IMI

Preprint Series

Industrial Mathematics Institute

2001:05

Polynomial expansions of continuous functions on the sphere and on the disk

M. Skopina

Department of Mathematics
University of South Carolina

Polynomial expansions of continuous functions on the sphere and on the disk

M. Skopina[1]

Abstract

Wavelet-type systems providing a uniformly convergent expansion for any continuous function on the sphere are found. This construction is transferred to the disk due to some special connections between polynomial bases on the sphere and on the disk.

1. In [1], a polynomial basis for the space of continuous functions on the disk is constructed. This basis is not good for implementations. First, the growth of the degrees of polynomials is too rapid. Second, finding of the coefficient functionals for a concrete function is quite difficult. In the one-dimensional case, construction of polynomial bases was actively studied by many mathematicians for almost forty years. An optimal (regarding the growth of the degrees of polynomials) orthogonal basis for $C[a,b]$ was presented in [2]. This basis is a wavelet system with respect to some special "shift"-operators. Though this construction can be realized in any Hilbert space with a polynomial orthonormal basis, it is not clear if the Lebesgue functions of the wavelet Fourier sums are bounded in general. Wavelet-type systems for the sphere was proposed by W. Freeden [3]. In contrast to classical wavelet bases these systems are not orthogonal. Moreover, they are even not L_2 - bases. Nevertheless, expansions with respect to such systems are very alike usual wavelet series. In the present paper we show that some special cases of Freeden's "wavelets" provide a uniformly convergent polynomial expansion for any continuous function on the sphere. This construction can be transferred to the disk due to connections between weighted orthonormal polynomial bases on the disk and Laplace series.

2. The following notations will be used throughout the paper:
$x \cdot y = x_1y_1 + \ldots + x_dy_d$, $|x| = \sqrt{x \cdot x}$ for $x, y \in \mathbb{R}^d$, π_n^d is the space of polynomials in d variables of degree at most n. X_n denotes the n-th Legendre normalized polynomial, $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. Let $B = \{x \in \mathbb{R}^2 : |x| \le 1\}$,

[1] This research was supported by ARO-DEPSCoR Grant DAAG55-98-1-0002 and the grant 00-01-00467 of Russian Foundation for Basic Research.

$w(x) = \pi^{-1}(1-|x|^2)^{-1/2}$ for $x \in B$, we consider the weighted space $L_{2,w}$ of functions defined on B with the inner product $\langle f, g\rangle = \int\limits_B fgw$. For functions $F, G \in L_2(S^2)$ the inner product is $\langle F, G\rangle = \int\limits_{S^2} FG$.

Set $\mathcal{P}_n = \pi_n^2 \ominus \pi_{n-1}^2$ in $L_{2,w}$. It is clear that the spaces $\mathcal{P}_n$ are mutually orthogonal, $\dim \mathcal{P}_n = n+1$. For each n, $\mathcal{P}_n$ consists of the polynomials in two variables of degree exactly n. Let $\{p_{nk}\}_{k=0}^n$ be an orthonormal basis for $\mathcal{P}_n$. Two different explicit formulas for p_{nk} were presented in [1] and [4], but we will not need them. The entire collection $\{p_{nk}\}_{k,n}$ constitute an orthonormal polynomial complete system in $L_{2,w}$.

An orthonormal basis for $L_2(S^2)$ has a similar structure (see [4]): it consists of spherical harmonics Y_{nk}, $k = 0, \ldots, n$, $n = 0, 1, \ldots$; the spaces $\mathcal{H}_n = \operatorname{span}\{Y_{nk}, k = 0, \ldots, n\}$ are mutually orthogonal; the space $H_N = \oplus_{n=0}^N \mathcal{H}_n$ comprises the restriction to S^2 of π_n^3. One has the following addition formula [4]:

$$2\pi \sum_{k=0}^{n} Y_{nk}(x)Y_{nk}(y) = X_n(1)X_n(x\cdot y) \tag{1}$$

for all $x, y \in S^2$ and for all $n = 0, 1, \ldots$.

The following statement summarizes the quadrature formula given in [6] (see also [7], Theorem 2.1).

Theorem 1 *For any finite set $\{\eta_\ell\}_{\ell\in\Omega}$ of distinct points $\eta_\ell \in S^2$ and for any positive integer N satisfying*

$$N \max_{x\in S^2} \min_{\ell\in\Omega} |x-\eta_\ell| \le A,$$

where A is an absolute constant, there exist nonnegative weights a_ℓ, $\ell \in \Omega$, such that

$$\int\limits_{S^2} P = \sum_{\ell\in\Omega} a_\ell P(\eta_\ell)$$

for all $P \in \pi_N^3$.

Due to this theorem, to each nonnegative integer j we can assign a set $\{\eta_\ell^{(j)}\}_{\ell\in\Omega_j}$ of distinct points $\eta_\ell^{(j)} \in S^2$ and a set $\{a_\ell^{(j)}\}_{\ell\in\Omega_j}$ of nonnegative weights with the following properties: *card* $\Omega_j \sim 2^{2j}$; for any $\ell \in \Omega_j$ there exists $\ell' \in \Omega_j$, $\ell' \neq \ell$, such that $\eta_\ell^{(j)}$ and $\eta_{\ell'}^{(j)}$ are symmetric with respect to the plane $\{x \in \mathbb{R}^3 : \ x_3 = 0\}$; for any $P \in \pi_{2^j}^3$

$$\int_{S^2} P = \sum_{\ell \in \Omega_j} a_\ell^{(j)} P(\eta_\ell^{(j)}). \tag{2}$$

3. Let

$$h_j(n) = \frac{\begin{pmatrix} 2^j - n + 2 \\ 2^j - n \end{pmatrix}}{\begin{pmatrix} 2^j + 2 \\ 2^j \end{pmatrix}}$$

for $n = 0, \ldots, 2^j$, $h_j(n) = 0$ for $n = 2^j + 1, 2^j + 2, \ldots$. One can easily recognize the factors of the $(C, 2)$ means of order 2^j. It is well known (see, e.g., [8]) that the Fourier-Legendre series is $(C, 2)$-summable in $C[-1, 1]$. Moreover, an inequality due to Kogbetliantz [9, p. 71] states that

$$\sum_{n=0}^{N} \begin{pmatrix} N - n + 2 \\ N - n \end{pmatrix} X_n(1) X_n(t) \geq 0 \tag{3}$$

for all $t \in [-1, 1]$. Set $g_j(n) = h_j(n) + h_{j-1}(n)$, $\widetilde{g}_j(n) = h_j(n) - h_{j-1}(n)$ for $j = 1, 2, \ldots$, $n = 0, 1, \ldots$, $g_0(0) = h_0(0) + 1$, $\widetilde{g}_0(0) = h_0(0) - 1$, $g_0(n) = \widetilde{g}_0(n) = 0$ for $n = 1, 2, \ldots$. For each nonnegative integer j and for each $\ell \in \Omega_{j+1}$ define the functions

$$\begin{aligned}
\Psi_{j\ell}(x) &= \sum_{n \in \mathbb{Z}_+} g_j(n) X_n(1) X_n(\eta_\ell^{(j+1)} \cdot x), \\
\widetilde{\Psi}_{j\ell}(x) &= \sum_{n \in \mathbb{Z}_+} \widetilde{g}_j(n) X_n(1) X_n(\eta_\ell^{(j+1)} \cdot x), \\
\Phi_{(j+1)\ell}(x) &= \sum_{n \in \mathbb{Z}_+} h_j(n) X_n(1) X_n(\eta_\ell^{(j+1)} \cdot x).
\end{aligned}$$

Complete this collection by the function $\Phi_0 \equiv 1$.

For $F \in C(S^2)$, we will study the convergence of the series

$$\langle F, \Phi_0 \rangle \Phi_0 + \sum_{i=0}^{\infty} \sum_{\ell \in \Omega_{i+1}} a_\ell^{(j+1)} \langle F, \widetilde{\Psi}_{i\ell} \rangle \Psi_{i\ell}.$$

Set

$$\Lambda_{j,\omega}(F) = \langle F, \Phi_0\rangle\Phi_0 + \sum_{i=0}^{j-1}\sum_{\ell\in\Omega_{i+1}} a_\ell^{(i+1)}\langle F, \widetilde{\Psi}_{i\ell}\rangle\Psi_{i\ell} + \sum_{\ell\in\omega} a_\ell^{(j+1)}\langle F, \widetilde{\Psi}_{j\ell}\rangle\Psi_{j\ell},$$

where ω is a subset of Ω_{j+1}.

Lemma 2 *For any $F \in C(S^2)$,*

$$\langle F, \Phi_0\rangle\Phi_0 + \sum_{i=0}^{j-1}\sum_{\ell\in\Omega_{i+1}} a_\ell^{(i+1)}\langle F, \widetilde{\Psi}_{i\ell}\rangle\Psi_{i\ell} = \sum_{\ell\in\Omega_j} a_\ell^{(j)}\langle F, \Phi_{j\ell}\rangle\Phi_{j\ell}. \tag{4}$$

`Proof`. On the basis of (2),

$$\sum_{\ell\in\Omega_j} a_\ell^{(j)}\int_{S^2} F(t)\sum_{n\in\mathbb{Z}_+} h_{j-1}(n)X_n(1)X_n(\eta_\ell^{(j)}\cdot t)\,ds(t)\sum_{k\in\mathbb{Z}_+} h_{j-1}(k)X_k(1)X_k(\eta_\ell^{(j)}\cdot x) =$$
$$\int_{S^2} ds(t)F(t)\sum_{n\in\mathbb{Z}_+}\sum_{k\in\mathbb{Z}_+} h_{j-1}(n)h_{j-1}(k)\int_{S^2} ds(\eta)X_n(1)X_n(\eta\cdot t)X_k(1)X_k(\eta\cdot x).$$

¿From this, using (1) and taking into account the orthonormality of $\{Y_{nk}\}_{n,k}$, we have

$$\sum_{\ell\in\Omega_j} a_\ell^{(j)}\langle F, \Phi_{j\ell}\rangle\Phi_{j\ell}(x) = 2\pi\int_{S^2} F(t)\sum_{n\in\mathbb{Z}_+} h_{j-1}^2(n)X_n(1)X_n(t\cdot x)\,ds(t). \tag{5}$$

Similarly,

$$\sum_{\ell\in\Omega_{i+1}} a_\ell^{(i+1)}\langle F, \widetilde{\Psi}_{i\ell}\rangle\Psi_{i\ell}(x) = 2\pi\int_{S^2} F(t)\sum_{n\in\mathbb{Z}_+} \widetilde{g}_i(n)g_i(n)X_n(1)X_n(t\cdot x)\,ds(t). \tag{6}$$

Since $\widetilde{g}_i(n)g_i(n) = h_i^2(n) - h_{i-1}^2(n)$, it follows that

$$\sum_{\ell\in\Omega_i} a_\ell^{(i)}\langle F, \Phi_{i\ell}\rangle\Phi_{i\ell} = \sum_{\ell\in\Omega_{i-1}} a_\ell^{(i-1)}\langle F, \Phi_{(i-1)\ell}\rangle\Phi_{(i-1)\ell} + \sum_{\ell\in\Omega_i} a_\ell^{(i)}\langle F, \widetilde{\Psi}_{(i-1)\ell}\rangle\Psi_{(i-1)\ell}.$$

Summing these equalities over all $i = 1, \ldots, j$ we obtain (4). $\diamond$

Lemma 3 *Let χ be an integrable function on $[-1,1]$, then*

$$\int\limits_{S^2} \chi(x \cdot t)\, ds(t) = 2\pi \int\limits_{-1}^{1} \chi$$

for all $x \in S^2$.

`Proof.` Fix a point $x \in S^2$ and chose a Descartes coordinate system such that $x = (0,0,1)$. We have

$$\begin{aligned} \int\limits_{S^2} \chi(x \cdot t)\, ds(t) &= \int\limits_{S^2} \chi(t_3)\, ds(t) = \\ & \int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{\pi} \chi(\cos\theta) \sin\theta\, d\theta = 2\pi \int\limits_{-1}^{1} \chi(\tau)\, d\tau. \quad \diamond \end{aligned}$$

Theorem 4 *For any $F \in C(S^2)$,*

$$\lim_{j\to\infty} \|F - \Lambda_{j,\omega}(F)\|_\infty = 0, \tag{7}$$

where the convergence is uniform over all $\omega \subset \Omega_{j+1}$.

`Proof.` First we will prove that the operators $\Lambda_{j,\omega}$ taking $C(S^2)$ to $C(S^2)$ are uniformly bounded. By (4), (2) and (3),

$$|\Lambda_{j,\emptyset}(F,x)| = \left| \sum_{\ell\in\Omega_j} a_\ell^{(j)} \langle F, \Phi_{j\ell}\rangle \Phi_{j\ell} \right| =$$

$$\left| \int\limits_{S^2} ds(\eta) \sum_{k\in\mathbb{Z}_+} h_{j-1}(k) X_k(1) X_k(\eta\cdot x) \int\limits_{S^2} ds(t) F(t) \sum_{n\in\mathbb{Z}_+} h_{j-1}(n) X_n(1) X_n(\eta\cdot t) \right| \leq$$

$$\|F\|_\infty \int\limits_{S^2} ds(\eta) \sum_{k\in\mathbb{Z}_+} h_{j-1}(k) X_k(1) X_k(\eta\cdot x) \int\limits_{S^2} ds(t) \sum_{n\in\mathbb{Z}_+} h_{j-1}(n) X_n(1) X_n(\eta\cdot t).$$

Since, due to 4,

$$\int\limits_{S^2} \sum_{n\in\mathbb{Z}_+} h_{j-1}(n) X_n(1) X_n(y\cdot t)\, ds(t) = 2\pi \int\limits_{-1}^{1} \sum_{n\in\mathbb{Z}_+} h_{j-1}(n) X_n(1) X_n(\tau)\, d\tau = 2\pi,$$

we obtain

$$\|\Lambda_{j,\emptyset}\| \le 4\pi^2. \tag{8}$$

Similarly, taking into account that $a_\ell^{(j)} \ge 0$, we have

$$\left|\sum_{\ell\in\omega} a_\ell^{(j+1)} \langle F, \widetilde{\Psi}_{j\ell}\rangle \Psi_{j\ell}(x)\right| \le$$

$$\sum_{\ell\in\omega} a_\ell^{(j+1)} \int\limits_{S^2} |F(t)| \sum_{s=j-1}^{j} \sum_{n\in\mathbb{Z}_+} h_s(n) X_n(1) X_n(t\cdot \eta_\ell^{(j+1)})\, ds(t)\, \cdot$$

$$\sum_{r=j-1}^{j} \sum_{k\in\mathbb{Z}_+} h_r(k) X_k(1) X_k(x\cdot\eta_\ell^{(j+1)}) \le$$

$$\|F\|_\infty \sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)} \int\limits_{S^2} \sum_{s=j-1}^{j} \sum_{n\in\mathbb{Z}_+} h_s(n) X_n(1) X_n(t\cdot\eta_\ell^{(j+1)})\, ds(t)\, \cdot$$

$$\sum_{r=j-1}^{j} \sum_{k\in\mathbb{Z}_+} h_r(k) X_k(1) X_k(x\cdot\eta_\ell^{(j+1)}) =$$

$$\|F\|_\infty \int\limits_{S^2} ds(\eta) \int\limits_{S^2} ds(t) \sum_{s=j-1}^{j} \sum_{n\in\mathbb{Z}_+} h_s(n) X_n(1) X_n(t\cdot\eta)\, \cdot$$

$$\sum_{r=j-1}^{j} \sum_{k\in\mathbb{Z}_+} h_r(k) X_k(1) X_k(x\cdot\eta) = 16\pi^2 \|F\|_\infty.$$

This and (8) give $\|\Lambda_{j,\omega}\| \le 20\pi^2$. Now, by the Banach-Steinhaus theorem, it suffices to check that (7) holds on the set of spherical polynomials. Let $F = \sum\limits_{n=0}^{N} \sum\limits_{\nu=0}^{n} \alpha_{n\nu} Y_{n\nu}$, due to the orthonormality of $\{Y_{nk}\}_{n,k}$ and (1), it follows from (4) and (5), that

$$\Lambda_{j,\emptyset}(F) = \sum_{n=0}^{N} h_{j-1}^2(n) \sum_{\nu=0}^{n} \alpha_{n\nu} Y_{n\nu}.$$

Since

$$\lim_{j\to\infty} h_j(n) = 1, \tag{9}$$

whenever n is fixed, we obtain (7) for $\omega = \emptyset$. Again, due to the orthonormality of $\{Y_{nk}\}_{n,k}$ and (1),

$$\left|\sum_{\ell\in\omega} a_\ell^{(j+1)} \langle F, \widetilde{\Psi}_{j\ell}\rangle \Psi_{j\ell}(x)\right| \leq \left|\sum_{\ell\in\omega} a_\ell^{(j+1)} \sum_{n=0}^{N} \widetilde{g}_j(n) \sum_{\nu=0}^{n} \alpha_{n\nu} Y_{n\nu}(\eta_\ell^{(j+1)}) \Psi_{j\ell}(x)\right| \leq$$

$$\max_{0\leq n\leq N} |h_j(n) - h_{j-1}(n)| \sum_{\nu=0}^{n} |\alpha_{n\nu}| \|Y_{n\nu}\|_\infty \sum_{\ell\in\omega} \left|a_\ell^{(j+1)} \Psi_{j\ell}(x)\right| . \tag{10}$$

On the basis of (2) and Lemma 3, taking into account the positivity of $a_\ell^{(j)}$ and (3), we have

$$\sum_{\ell\in\omega} \left|a_\ell^{(j+1)} \Psi_{j\ell}(x)\right| \leq$$

$$\sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)} \left(\sum_{n\in\mathbb{Z}_+} h_j(n) X_n(1) X_n(t\cdot\eta_\ell^{(j+1)}) + \sum_{k\in\mathbb{Z}_+} h_{j-1}(k) X_k(1) X_k(t\cdot\eta_\ell^{(j+1)}) \right) =$$

$$\int_{S^2} ds(\eta) \left(\sum_{n\in\mathbb{Z}_+} h_j(n) X_n(1) X_n(t\cdot\eta) + \sum_{k\in\mathbb{Z}_+} h_{j-1}(k) X_k(1) X_k(t\cdot\eta) \right) = 4\pi.$$

Combining this with (9) and (10), we obtain

$$\lim_{j\to\infty} \max_{\omega\subset\Omega_{j+1}} \left\| \sum_{\ell\in\omega} a_\ell^{(j+1)} \langle F, \widetilde{\Psi}_{j\ell}\rangle \Psi_{j\ell} \right\|_\infty = 0,$$

what remained to be proved. $\diamond$

4. To any $f \in C(B)$ we assign an associated function F defined on S^2 by: $F(x_1, x_2, x_3) = f(x_1, x_2)$ for all $x \in S^2$.

Theorem 5 ([1]) *Let $\{p_{nk}\}_{k=0}^{n}$ be an orthonormal basis for $\mathcal{P}_n$ in $L_{2,w}$, $f \in C(B)$ and let F be the function associated with f, then*

$$2\pi \sum_{k=0}^{n} \langle f, p_{nk} \rangle p_{nk}(x_1, x_2) = X_n(1) \int_{S^2} F(t) X_n(t \cdot x)\, ds(t)$$

for all $x \in S^2$.

Denote by $\xi_\ell^{(j)}$ the orthogonal projection of $\eta_\ell^{(j)}$ onto the plane $\{x \in \mathbb{R}^3 : x_3 = 0\}$. For each nonnegative integer j and for each $\ell \in \Omega_{j+1}$ define the functions

$$\begin{aligned}
\psi_{j\ell}(x_1, x_2) &= \sum_{n \in \mathbb{Z}_+} g_j(n) \sum_{k=0}^{n} p_{nk}(x_1, x_2) p_{nk}(\xi_\ell^{(j+1)}), \\
\widetilde{\psi}_{j\ell}(x_1, x_2) &= \sum_{n \in \mathbb{Z}_+} \widetilde{g}_j(n) \sum_{k=0}^{n} p_{nk}(x_1, x_2) p_{nk}(\xi_\ell^{(j+1)}), \\
\varphi_{(j+1)\ell}(x_1, x_2) &= \sum_{n \in \mathbb{Z}_+} h_j(n) \sum_{k=0}^{n} p_{nk}(x_1, x_2) p_{nk}(\xi_\ell^{(j+1)}).
\end{aligned}$$

Complete this collection by the function $\varphi_0 \equiv 1$.

For $f \in C(B)$, we will study the convergence of the series

$$\langle f, \varphi_0 \rangle \varphi_0 + \sum_{i=0}^{\infty} \sum_{\ell \in \Omega_{i+1}} a_\ell^{i+1} \langle f, \widetilde{\psi}_{i\ell} \rangle \psi_{i\ell}.$$

Set

$$\lambda_{j,\omega}(f) = \langle f, \varphi_0 \rangle \varphi_0 + \sum_{i=0}^{j-1} \sum_{\ell \in \Omega_{i+1}} a_\ell^{i+1} \langle f, \widetilde{\psi}_{i\ell} \rangle \psi_{i\ell} + \sum_{\ell \in \omega} a_\ell^{(j+1)} \langle f, \widetilde{\psi}_{j\ell} \rangle \psi_{j\ell},$$

where ω is a subset of Ω_{j+1}.

Lemma 6 *If $f \in C(B)$, F is the function associated with f, then*

$$2\pi \sum_{\ell \in \Omega_{j+1}} a_\ell^{(j+1)} \langle f, \widetilde{\psi}_{j\ell} \rangle \psi_{j\ell}(x_1, x_2) = \sum_{\ell \in \Omega_{j+1}} a_\ell^{(j+1)} \langle F, \widetilde{\Psi}_{j\ell} \rangle \Psi_{j\ell}(x), \tag{11}$$

$$2\pi \sum_{\ell \in \Omega_j} a_\ell^{(j)} \langle f, \varphi_{j\ell} \rangle \varphi_{j\ell}(x_1, x_2) = \sum_{\ell \in \Omega_j} a_\ell^{(j)} \langle F, \Phi_{j\ell} \rangle \Phi_{j\ell}(x) \tag{12}$$

for all $x \in S^2$, $j = 0, 1, \ldots$.

`Proof.` Let P_{nk} be a polynomial in three variables defined by $P_{nk}(x_1,x_2,x_3) = p_{nk}(x_1,x_2)$. Using (2), we have

$$\sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)}\langle f,\widetilde{\psi}_{j\ell}\rangle\psi_{j\ell}(x_1,x_2) =$$

$$\sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)} \sum_{n\in\mathbb{Z}_+}\widetilde{g}_j(n)\sum_{k=0}^{n}\langle f,p_{nk}\rangle P_{nk}(\eta_\ell^{(j+1)})\cdot$$

$$\sum_{n'\in\mathbb{Z}_+}\widetilde{g}_j(n')\sum_{k'=0}^{n'} p_{n'k'}(x_1,x_2)P_{n'k'}(\eta_\ell^{(j+1)}) =$$

$$\int\limits_{S^2} ds(\eta)\sum_{n\in\mathbb{Z}_+}\widetilde{g}_j(n)\sum_{k=0}^{n}\langle f,p_{nk}\rangle P_{nk}(\eta)\sum_{n'\in\mathbb{Z}_+}\widetilde{g}_j(n')\sum_{k'=0}^{n'} p_{n'k'}(x_1,x_2)P_{n'k'}(\eta) =$$

$$2\int\limits_{B}\frac{d\xi}{\sqrt{1-|\xi|^2}}\sum_{n\in\mathbb{Z}_+}\widetilde{g}_j(n)\sum_{k=0}^{n}\langle f,p_{nk}\rangle p_{nk}(\xi)\sum_{n'\in\mathbb{Z}_+}\widetilde{g}_j(n')\sum_{k'=0}^{n'} p_{n'k'}(x_1,x_2)p_{n'k'}(\xi).$$

Due to the orthonormality of $\{p_{nk}\}_{n,k}$ in $L_{2,w}$, we reduce this to

$$2\pi\sum_{n\in\mathbb{Z}_+}\widetilde{g}_j(n)g_j(n)\sum_{k=0}^{n}\langle f,p_{nk}\rangle p_{nk}(x_1,x_2). \tag{13}$$

On the other hand, by (6),

$$\frac{1}{2\pi}\sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)}\langle F,\widetilde{\Psi}_{i\ell}\rangle\Psi_{i\ell}(x) = \int\limits_{S^2} ds(t)F(t)\sum_{n\in\mathbb{Z}_+}\widetilde{g}_j(n)g_j(n)X_n(1)X_n(t\cdot x)$$

The right hand side is equal to (13), on the basis of Theorem 5. So, (11) is proved, (12) can be proved similarly. $\diamond$

Corollary 7 *For any $f\in C(B)$,*

$$\langle f,\varphi_0\rangle\varphi_0 + \sum_{i=0}^{j-1}\sum_{\ell\in\Omega_{i+1}} a_\ell^{i+1}\langle f,\widetilde{\psi}_{i\ell}\rangle\psi_{i\ell} = \sum_{\ell\in\Omega_j} a_\ell^{(j)}\langle f,\varphi_{j\ell}\rangle\varphi_{j\ell}.$$

The proof follows immediately from Lemmas 2 and 6.

Theorem 8 *For any* $f \in C(B)$,

$$\lim_{j\to\infty} \|f - \lambda_{j,\omega}(f)\|_\infty = 0, \tag{14}$$

where the convergence is uniform over all $\omega \subset \Omega_{j+1}$.

`Proof.` First we will prove that the operators $\lambda_{j,\omega}$ taking $C(B)$ to $C(B)$ are uniformly bounded. Let F and P_{nk} be the functions associated respectively with f and p_{nk} . By Corollary 7,

$$\lambda_{j,\emptyset}(f) = \sum_{\ell\in\Omega_j} a_\ell^{(j)} \langle f, \varphi_{j\ell}\rangle \varphi_{j\ell}.$$

It follows from (12) and (8) that

$$|\lambda_{j,\emptyset}(f, x_1, x_2)| \le 2\pi \|F\|_\infty = 2\pi \|f\|_\infty. \tag{15}$$

In [4, Theorem 4.1], it is proved that

$$\sum_{n\in\mathbb{Z}_+} h_j(n) \sum_{k=0}^{n} p_{nk}(u) p_{nk}(v) \ge 0 \tag{16}$$

for all $u, v \in B$. This also follows immediately from Theorem 5 and (3). By this, taking into account the positivity of $a_\ell^{(j)}$ and the orthonormality of of $\{p_{nk}\}_{n,k}$ in $L_{2,w}$, we have

$$\left| \sum_{\ell\in\omega} a_\ell^{(j+1)} \langle f, \widetilde{\psi}_{j\ell}\rangle \psi_{j\ell}(x_1, x_2) \right| \le$$

$$\sum_{\ell\in\omega} a_\ell^{(j+1)} \int_B |f(t_1, t_2)| \sum_{s=j-1}^{j} \sum_{n\in\mathbb{Z}_+} h_s(n) \sum_{k=0}^{n} p_{nk}(\xi_\ell^{(j+1)}) p_{nk}(t_1, t_2)\, w(t_1, t_2) dt_1\, dt_2 \cdot$$

$$\sum_{r=j-1}^{j} \sum_{n'\in\mathbb{Z}_+} h_r(n') \sum_{k'=0}^{n'} p_{n'k'}(\xi_\ell^{(j+1)}) p_{n'k'}(x_1, x_2) \le$$

$$\|f\|_\infty \sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)} \int_B \sum_{s=j-1}^{j} \sum_{n\in\mathbb{Z}_+} h_s(n) \sum_{k=0}^{n} p_{nk}(\xi_\ell^{(j+1)}) p_{nk}(t_1, t_2)\, w(t_1, t_2) dt_1\, dt_2 \cdot$$

$$\sum_{r=j-1}^{j}\sum_{n'\in\mathbb{Z}_+} h_r(n')\sum_{k'=0}^{n'} p_{n'k'}(\xi_\ell^{(j+1)})p_{n'k'}(x_1,x_2) =$$

$$2\|f\|_\infty \sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)} \sum_{r=j-1}^{j}\sum_{n'\in\mathbb{Z}_+} h_r(n')\sum_{k'=0}^{n'} P_{n'k'}(\eta_\ell^{(j+1)})p_{n'k'}(x_1,x_2).$$

By (2), the last sum over ℓ can be reduced to

$$\int_{S^2} ds(\eta) \sum_{r=j-1}^{j}\sum_{n'\in\mathbb{Z}_+} h_r(n')\sum_{k'=0}^{n'} P_{n'k'}(\eta)p_{n'k'}(x_1,x_2) =$$

$$2\int_B \frac{d\xi}{\sqrt{1-|\xi|^2}} \sum_{r=j-1}^{j}\sum_{n'\in\mathbb{Z}_+} h_r(n')\sum_{k'=0}^{n'} p_{n'k'}(\xi)p_{n'k'}(x_1,x_2) = 4\pi.$$

Hence

$$\left|\sum_{\ell\in\omega} a_\ell^{(j+1)}\langle f,\widetilde{\psi}_{j\ell}\rangle\psi_{j\ell}(x_1,x_2)\right| \le 8\pi.$$

Together with (15), this gives $\|\lambda_{j\omega}\| \le 10\pi$.

Now, by the Banach-Steinhaus theorem, it suffices to check that (14) holds on the polynomials. Let $f=\sum\limits_{n=0}^{N}\sum\limits_{\nu=0}^{n}\alpha_{n\nu}p_{n\nu}$, due to the orthonormality of $\{p_{nk}\}_{n,k}$ and Corollary 7, by means of manipulations similar to the proof of Lemma 6, it is not difficult to show that

$$\lambda_{j,\emptyset}(f) = \sum_{n=0}^{N} h_{j-1}^2(n)\sum_{\nu=0}^{n}\alpha_{n\nu}p_{n\nu}.$$

Since

$$\lim_{j\to\infty} h_j(n) = 1, \tag{17}$$

whenever n is fixed, we obtain (14) for $\omega=\emptyset$. Again, due to the orthonormality of $\{p_{nk}\}_{n,k}$,

$$\left|\sum_{\ell\in\omega} a_\ell^{(j+1)}\langle f,\widetilde{\psi}_{j\ell}\rangle\psi_{j\ell}\right| = \left|\sum_{\ell\in\omega} a_\ell^{(j+1)}\sum_{n=0}^{N}\widetilde{g}_j(n)\sum_{\nu=0}^{n}\alpha_{n\nu}p_{n\nu}(\xi_\ell^{(j+1)})\psi_{j\ell}\right| \le$$

$$\max_{0\le n\le N}|h_j(n)-h_{j-1}(n)|\sum_{\nu=0}^{n}|\alpha_{n\nu}|\|p_{n\nu}\|_\infty \sum_{\ell\in\omega}\left|a_\ell^{(j+1)}\psi_{j\ell}\right|. \tag{18}$$

On the basis of (2), taking into account the positivity of $a_\ell^{(j)}$ and (16), we have

$$\sum_{\ell\in\omega}\left|a_\ell^{(j+1)}\psi_{j\ell}(x_1,x_2)\right| \le \sum_{\ell\in\omega} a_\ell^{(j+1)}\left|\sum_{m\in\mathbb{Z}_+} g_j(m)\sum_{k=0}^{m} p_{mk}(x_1,x_2)p_{mk}(\xi_\ell^{(j+1)}).\right| \le$$

$$\sum_{\ell\in\Omega_{j+1}} a_\ell^{(j+1)}\sum_{s=j-1}^{j}\sum_{m\in\mathbb{Z}_+} h_s(m)\sum_{k=0}^{m} p_{mk}(x_1,x_2)P_{mk}(\eta_\ell^{(j+1)}) =$$

$$\int\limits_{S^2} ds(\eta)\sum_{s=j-1}^{j}\sum_{m\in\mathbb{Z}_+} h_s(m)\sum_{k=0}^{m} p_{mk}(x_1,x_2)P_{mk}(\eta) =$$

$$2\int\limits_{B}\frac{d\xi}{\sqrt{1-|\xi|^2}}\sum_{s=j-1}^{j}\sum_{m\in\mathbb{Z}_+} h_s(m)\sum_{k=0}^{m} p_{mk}(x_1,x_2)p_{mk}(\xi) = 4\pi.$$

Combining this with (18) and (17), we obtain

$$\lim_{j\to\infty}\max_{\omega\subset\Omega_{j+1}}\left\|\sum_{\ell\in\omega} a_\ell^{(j+1)}\langle F,\widetilde{\Psi}_{j\ell}\rangle\Psi_{j\ell}\right\|_\infty = 0,$$

what remained to be proved. $\diamond$

References

[1] M. Skopina *Ridge expansions of continuous functions on the disk* Preprint. Department of Mathematics, University of South Carolina. 2000:24

[2] M. Skopina *On polynomial bases for the space C[-1,1]* (in Russian) Zap. nauchn. sem. POMI. 1999. . 262. P. 223-226.

[3] W. Freeden and M. Schreiner *Orthogonal and Non-orthogonal Multiresolution Analysis, Scale Discrete and Exact Fully Discrete Wavelet Transform on the Sphere* Constructive Approximation 14 (1998). P. 493-515

[4] Y. Xu *Summability of Fourier orthogonal series for Jacobi weight on a ball in* $\mathbb{R}^d$

[5] C. Müller *Spherical harmonics* Lecture Notes in Mathematics. Vol. 17. Springer Verlag. Berlin. 1966.

[6] H.N. Mhaskar, F.J. Narcowich and J.D. Ward *Quadrature formulas on spheres using scattered date* Math. Comp., to appear

[7] H.N. Mhaskar, F.J. Narcowich, J.D. Ward and J. Prestin *Polynomial frames on the sphere* Comp. Math., to appear

[8] R. Askey *Orthogonal polynomials and spherical functions* SIAM Philadelphia. 1975. MR 58:1288

[9] G. Szegö Orthogonal polynomials. AMS, New York. 1959.

Maria Skopina, Department of Applied Mathematics - Control Processes, Saint-Petersburg State University, Russia
e-mail: skopina@sk.usr.lgu.spb.su