/2}(a_{n-2} - b_{n-2} - 2^{1/2}\gamma_{n-1}) \ge$$
$$2^{-1/2}(2\sqrt{2}\gamma_{n-1}) = 2\gamma_{n-1}.$$

Thus, if $2\gamma_{n-1} \ge \gamma_n$ then the above described Equalizer is a WGA with weakness sequence $\gamma_1, \dots, \gamma_n$.

At regular step l we reduce the $\|\cdot\|^2$ by γ_l^2. At the final step we reduce the $\|\cdot\|^2$ by

$$\frac{1}{2}(a_{n-1} - b_{n-1})^2 < 9\gamma_n^2.$$

We also have

$$a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n$$

and

$$a_{n-1} - b_{n-1} \ge 1 - \sqrt{2}\sum_{j=1}^{n-1}\gamma_j.$$

Thus,

$$\sqrt{2}\sum_{j=1}^{n-1}\gamma_j + 3\sqrt{2}\gamma_n > 1.$$

On the other hand

$$a_l - b_l \le a_{l-1} - b_{l-1} - \gamma_l.$$

Therefore,

$$0 \le a_{n-1} - b_{n-1} \le 1 - \sum_{l=1}^{n-1}\gamma_l$$

and

$$\sum_{l=1}^{n-1}\gamma_l \le 1. \tag{3.3}$$

In order to apply the above Equalizer we need to have the inequality $2\gamma_{n-1} \ge \gamma_n$ satisfied. Let us use the following regularization procedure.

Regularization. For a given $\tau = \{t_k\}_{k=1}^{\infty}$, $\tau \in l_\infty$, we define $\tau^R := \{t_k^R\}_{k=1}^{\infty}$ with

$$t_k^R := \sum_{m=0}^{\infty} 2^{-m} t_{n+m}.$$

Lemma 3.1. *If $\tau \in \mathcal{V} \cap l_\infty$ then $\tau^R \in \mathcal{V} \cap l_\infty$.*

Proof. Assumption $\tau \in \mathcal{V}$ implies

$$\sum_s 2^{-s} \sum_{k=1}^{q_s} t_k^2 < \infty. \tag{3.4}$$

We will prove that

$$\sum_s 2^{-s} \sum_{k=1}^{q_s} (t_k^R)^2 < \infty \tag{3.5}$$

with the same $q_s = q_s(\tau)$ as above. Thus (3.5) will imply $\tau^R \in \mathcal{V}$. Let us prove (3.5). We have for any N

$$\sum_{k=1}^{N}(t_k^R)^2 = \sum_{k=1}^{N}(\sum_{m=0}^{\infty} 2^{-m} t_{n+m})^2 = \sum_{k=1}^{N}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{-m-n} t_{k+m} t_{k+n} =$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{-m-n} \sum_{k=1}^{N} t_{k+m} t_{k+n} \le \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{-m-n} (\sum_{k=1}^{N} t_{k+m}^2)^{1/2} (\sum_{k=1}^{N} t_{k+n}^2)^{1/2} =$$

$$(\sum_{m=0}^{\infty} 2^{-m} (\sum_{k=1}^{N} t_{k+m}^2)^{1/2})^2 \le (\sum_{m=0}^{\infty} 2^{-m})(\sum_{m=0}^{\infty} 2^{-m} \sum_{k=1}^{N} t_{k+m}^2).$$

Next,

$$\sum_{k=1}^{N} t_{k+m}^2 \le \sum_{k=1}^{N} t_k^2 + m\|\tau\|_\infty^2$$

and

$$\sum_{m=0}^{\infty} 2^{-m}(\sum_{j=1}^{N} t_j^2 + m\|\tau\|_\infty^2) \le 2\sum_{j=1}^{N} t_j^2 + C(\tau).$$

Therefore we got

$$\sum_{j=1}^{N}(t_j^R)^2 \le 2\sum_{j=1}^{N} t_j^2 + C(\tau)$$

and

$$\sum_s 2^{-s} \sum_{k=1}^{q_s} (t_k^R)^2 \le 2\sum_s 2^{-s} \sum_{k=1}^{q_s} t_k^2 + C(\tau) < \infty.$$

It is easy to see that $\|\tau^R\|_\infty \le 2\|\tau\|_\infty$.

Lemma 3.1 is proved now.

Thus for any $\tau \in \mathcal{V} \cap l_\infty$ we have $\tau^R \in \mathcal{V}$, $\|\tau^R\|_\infty \le 2\|\tau\|_\infty$, and

$$2t^R_{n-1} \ge t^R_n, \quad n = 2, 3, \dots .$$

Clearly, we also have for all n

$$t_n \le t^R_n.$$

One more restriction in the Equalizer is $\gamma_n \le 1/5$. Define a new sequence τ' by

$$t'_n := \min\{t^R_n, 1/5\}.$$

It is clear that $\tau' \in \mathcal{V}$ and also satisfies

$$2t'_{n-1} \ge t'_n.$$

Let $\{q_s\} := \{q_s(\tau')\}$ be the sequence for τ' from the definition of $\mathcal{V}$:

$$\sum_s \frac{2^s}{\Delta q_s} < \infty, \quad \sum_s 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \infty.$$

Let ϵ be a small number which we will specify later and s_0 be such that

$$\sum_{s \ge s_0} \frac{2^s}{\Delta q_s} < \epsilon, \quad \sum_{s \ge s_0} 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \epsilon.$$

Consider the function

$$f_{s_0} := 2^{-s_0/2}(e_1 + \dots + e_{2^{s_0}}).$$

We have $\|f_{s_0}\| = 1$. Define

$$t''_k := \max\{t'_k, 2^{s_0+2}(\Delta q_{s_0})^{-1}\}.$$

We apply a mixture of Equalizer with schedule $\{t''_k\}$ to vectors e_i, $i \le 2^{s_0}$, and the PGA to the corresponding residual of f_{s_0}. We do this in the following way. If $t''_1 \ge 1/5$ we use PGA and throw away, say, $2^{-s_0/2}e_{2^{s_0}}$. If $t''_1 < 1/5$ we start using the Equalizer with schedule $\{t''_k\}$ to vectors e_1 and $e_{2^{s_0}+1}$. If at some step $t''_k \ge 1/5$ then we use PGA what means throwing away one term of the form $2^{-s_0/2}e_j$, $j \in [1, 2^{s_0}]$. Applying the Equalizer to the very last term of the form $2^{-s_0/2}e_m$ we may incounter with $t''_k \ge 1/5$. In such a case we apply PGA and stop. As a result we get

$$f_{s_0+1} := \sum_{k \in F_{s_0+1}} c^{s_0+1}_k e_k.$$

It is clear that for all $k \in F_{s_0+1}$ we have

$$(c^{s_0+1}_k)^2 \le 2^{-s_0-1}$$

and also

$$|F_{s_0+1}| \le 2^{s_0+1}.$$

Assume that $\epsilon < 1/20$. Then $2^{s_0+2}(\Delta q_{s_0})^{-1} < 1/5$ and $t''_k \geq 1/5$ is equivalent to $t''_k = t'_k = 1/5$. If $t''_k < 1/5$ then $t'_k < 1/5$ and $t_k \leq t'_k$. Therefore, at all Equalizer steps we have a WGA with weakness parameters $\{t_k\}$. If $t''_k = 1/5$ we apply PGA what is a WGA with any t_k at this step. During this procedure which we call "working on s_0-level" we perform $M^w_{s_0}$ steps of Equalizer and $M^G_{s_0}$ steps of PGA. Let us estimate $M^w_{s_0}$ and $M^G_{s_0}$. It is clear that $M^G_{s_0} \leq 2^{s_0}$. We have applied the Equalizer to terms of the form $2^{-s_0/2}e_j$ at most 2^{s_0} times. For each Equalizer application we have $\sum \gamma_j \leq 2$ (see (3.3)). Thus denoting $E(s_0) := \{k \quad : \quad t''_k < 1/5\}$ we get

$$\sum_{k\in E(s_0)} t''_k \leq 2^{s_0+1}.$$

On the other hand we have

$$\sum_{k\in E(s_0)} t''_k \geq M^w_{s_0} 2^{s_0+2}(\Delta q_{s_0})^{-1}$$

and

$$M^w_{s_0} \leq \Delta q_{s_0}/2.$$

Therefore,

$$N_{s_0} := M^w_{s_0} + M^G_{s_0} \leq \Delta q_{s_0}/2 + 2^{s_0} \leq \Delta q_{s_0}.$$

At each Equalizer step we reduced the $\|\cdot\|^2$ by at most $9(t''_k)^2 2^{-s_0}$ and at each PGA by at most $25(t'_k)^2 2^{-s_0}$. Thus the total reduction δ_{s_0} for the s_0-level does not exceed

$$25(2^{-s_0})\sum_{k=1}^{q_{s_0}}(t'_k)^2 + 9(2^{s_0+4})(\Delta q_{s_0})^{-1}.$$

We are on the (s_0+1)-level now and perform the similar procedure. We describe it for the general case of an s-level. Assume we have after $N_{s-1} \leq q_{s-1}$ steps of our WGA the function

$$f_s = \sum_{k\in F_s} c^s_k e_k$$

with

$$(c^s_k)^2 \leq 2^{-s}, \quad |F_s| \leq 2^s.$$

Define now

$$t''_k := \max\{t'_k, 2^{s+2}(\Delta q_s)^{-1}\}, \quad k > N_{s-1}.$$

We pick $c^s_k e_k$ with the biggest c^s_k out of $\{c^s_k, k \in F_s\}$ and throw it away if $t''_{N_{s-1}+1} = 1/5$ (we remind that assumption $\epsilon < 1/20$ implies $2^{s+2}(\Delta q_s)^{-1} < 1/5$) and apply the Equalizer with schedule $\{t''_n\}$ otherwise. We continue to perform the above described procedure (the mixture of Equalizer and PGA steps) untill we get

$$f_{s+1} = \sum_{k\in F_{s+1}} c^{s+1}_k e_k$$

with

$$(c^{s+1}_k)^2 \leq 2^{-s-1}.$$

It is clear that then $|F_{s+1}| \le 2^{s+1}$. Similarly to the above estimates of $M^w_{s_0}$ and $M^G_{s_0}$ we get

$$M^G_s \le 2^s$$

and

$$M^w_s 2^{s+2}(\Delta q_s)^{-1} \le \sum_{k\in E(s)} t''_k \le 2^{s+1}.$$

Thus

$$M^w_s + M^G_s \le \Delta q_s/2 + 2^s \le \Delta q_s$$

and

$$N_s := N_{s-1} + M^w_s + M^G_s \le q_s.$$

The total reduction δ_s of the $\|\cdot\|^2$ from working on the s-level does not exceed

$$25(2^{-s})\sum_{k=1}^{q_s}(t'_k)^2 + 9(2^{s+4})(\Delta q_s)^{-1}.$$

We continue this process and get that the $\|\cdot\|^2$ will be reduced by at most

$$\sum_{s=s_0}^{\infty}\delta_s \le 25(\sum_{s=s_0}^{\infty} 2^{-s}\sum_{k=1}^{q_s}(t'_k)^2) + 144\sum_{s=s_0}^{\infty} 2^s(\Delta q_s)^{-1} \le 169\epsilon.$$

Choosing ϵ small enough, say, $\epsilon = 0.005$ we get divergent WGA with the weakness sequence τ. This completes the construction of the counterexample.

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