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CIRCLES MINIMIZE MOST KNOT ENERGIES

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Abstract. We show that circles uniquely minimize most of O'Hara's knot energies, including those conjectured by Freedman, He, and Wang. The proof is based on a theorem of Lükő on average chord lengths of closed curves. We also prove this result for a broader class of energy functionals and find energies not minimized by a round circle.

1. Introduction

For the past decade, there has been a great deal of interest in defining new knot invariants by minimizing various functionals on the space of curves of a given knot type. For example, imagine a loop of string bearing a uniformly distributed electric charge, floating in space. The loop will repel itself, and settle into some least energy configuration. If the loop is knotted, the potential energy of this configuration will provide a measure of the complexity of the knot.

In 1991 Jun O'Hara began to formalize this picture [12, 13] by proposing a family of energy functionals e_j^p (for j, $p > 0$) which are based on the physicists' concept of renormalization, and are defined by $e_j^p[c] := (1/j)(E_j^p[c])^{1/p}$, where

$$E_j^p[c] := \iint \left(\frac{1}{|c(t) - c(s)|^j} - \frac{1}{d(t,s)^j} \right)^p dt\, ds, \tag{1.1}$$

$c\colon S^1 \to \mathbf{R}^3$ is a unit-speed curve, $|c(t) - c(s)|$ is the distance between $c(t)$ and $c(s)$ in space, and $d(t, s)$ is the shortest distance between $c(t)$ and $c(s)$ along the curve. O'Hara showed [14] that these integrals converge if the curve c is smooth and embedded, $j < 2 + 1/p$, and that a minimizing curve exists in each isotopy class when $jp > 2$.

It was then natural to examine the shapes of these energy minimizing curves, and to conjecture that they must be the (round) circle. Two pieces of evidence had supported this: Freedman, He, and Wang [4] had intensively studied the energy e_2^1, proving that it was Möbius-invariant, and hence that the minimizer in this case was the circle, while O'Hara had shown that the limit of e_j^p as $p \to \infty$ and $j \to 0$ was the logarithm of Gromov's *distortion*, which was known to be minimized for the circle as well (see [10] for a simple proof). The main result of this paper is that this conjecture is true for $p \geq 1$ (c.f. [4, p. 48])

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Theorem 1.1. *Suppose $0 < j < 2 + 1/p$, while $p \geq 1$. Then for every closed unit-speed curve c in $\mathbf{R}^n$ with length 2π,*

$$E_j^p[c] \geq 2^{3-jp}\pi \int_0^{\frac{\pi}{2}} \left(\left(\frac{1}{\sin s}\right)^j - \left(\frac{1}{s}\right)^j \right)^p ds. \tag{1.2}$$

with equality if and only if c is the circle.

We must include the condition $j < 2 + 1/p$ in our theorem, for otherwise the integral defining E_j^p does not converge. We do not know whether the condition $p \geq 1$ is sharp, since the energies are well-defined for $0 < p < 1$, but it is required for our proof.

Our arguments apply to the much broader class of energies of the form

$$\iint F(s, |c(t+s) - c(t)|^2)\, dt\, ds,$$

where $F(s, \cdot)$ is decreasing and convex (Theorem 4.1). We use several ideas from a prophetic paper of Lükő Gábor [11], written almost thirty years before the conjecture of Freedman, He, and Wang was made. Lükő[1] showed that among closed, unit-speed planar curves of length 2π, circles are the only maximizers of any functional in the form

$$\iint f(|c(s) - c(t)|^2)\, ds\, dt, \tag{1.3}$$

where f is increasing and concave.

Our arguments are modeled in part on Hurwitz's proof of the planar isoperimetric inequality [8] [3, p. 111]. In Section 2, we derive a Wirtinger-type inequality (Theorem 2.2), which we use in Section 3 to generalize Lükő's theorem (Theorem 3.1). We then apply this result to obtain sharp integral inequalities for average chord lengths and distortions. In the process, we find another proof that the curve of minimum distortion is a circle. In Section 4, we formulate precisely (Theorem 4.1) the conditions which an energy functional must satisfy to complete the proof of Theorem 1.1.

All our methods depend on the concavity of f in functionals of the form of Equation 1.3. In Section 5, we consider the case where f is convex, as in the case of the functional

$$\iint |c(s) - c(t)|^p\, ds\, dt \tag{1.4}$$

for $p > 2$. Numerical experiments suggest that the maximizing curve for this functional remains a circle for $p < \alpha$, with $3.3 < \alpha < 3.5721$, while for $p > 3.5721$, the maximizers form a family of stretched ovals converging to a doubly-covered line segment as $p \to \infty$.

2. A Wirtinger type inequality

Definition 2.1. Let $\lambda\colon \mathbf{R} \to \mathbf{R}$ be given by

$$\lambda(s) := 2\sin\frac{s}{2}. \tag{2.1}$$

[1]There are references in the literature to papers authored both by Lükő Gábor and by Gábor Lükő. We are informed that these people are identical and that Lükő is the family name; the confusion likely results from the Hungarian convention of placing the family name first.

For $0 \le s \le 2\pi$, $\lambda(s)$ is the length of the chord connecting the end points of an arc of length s in the unit circle.

Our main aim in this section is to prove the following inequality, modeled after a well known lemma of Wirtinger [3, p. 111]. For simplicity, we restrict our attention to closed curves of length 2π in $\mathbf{R}^n$.

Theorem 2.2. *Let $c\colon S^1 := \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{R}^n$ be an absolutely continuous function. If $c'(t)$ is square integrable, then for any $s \in \mathbf{R}$*

$$\int |c(t+s) - c(t)|^2\,dt \le \lambda^2(s) \int |c'(t)|^2\,dt, \tag{2.2}$$

with equality if and only if s is an integral multiple of 2π or

$$c(t) = a_0 + (\cos t)\,a + (\sin t)\,b \tag{2.3}$$

for some vectors $a_0, a, b \in \mathbf{R}^n$.

We give two proofs of this result, one based on the elementary theory of Fourier series, and one based on the maximum principle for ordinary differential equations.

Fourier series proof. We assume that $c\colon S^1 \to \mathbf{R}^n \subset \mathbf{C}^n$, as the complex form of the Fourier series is more convenient. $\mathbf{C}^n$ is equipped with its standard positive definite Hermitian inner product $\langle v, w\rangle = \sum_{k=1}^n z_k \overline{w}_k$ where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. This agrees with the usual inner product on $\mathbf{R}^n \subset \mathbf{C}^n$. The norm of $v \in \mathbf{C}^n$ is given by $|v| := \sqrt{\langle v, v\rangle}$, and $i := \sqrt{-1}$.

The facts about Fourier series required for the proof are as follows. If $\phi\colon S^1 \to \mathbf{C}^n$ is locally square integrable then it has a Fourier expansion

$$\phi(t) = \sum_{k=-\infty}^{\infty} \phi_k e^{kti},$$

(the convergence is in L^2 and the series may not converge pointwise). The L^2 norm of ϕ is given by

$$\int |\phi(t)|^2\,dt = 2\pi \sum_{k=-\infty}^{\infty} |\phi_k|^2. \tag{2.4}$$

If ϕ is absolutely continuous and ϕ' is locally square integrable then ϕ' has the Fourier expansion $\phi'(t) = i\sum_{k=-\infty}^{\infty} k\phi_k e^{kti}$ and therefore

$$\int |\phi'(t)|^2\,dt = 2\pi \sum_{k=-\infty}^{\infty} k^2|\phi_k|^2 = 2\pi \sum_{k=1}^{\infty} k^2(|\phi_{-k}|^2 + |\phi_k|^2), \tag{2.5}$$

as the contribution to the middle sum from the term $k = 0$ is zero.

Let $\sum_{k=-\infty}^{\infty} a_k e^{kti}$ be the Fourier expansion of $c(t)$, where $a_k \in \mathbf{C}^n$. Then

$$\begin{aligned} c(t+s/2) - c(t-s/2) &= \sum_{k=-\infty}^{\infty} \left(e^{ksi/2} - e^{-ksi/2}\right) a_k e^{kti} \\ &= 2i \sum_{k=-\infty}^{\infty} \left(\sin\frac{ks}{2}\right) a_k e^{kti}. \end{aligned}$$

Therefore, using (2.4), we have

$$\begin{aligned}\int |c(t+s)-c(t)|^2\,dt &= \int \left|c\left(t+\frac{s}{2}\right)-c\left(t-\frac{s}{2}\right)\right|^2\,dt\\ &= 2\pi|2i|^2\sum_{k=-\infty}^{\infty}\left(\sin^2\frac{ks}{2}\right)|a_k|^2\\ &= 8\pi\sum_{k=1}^{\infty}\left(\sin^2\frac{ks}{2}\right)\left(|a_{-k}|^2+|a_k|^2\right). \qquad (2.6)\end{aligned}$$

Also, by (2.5) and (2.1),

$$\begin{aligned}\lambda^2(s)\int |c'(t)|^2\,dt &= \left(4\sin^2\frac{s}{2}\right)\left(2\pi\sum_{k=1}^{\infty}k^2(|a_k|^2+|a_{-k}|^2)\right)\\ &= 8\pi\sum_{k=1}^{\infty}\left(k^2\sin^2\frac{s}{2}\right)(|a_k|^2+|a_{-k}|^2). \qquad (2.7)\end{aligned}$$

Subtracting (2.6) from (2.7), we set

$$\begin{aligned}\rho_c(s) &:= \lambda^2(s)\int |c'(t)|^2\,dt - \int |c(t+s)-c(t)|^2\,dt\\ &= 8\pi\sum_{k=2}^{\infty}\left(k^2\sin^2\frac{s}{2}-\sin^2\frac{ks}{2}\right)(|a_{-k}|^2+|a_k|^2).\end{aligned}$$

Lemma 2.3 (below) implies that $\rho_c(s)\geq 0$ with equality if and only if s is a multiple of 2π, or $a_k=a_{-k}=0$ for all $k\geq 2$. The latter occurs if and only if

$$c(t)=a_{-1}e^{-it}+a_0+a_1e^{it}=a_0+(\cos t)\,a+(\sin t)\,b \qquad (2.8)$$

where $a:=a_1+a_{-1}$ and $b:=i(a_1-a_{-1})$. □

Lemma 2.3. *Let $k\geq 2$ be an integer. Then*

$$\sin^2(k\theta)\leq k^2\sin^2(\theta), \qquad (2.9)$$

with equality if and only if $\theta=m\pi$ for some integer m

Proof. If $\theta=m\pi$, for some integer m, then equality holds in (2.9). If $\theta\neq m\pi$, set $q_k(\theta):=|\sin(k\theta)/\sin(\theta)|$. Then $|\cos(\theta)|<1$, and the addition formula for sine yields

$$q_{k+1}(\theta)=|\cos(\theta)\,q_k(\theta)+\cos(k\theta)|<q_k(\theta)+1, \qquad (2.10)$$

Since $q_1(\theta)\equiv 1$, we then have $q_k(\theta)<k$ by induction, which completes the proof. □

Maximum principle proof. This method is an adaptation of Lükő's original approach [11]. In that paper, he solves a discrete version of the problem, showing that the average squared distance between the vertices of an n-gon of constant side length is maximized by the regular n-gon. He then obtains the main result by approximation. We go directly to the continuum case, which turns out to be simpler.

To simplify notation, let $L := \int |c'(t)|^2\,dt$, and set

$$f(s) := \int |c(t+s) - c(t)|^2\,dt,$$
$$\Lambda(s) := \lambda^2(s) \int |c'(t)|^2 = L\lambda^2(s).$$

We claim that f is C^2 with

$$f'(s) = 2 \int \langle c(t) - c(t-s), c'(t)\rangle\,dt,$$
$$f''(s) = 2 \int \langle c'(t-s), c'(t)\rangle\,dt,$$

and initial conditions

$$f(0) = 0, \qquad f'(0) = 0, \qquad f''(0) = 2 \int |c'(t)|^2\,dt = 2L. \tag{2.11}$$

These formulas are clear when c is C^2 and hold in the general case by approximating by C^2 functions. The explicit formula for f'' makes it clear that f is C^2.

Next we derive a differential inequality for f, using an elementary geometric fact (which appears in a slightly different form in Lükő's paper as Lemma 7):

Lemma 2.4. *For any tetrahedron A, B, C, D in $\mathbf{R}^n$,*

$$|AC|^2 + |BD|^2 \le |BC|^2 + |AD|^2 + 2|AB|\,|CD|, \tag{2.12}$$

with equality if and only if AB and DC are parallel as vectors.

Proof. Denote the vectors AB, BC, CD, DA by v_1, v_2, v_3, v_4. Then $\sum v_i = 0$, and

$$\begin{aligned}
|AC|^2 + |BD|^2 &= \frac{1}{2}\left(|v_1+v_2|^2 + |v_2+v_3|^2 + |v_3+v_4|^2 + |v_4+v_1|^2\right) \\
&= \sum_{i=1}^{4} |v_i|^2 + \langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_4\rangle + \langle v_4, v_1\rangle \\
&= \sum_{i=1}^{4} |v_i|^2 + \langle v_1 + v_3, v_2 + v_4\rangle \\
&= \sum_{i=1}^{4} |v_i|^2 - |v_1 + v_3|^2 \\
&\le \sum_{i=1}^{4} |v_i|^2 - (|v_1| - |v_3|)^2 \\
&= |v_2|^2 + |v_4|^2 + 2|v_1||v_3| = |BC|^2 + |AD|^2 + 2|AB|\,|CD|.
\end{aligned}$$

Equality holds if and only if $v_3 = -\rho v_1$ for some $\rho > 0$, which is equivalent to AB and DC being parallel as vectors. ∎

For any t, s and h, we can apply Lemma 2.4 to the tetrahedron $c(t)$, $c(t+s+h)$, $c(t+s)$, $c(t+h)$ to derive the equation

$$\begin{aligned}|c(t+s)-c(t)|^2 + |c(t+s+h)-c(t+h)|^2 \\ \leq |c(t+s+h)-c(t+s)|^2 + |c(t+h)-c(t)|^2 \\ + 2|c(t+s+h)-c(t)|\,|c(t+s)-c(t+h)|.\end{aligned}$$

Holding s, h fixed and integrating with respect to t,

$$\begin{aligned}2f(s) &\leq 2f(h) + 2\int |c(t+s+h)-c(t)|\,|c(t+s)-c(t+h)|\,dt \\ &\leq 2f(h) + 2\sqrt{f(s+h)f(s-h)}\end{aligned}$$

by the Cauchy-Schwartz inequality. Therefore $f(s) \leq f(h) + \sqrt{f(s+h)f(s-h)}$. For any fixed s, this can be rewritten

$$g(h) := \frac{1}{2}\big(\log f(s+h) + \log f(s-h)\big) - \log\big(f(s) - f(h)\big) \geq 0.$$

When s is not a multiple of 2π, $f(s) > 0$ and g is well-defined for small h. Further, g has a local minimum at $h = 0$, and so the second derivative of g is non-negative at zero. Using (2.11), this tells us that

$$\frac{d^2}{ds^2}\log f(s) \geq \frac{-2L}{f(s)}. \tag{2.13}$$

Meanwhile, $\Lambda(s)$ satisfies the differential equation

$$\frac{d^2}{ds^2}\log \Lambda(s) = \frac{-2L}{\Lambda(s)}. \tag{2.14}$$

We are trying to show that $f(s) \leq \Lambda(s)$ and that if equality holds for any $s \in (0, 2\pi)$, then $f(s) \equiv \Lambda(s)$. Let

$$u(s) = \log\frac{f(s)}{\Lambda(s)} = \log f(s) - \log \Lambda(s).$$

In these terms, we want to show that $u(s) \leq 0$ and that if $u(s) = 0$ for some $s \in (0, 2\pi)$ then $u \equiv 0$. Using (2.13) and (2.14),

$$u''(s) \geq \frac{-2L}{f(s)} + \frac{2L}{\Lambda(s)} = \frac{2L}{f(s)}\left(\frac{f(s)}{\Lambda(s)} - 1\right) = \frac{2L}{f(s)}\left(e^{u(s)} - 1\right) \geq \frac{2L}{f(s)}u(s).$$

By two applications of L'Hospital's rule, we compute $\lim_{s\to 0} u(s) = 0$. It follows that $\lim_{s\to 2\pi} u(s) = 0$ as well. So if u is ever positive, it will have a positive local maximum at some point $s_0 \in (0, 2\pi)$. At that point,

$$0 \geq u''(s_0) \geq \frac{2L}{f(s_0)}u(s_0) > 0,$$

which is a contradiction. So u is non-positive on $(0, 2\pi)$. Further, if u is zero at any point in $(0, 2\pi)$, the strong maximum principle [20, Thm 17 p. 183] implies that u vanishes on the entire interval. Thus $f(s) \leq \Lambda(s)$ with equality at any point of $(0, 2\pi)$ if and only if $f(s) \equiv \Lambda(s)$.

Last, we show that if $f(s) = \int |c(t+s) - c(t)|^2\,dt \equiv \lambda^2(s)\int |c'(t)|^2\,dt = \Lambda(s)$, then c is an ellipse. By our work above, if $f = \Lambda$, then for each fixed s, c maximizes

$\int |c(t+s)-c(t)|^2\,dt$ subject to the constraint that $\int |c'(t)|^2\,dt$ is held constant. The Lagrange multiplier equation for this variational problem is

$$c''(t) = M\big(c(t+s) - 2c(t) + c(t-s)\big)$$

where M is a constant depending on s. When $s = \pi$ we can use the fact that c has period 2π and this becomes

$$c''(t) = 2M\big(c(t+\pi) - c(t)\big).$$

Differentiating twice with respect to t, and using both the periodicity and the equation,

$$\begin{aligned} c''''(t) &= 2M\big(c''(t+\pi) - c''(t)\big) \\ &= 4M^2\big(c(t) - c(t-\pi) - c(t+\pi) + c(t)\big) \\ &= -8M^2\big(c(t+\pi) - c(t)\big) \\ &= -4Mc''(t). \end{aligned}$$

So c'' satisfies the equation $g'' = -4Mg$ and has period 2π. This implies that $4M = k^2$ for some $k \in \mathbf{Z}$, and $c''(t) = (\cos kt)V + (\sin kt)W$ with V and W in $\mathbf{R}^n$. But $k = \pm 1$, for otherwise $f(2\pi/k) = 0 \neq \Lambda(2\pi/k)$, a contradiction. Taking two antiderivatives,

$$c(t) = a_0 + tb_0 + (\cos t)\ a + (\sin t)\ b, \tag{2.15}$$

with a_0, b_0, a, b in $\mathbf{R}^n$. Periodicity implies that $b_0 = 0$, completing the proof. □

Remark 2.5. By equation (2.8), extremals for the inequality of Theorem 2.2 are either ellipses or double coverings of line segments, depending on whether a and b are linearly independent. Thus the set of extremal curves is invariant under affine maps of $\mathbf{R}^n$. When the extremal is an ellipse, the parameterization is a constant multiple of the *special affine arclength* (c.f. [2, p. 7], [19, p. 56]). It would be interesting to find an affine invariant interpretation of inequality (2.2) or of the deficit $\rho_c(s)$ used in the first proof—especially when c is a convex planar curve.

3. Inequalities for Concave Functionals

We now apply Theorem 2.2 to obtain the following inequalities. Inequality (3.2) corresponds to the theorem of Lükő [11] mentioned in the introduction.

Theorem 3.1. *Let c be a closed, unit-speed curve of length 2π in $\mathbf{R}^n$. For $0 < s < 2\pi$, if $f\colon \mathbf{R} \to \mathbf{R}$ is increasing and concave on $(0, s^2) \cap (0, (2\pi - s)^2)$,*

$$\frac{1}{2\pi}\int f\left(|c(t+s) - c(t)|^2\right)\,dt \le f\left(\lambda^2(s)\right) \tag{3.1}$$

and equality holds if and only if c is the unit circle. Further, if $f\colon \mathbf{R} \to \mathbf{R}$ is increasing and concave on $(0, \pi^2)$,

$$\frac{1}{4\pi^2}\iint f\left(|c(t) - c(s)|^2\right)\,dt\,ds \le \frac{1}{2\pi}\int f\left(\lambda^2(s)\right)\,ds. \tag{3.2}$$

If $\frac{1}{2\pi}\int f\left(\lambda^2(s)\right)\,ds$ is finite, then equality holds if and only if c is the unit circle.

Proof. Since $c(t)$ and $c(t+s)$ are at distance s and $2\pi - s$ along the curve, $|c(t+s) - c(s)|^2$ is always in $(0, s^2) \cap (0, (2\pi - s)^2)$, except when $t = 0$ and possibly when $t = \min\{s^2, (2\pi - s)^2\}$. Being undefined at two points does not affect the existence of the integrals. Using Jensen's inequality for concave functions [15, p. 115], Theorem 2.2, that f is increasing, and that $|c'(t)| = 1$ for almost all t, we have

$$\begin{aligned} \frac{1}{2\pi}\int f\left(|c(t+s)-c(t)|^2\right)\,dt &\leq f\left(\frac{1}{2\pi}\int |c(t+s)-c(t)|^2\,dt\right) \\ &\leq f\left(\frac{\lambda^2(s)}{2\pi}\int |c'(t)|^2\,dt\right) \\ &= f\left(\lambda^2(s)\right). \end{aligned}$$

If equality holds in (3.1), then the above string of inequalities implies that equality holds between the two middle terms, i.e., equality holds in (2.2). Thus, since $0 < s < 2\pi$, we may apply Theorem 2.2 to conclude that $c(t)$ must be as in (2.3). Since c has unit speed, it follows that

$$c'(t) = -(\sin t)\ a + (\cos t)\ b$$

is a unit vector for all t, which forces the vectors a and b to be orthonormal, and so implies that c is the unit circle. Conversely, if c is the unit circle, then $|c(t+s) - c(t)| = \lambda(s)$ for all t (recall Definition 2.1) and therefore equality holds in (3.1).

It is clear that $(0, s^2) \cap (0, (2\pi - s)^2) \subset (0, \pi^2)$ for all $s \in (0, 2\pi)$. The inequality (3.2) now follows easily by a change of variables and applying (3.1):

$$\begin{aligned} \frac{1}{4\pi^2}\iint f\left(|c(t)-c(s)|^2\right)\,dt\,ds &= \frac{1}{4\pi^2}\iint f\left(|c(t+s)-c(t)|^2\right)\,dt\,ds \\ &\leq \frac{1}{2\pi}\int f\left(\lambda^2(s)\right)\,ds. \end{aligned}$$

This shows that if the last of these integrals finite, then equality holds in (3.2) if and only if equality holds in (3.1), which can only happen when c is the unit circle. $\square$

Letting $f(x) = \sqrt{x}$ in Theorem 3.1, we obtain the following inequalities for the average chord length of space curves:

Corollary 3.2. *Let c be a closed, unit-speed curve of length 2π in $\mathbf{R}^n$. Then for any $s \in (0, 2\pi)$,*

$$\frac{1}{2\pi}\int |c(t+s)-c(t)|\,dt \leq \lambda(s), \tag{3.3}$$

$$\frac{1}{4\pi^2}\iint |c(t)-c(s)|\,dt\,ds \leq \frac{4}{\pi}, \tag{3.4}$$

with equalities if and only if c is the unit circle. $\square$

Next we apply Theorem 3.1 to obtain sharp inequalities for Gromov's *distortion* [6, 10]. By definition, the distortion of a curve is the maximum value of the ratio of the distance in space to the distance along the curve for all pairs of points on the curve. As we mentioned above, distortion is a limit of O'Hara energies: $\exp(e_0^\infty(c)) = \mathrm{distort}(c)$ [14, p. 150].

The inequality (3.6) is due to Gromov [7, pp. 11–12], [10]. As always, while we state our results for curves of length 2π, the corresponding result holds for curves of arbitrary length.

Corollary 3.3. *For every closed, unit-speed curve c of length 2π in $\mathbf{R}^n$*

$$\operatorname{distort}_s(c) := \sup_{t\in\mathbf{R}} \frac{s}{|c(t+s)-c(t)|} \geq \frac{s}{\lambda(s)}, \tag{3.5}$$

$$\operatorname{distort}(c) := \sup_{s\in(0,\pi]} \sup_{t\in\mathbf{R}} \frac{s}{|c(t+s)-c(t)|} \geq \frac{\pi}{2}, \tag{3.6}$$

with equalities if and only if c is the unit circle.

Proof. In both cases equality is clear for the unit circle. By the mean value property of integrals and inequality (3.3),

$$\frac{1}{\operatorname{distort}_s(c)} = \inf_{t\in\mathbf{R}} \frac{|c(t+s)-c(t)|}{s} \leq \frac{1}{2\pi s} \int |c(t+s)-c(t)|\, dt \leq \frac{\lambda(s)}{s},$$

establishing (3.5). Further, equality in (3.5) implies equality in (3.3), which, by Theorem 3.1, happens if and only if c is the unit circle.

The proof of (3.6) follows easily from (3.5):

$$\operatorname{distort}(c) = \sup_{s\in(0,\pi]} \operatorname{distort}_s(c) \geq \operatorname{distort}_\pi(c) \geq \frac{\pi}{\lambda(\pi)} = \frac{\pi}{2},$$

and again equality implies in particular that $\operatorname{distort}_\pi(c) = \pi/\lambda(\pi)$, which, by (3.5), happens if and only if c is the unit circle. $\square$

For general maps $f\colon M \to \mathbf{R}^n$ of a compact Riemannian manifold to Euclidean space Gromov [6, p. 115] has given, by methods related to ours, lower bounds—which are not sharp—for the distortion of f in terms of the first eigenvalue of M and the average square distance, $\operatorname{Vol}(M)^{-2} \iint_{M\times M} d(x,y)^2\, dx\, dy$, between points of M (where d is the Riemannian distance).

4. Proof of the Inequality for Energies

We have stated Theorem 1.1 only for O'Hara's energies, but the techniques developed here apply to a much broader class of energy functionals. The following theorem precisely formulates the conditions we need to apply Theorem 3.1 and prove that a knot energy is minimized by circles.

Theorem 4.1. *Let c be a closed, unit-speed curve of length 2π in $\mathbf{R}^n$, and $F(s,x)$ be a function from $\mathbf{R}^2$ to $\mathbf{R}$. If $F(s,\cdot)$ is increasing and concave on $(0,s^2)\cap(0,(2\pi-s)^2)$ for all $s\in(0,2\pi)$ and $\int F\left(s,\lambda^2(s)\right)\, ds < \infty$, then*

$$\frac{1}{4\pi^2} \iint F\left(s, |c(t+s)-c(t)|^2\right)\, dt\, ds \leq \frac{1}{2\pi} \int F\left(s,\lambda^2(s)\right)\, ds \tag{4.1}$$

with equality if and only if c is the unit circle.

Proof. Using our hypotheses, $f(x) := F(s,x)$ satisfies the conditions of Theorem 3.1 and so for $s\in(0,2\pi)$

$$\frac{1}{2\pi} \int F\left(s, |c(t+s)-c(t)|^2\right)\, dt \leq F\left(s,\lambda^2(s)\right) \tag{4.2}$$

with equality if and only if c is the unit circle. This can be integrated from $s=0$ to $s=2\pi$ to give inequality (4.1) with equality if and only if equality (4.2) holds for almost all $s \in [0, 2\pi]$. But if equality holds for any $s \in (0, 2\pi)$, then c is the unit circle. $\square$

We now complete the proof of the main theorem.

Proof of Theorem 1.1. Set $m(s) := \min\{s, 2\pi - s\}$ and

$$F(s, x) := -\left(\frac{1}{x^{j/2}} - \frac{1}{m(s)^j}\right)^p.$$

Then using (1.2), we have

$$-E_j^p[c] = \iint F\left(s, |c(t+s) - c(t)|^2\right)\, dt\, ds.$$

When $p \geq 1$, $s \neq 0$, and $x \in (0, m(s)^2)$,

$$\frac{\partial F}{\partial x} = \frac{jp}{2x^{(j+2)/2}}\left(\frac{1}{x^{j/2}} - \frac{1}{m(s)^j}\right)^{p-1} > 0,$$

and

$$\frac{\partial^2 F}{\partial x^2} = -\frac{j(j+2)p}{4x^{(j+4)/2}}\left(\frac{1}{x^{j/2}} - \frac{1}{m(s)^j}\right)^{p-1} - \frac{j^2 p(p-1)}{4x^{(j+2)}}\left(\frac{1}{x^{j/2}} - \frac{1}{m(s)^j}\right)^{p-2} < 0.$$

Since $x^{j/2}$ can be arbitrarily close to $m(s)^j$ if the curve is nearly straight, examining this equation shows that the condition $p \geq 1$ is required to enforce the concavity of $F(s, \cdot)$.

So for every $s \neq 0$, $F(s, \cdot)$ is increasing and concave on $(0, m(s)^2)$. Further, a direct calculation shows that $\int F\left(s, \lambda^2(s)\right)\, ds < \infty$ when $j < 2 + 1/p$. Thus F satisfies the hypotheses of Theorem 4.1. Changing the variable $s \mapsto 2s$ and noting that the resulting integrand is symmetric about $s = \pi/2$, we have

$$\begin{aligned} -E_j^p[c] &\leq 2\pi \int F\left(s, \lambda^2(s)\right)\, ds \\ &= -2^{2-jp}\pi \int_0^\pi \left(\left(\frac{1}{\sin s}\right)^j - \left(\frac{1}{\min\{s, \pi - s\}}\right)^j\right)^p ds \\ &= -2^{3-jp}\pi \int_0^{\pi/2} \left(\left(\frac{1}{\sin s}\right)^j - \left(\frac{1}{s}\right)^j\right)^p ds \end{aligned}$$

with equality if and only if c is the unit circle. $\square$

5. Convex functionals and numerical experiments

All of our work so far has depended on the hypotheses of Theorem 4.1: our energy integrands must be increasing, *concave* functions of squared chord length. To take a concrete example, if $0 < p < 2$, then $f(x) = x^{p/2}$ is increasing and concave; so Theorem 3.1 implies that among closed, unit speed curves of length 2π in $\mathbf{R}^n$,

$$A_p[c] := \left(\frac{1}{4\pi^2}\iint |c(t) - c(s)|^p\, dt\, ds\right)^{\frac{1}{p}} \leq \left(\frac{1}{2\pi}\int (\lambda(s))^p\, ds\right)^{\frac{1}{p}},$$

where equality holds if and only if c is the unit circle. It is natural to ask:

Question 5.1. *Which closed, unit speed curves of length 2π maximize A_p for $p > 2$?*

We begin by sketching a proof that such a maximizing curve exists for $p > 0$.

Proposition 5.2. *Let $A_p[c]$ be defined as above. For $p > 0$, there exists a closed, unit-speed curve of length 2π maximizing $A_p[c]$. Further, every maximizer of $A_p[c]$ is convex and planar.*

Proof. Sallee's stretching theorem [16] (see also [5]) says that for any closed unit-speed space curve c of length 2π, there exists a corresponding closed, convex, unit-speed plane curve c^* of length 2π such that for every s, t in $[0, 2\pi]$,

$$|c(t) - c(s)| \leq |c^*(t) - c^*(t)|, \tag{5.1}$$

with equality for all s and t iff c is convex and planar. Since the integrand defining $A_p[c]$ is an increasing function of chord length for $p > 0$, this implies that every maximizer of $A_p[c]$ must be convex and planar.

Let $\mathcal{U}$ denote the space of closed, convex, planar, unit-speed curves of length 2π which pass through the origin, with the C^0 norm. It now suffices to show that a maximizer of $A_p[c]$ exists in $\mathcal{U}$.

Blaschke's selection principle [18, p. 50] implies that this space of parameterized curves is compact in the C^0 norm. It easy to see that $A_p[c]$ is C^0-continuous for c in $\mathcal{U}$ (in fact, it is jointly continuous in p and c on the product $(0, \infty) \times \mathcal{U}$), completing the proof. □

We conjecture that these maximizers are unique (up to rigid motions), and depend continuously on p. It is easy to see the following:

Lemma 5.3. *As above, let $\mathcal{U}$ denote the space of closed, convex, planar, unit-speed curves of length 2π with the C^0 norm. Then*

$$\mathrm{Max} := \{(p, c_p) \mid c_p \textit{ is a maximizer of } A_p\} \subset (0, \infty) \times \mathcal{U}$$

is locally compact and projects onto $(0, \infty)$.

Proof. We know from the proof of Proposition 5.2 that A is a C^0-continuous functional on $(0, \infty) \times \mathcal{U}$. If we choose any (p_0, c_{p_0}), and choose a compact interval $I \subset \mathbf{R}$ containing p_0, then $\mathrm{Max}_I = \{(p, c_p) \in \mathrm{Max} \mid p \in I\}$ contains a neighborhood of (p_0, c_{p_0}). We now show Max_I is compact.

Take any sequence $(p_i, c_{p_i}) \in \mathrm{Max}_I$. Since I is compact, we may assume that the p_i converge to some p. Since $\mathcal{U}$ is C^0-compact (see the proof of Proposition 5.2), we may also assume that the c_{p_i} converge to some c. It remains to show that c is a maximizer for A_p.

If not, there exists some c_p with $A_p[c_p] > A_p[c]$. But then

$$\lim_{i\to\infty} A_{p_i}[c_p] = A_p[c_p] > A_p[c] = \lim_{i\to\infty} A_{p_i}[c_{p_i}],$$

since A_p is continuous in p. On the other hand, since the c_{p_i} are maximizers for the A_{p_i}, we have $A_{p_i}[c_{p_i}] \geq A_{p_i}[c_p]$ for each i, and so

$$\lim_{i\to\infty} A_{p_i}[c_p] \leq \lim_{i\to\infty} A_{p_i}[c_{p_i}].$$

□

Together with uniqueness, this would prove that the set Max was a single continuous family of curves depending on $p > 0$. As it stands, Lemma 5.3 tells us surprisingly little about the structure of Max. For example, the subset of $\mathbf{R}^2$ defined by

$$\left\{ \left(\sum_{i \geq N} \frac{a_i}{3^i}, \sum_{\{i \mid a_i = 1\}} \frac{a_i}{3^i} \right) \middle| \; a_i \in \{0,1,2\}, N \in \mathbf{Z} \right\}$$

is a locally compact set which projects onto the positive x-axis but is totally disconnected!

In any event, it is interesting to consider how the shape of the maximizers changes as we vary p. Since the limit of L^p norms as $p \to \infty$ is the supremum norm, we have

$$\lim_{p \to \infty} A_p[c] = \sup_{s,t} |c(t) - c(s)| \leq \pi$$

with equality if and only if c double covers a line segment of length π. So the c_p form a family of convex curves converging to the double-covered segment as $p \to \infty$, and to the circle as $p \to 2$. To illuminate this process, we numerically computed maximizers of A_p for values of p between 2 and 4 using Brakke's Evolver [1]. Figure 1 shows some of the c_p.

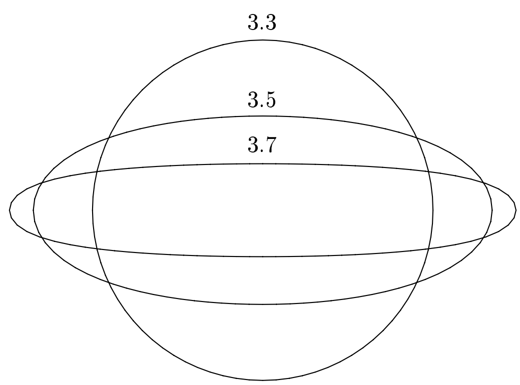

FIGURE 1. A collection of maximizers.

Since the double-covered segment has greater average chord length than the circle for $p > 3.5721$, there must be some critical value p^* of p between 2 and 3.5721 where "the symmetry breaks", and circles are no longer maximizers for A_p.

To find an approximate value for p^*, we computed the ratio $r(p)$ of the widest and narrowest projections of each of our computed minimizers for p between 2 and 4. Since all these curves are convex, a value close to unity indicates a curve close to a circle.

As Figure 2 shows, by this measure the computed minimizers are numerically very close to circles for $2 \leq p \leq 3.45$. To check this conclusion, we fit each minimizer to an ellipse using a least-squares procedure. Figure 3 shows the results of these computations.

To give a sense of the accuracy of our computations, this graph includes some computed minimizers for p between 1 and 2, for which we have proved that the unique minimizer is the circle. We also computed the eccentricities of each of the best-fit ellipses.

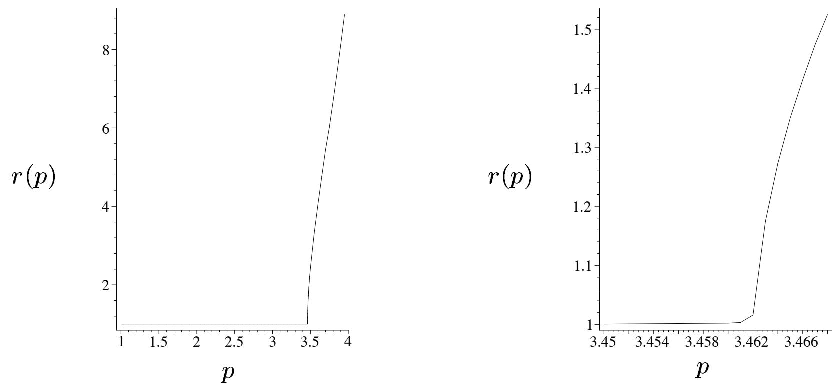

FIGURE 2. Two plots of $r(p)$ and p.

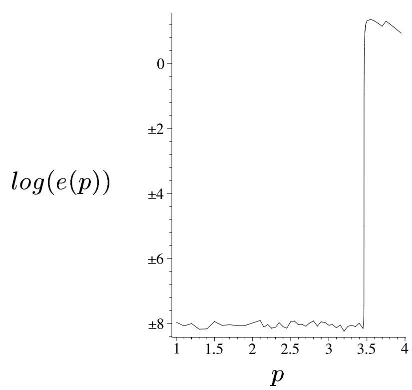

FIGURE 3. The logarithm of the error $e(p)$ in a least-squares fit of c_p to an ellipse, plotted against p.

A conservative reading of all this data supports the surprising conjecture that p^* is at least 3.3. Further, we note that for $p > p^*$, the maximizing curves do not seem to be ellipses, as one might have conjectured by looking at Theorem 2.2.

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