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Graph Distance-Dependent Labeling Related to Code Assignment in Computer Networks

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Abstract: For nonnegative integers d_1, d_2, and $L(d_1, d_2)$-*labeling* of a graph G, is a function $f : V(G) \rightarrow \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geq d_i$ whenever the distance between u and v is i in G, for $i = 1, 2$. The $L(d_1, d_2)$-*number* of G, $\lambda_{d_1,d_2}(G)$ is the smallest k such that there exists an $L(d_1, d_2)$-labeling with the largest label k. These labelings have an application to a computer code assignment problem. The task is to assign integer "control codes" to a network of computer stations with distance restrictions, which allow $d_1 \leq d_2$. In this article, we will study the labelings with $(d_1, d_2) \in \{(0, 1), (1, 1), (1, 2)\}$. © 2004 Wiley Periodicals, Inc. Naval Research Logistics 51: 000–000, 2004.

Keywords: graph labeling; distance; code assignment

1. INTRODUCTION

An interesting graph labeling problem comes from the radio channel assignment problem, as well as code assignment in computer networks. One version of the radio channel assignment problem [13] is to assign integer "channels" to a network of transmitters with distance restrictions, such that several levels of interference between nearby transmitters are avoided and the "span" of the labels used is minimized. An $L(d_1, d_2)$-*labeling* of a graph G is a function $f : V(G) \rightarrow \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geq d_i$ whenever the distance between u and v is i apart, $i \in \{1, 2\}$. We denote $\lambda_{d_1,d_2}(G)$ the minimum span of any such labeling of G, which means to minimize the largest label used in the labeling.

Since Griggs and Yeh [12] introduced these graph labeling, a large amount of literature (cf. [2], [4], [5], [7]–[12], [14]–[22]) has been contributed on efficient integer graph labeling with distance restrictions in the cases $d_1 > d_2 \geq 1$ (mostly, in the case $L(2, 1)$). Now it is reasonable to consider what if $d_1 \leq d_2$.

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A variation of the problem is code assignment in computer networks, i.e., to assign integer "control codes" to a network of computer stations with distance restrictions, which allow $d_1 \leq d_2$. Here we study the cases $(d_1, d_2) \in \{(0, 1), (1, 1), (1, 2)\}$.

Bertossi and Bonuccelli [1] introduced a kind of code assignment to avoid hidden terminal interference as follows. Since some modern computer networks consist of computers including mobile computers or computers displaced in wild areas, they need to use broadcast communication media such as radio frequencies. The computer network which communicates by radio frequencies called Packet Radio Network. It consists of computer stations (computers and transceivers), in which the transceivers broadcast outgoing message packets and listen for incoming message packets. Unconstrained transmission in broadcast media may lead to collision or interference, i.e., there is the time overlap of two or more incoming message packets received at the destination station. That results in damaged useless packets at the destination. Collided message packets must be retransmitted. That increases the time delay of the transmission, and, hence, lowers the system throughput. Several protocols have been devised to reduce or eliminate the collisions. They form the medium access control sublayer. For example, under Code Division Multiple Access protocol, the collision-free property is guaranteed by the use of proper assignment of orthogonal control codes to stations and spread spectrum communication techniques (e.g., hopping over different time slots or frequency bands).

We represent the network by a graph, such that all stations are vertices and two vertices are adjacent if the corresponding stations can hear each other. Hence, two stations are at distance two, if they are outside the hearing range of each other but can be received by the same destination station. There are two types of collisions or interference: direct collision, due to the transmission of adjacent stations, and hidden terminal collision, when stations at distance two transmit to the same receiving station at the same time.

To avoid hidden terminal interference, we assign a "control code" to each station in the software as follows. For one station, to avoid hidden terminal interference from its adjacent stations (which cannot hear each other) sending packets to it, we require distinct codes for its adjacent station, i.e., $d_2 = 1$. Here we suppose that there is little direct interference in the system, i.e., direct interference is so weak that we can ignore it. Apparently in the model of [1] there are some special hardware designs, which can avoid direct interference in the system. Hence, we allow the same code for two adjacent stations (which can hear each other), meaning $d_1 = 0$. Therefore, we have the $L(0, 1)$ case.

It is important to note that the $L(0, 1)$ problem is just a special case of ordinary graph coloring: Each feasible $L(0, 1)$-labeling of a graph $G = (V, E)$ yields a feasible coloring of the graph $G' = (V, E')$, where E' contains edges $\{u, v\}$ whenever u and v are distance two apart in G. Conversely, a coloring of G' becomes a feasible labeling of G by calling the colors 0, $1, \ldots, \chi(G') - 1$.

Although not discussed by Bertossi and Bonuccelli [1], it seems reasonable to consider this variation of their problem as follows. If we require distinct codes for any two adjacent stations, i.e., $d_1 = 1$, then to avoid direct interference, as well as to avoid hidden terminal interference as above, we will require larger code differences between any two stations at distance two (which cannot hear each other, but can both be received by the same stations), i.e., $d_2 \geq d_1$. Hence, we have the $L(1, 1)$ and $L(1, 2)$ cases.

The $L(1, 1)$-labeling has been studied by Yeh [21] and Liu and Yeh [16]. Similarly to $L(0, 1)$-labeling, $L(1, 1)$-labeling corresponds to coloring the graph G^2.

We will present results about $\lambda_{d_1,d_2}(G)$ for $(d_1, d_2) \in \{(0, 1), (1, 1), (1, 2)\}$ for some particular graphs, including the path P_n on n vertices, the cycle C_n on n vertices. We also consider two infinite graphs that model large regular transmitter networks: the triangular lattice and the square lattice, which will be defined in Section 3. Finally, we provide an upper bound on $\lambda_{1,2}(G) \leq 2\Delta^2 - \Delta$ for any graph G with the maximum degree Δ in Section 4.

2. BASIC RESULTS

By the definition, we obtain the following proposition.

PROPOSITION 1: Let G be any graph.

1. $\lambda_{d_1,d_2}(G) \leq \lambda_{p,q}(G)$, where $d_1 \leq p$ and $d_2 \leq q$.
2. $\lambda_{dd_1,dd_2}(G) = d\lambda_{d_1,d_2}(G)$, for positive integers d, d_1, d_2.
3. $\lambda_{0,1}(G) \leq \lambda_{1,1}(G) \leq \lambda_{1,2}(G) \leq \lambda_{2,2}(G) = 2\lambda_{1,1}(G)$.

Next we consider paths, cycles, and wheels. The results are easy to derive. Hence, we state them without proofs.

THEOREM 2: Let P_n be a path with $n \geq 2$ vertices. Then

1.

$$\lambda_{0,1}(P_n) = \begin{cases} 0 & \text{if } n = 2, \\ 1 & \text{if } n \geq 3 \end{cases} \qquad [1].$$

2.

$$\lambda_{1,1}(P_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n \geq 3 \end{cases} \qquad [16].$$

3.

$$\lambda_{1,2}(P_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n = 3, \\ 3 & \text{if } n \geq 4. \end{cases}$$

Notice that, for H being a subgraph of G, we have $\lambda_{d_1,d_2}(G) \geq \lambda_{d_1,d_2}(H)$ if H is an induced subgraph or $d_1 \geq d_2$, which may not hold for some other cases. For example, $\lambda_{1,2}(K_{1,3}) = 4 > 3 = \lambda_{1,2}(K_4)$ for $K_{1,3}$ being a subgraph of K_4.

When we join the first and the last vertex of a path, we have a *cycle.*

THEOREM 3: Let C_n be a cycle of order $n \geq 3$. Then

1.

$$\lambda_{0,1}(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 1 & \text{if } n \equiv 0 \pmod 4, \\ 2 & \text{otherwise} \end{cases} \qquad [1].$$

2.

$$\lambda_{1,1}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 3, \\ 4 & \text{if } n = 5, \\ 3 & \text{otherwise} \end{cases} \qquad [16].$$

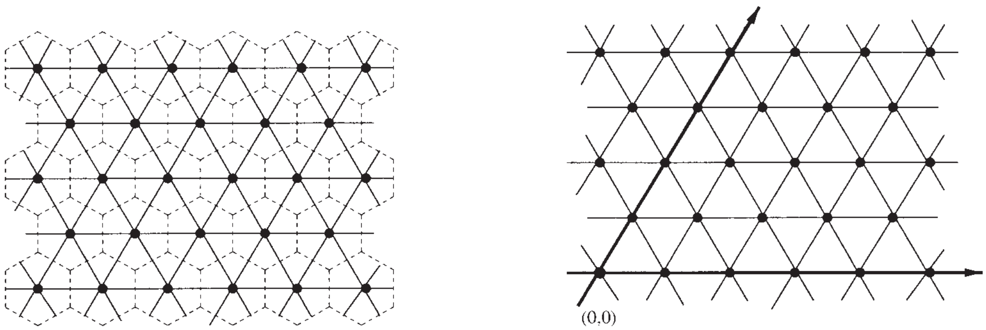

Figure 1. The hexagonal cells (left) and the triangular lattice Γ_Δ (right).

3.

$$\lambda_{1,2}(C_n) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n \equiv 0 \quad \pmod 4, \\ 4 & \text{otherwise.} \end{cases}$$

If we take a cycle C_n joined by a vertex, then we get a graph, $W_n = C_n \vee K_1$, called a *wheel with n spokes.* The next theorem considers the graph W_n, for $n \geq 4$.

THEOREM 4: (1) $\lambda_{0,1}(W_n) = \lfloor (n - 1)/2 \rfloor$, (2) $\lambda_{1,1}(W_n) = n$, and (3) $\lambda_{1,2}(W_n) = n$.

3. TRIANGULAR LATTICE AND SQUARE LATTICE

Define vectors $\epsilon_1 = (1, 0)$, $\epsilon_2 = (0, 1)$, and $\epsilon_3 = (1/2, \sqrt{3}/2)$ in the Euclidean plane. We denote by Γ_Δ the *triangular lattice* such that $V(\Gamma_\Delta) = \{i\epsilon_1 + j\epsilon_3 : i, j \in Z\}$ and $E(\Gamma_\Delta) = \{uv : u, v \in V(\Gamma_\Delta), d_E(u, v) = 1\}$, where $d_E(u, v)$ is the Euclidean distance between u and v. Similarly, we denote by $\Gamma_\square$ the *square lattice,* such that $V(\Gamma_\square) = \{i\epsilon_1 + j\epsilon_2 : i, j \in Z\}$ and $E(\Gamma_\square) = \{uv : u, v \in V(\Gamma_\square), d_E(u, v) = 1\}$. In both graphs, vertices are denoted by (i, j) for short.

The triangular lattice is important in radio communication such as radio broadcasting and mobile cellular networks. In a radio mobile network, large service areas are often covered by a network of congruent polygonal cells, with each transmitter in the center of a cell that it covers. A honeycomb of hexagonal cells (Fig. 1) provides the most economic covering, that is, the transmitters are placed in the triangular lattice [6]. If the areas are covered by a network of square cells, we may get all transmitters in the square lattice.

Notice that both graphs are infinite. See Figures 1 and Figure 3 for Γ_Δ and $\Gamma_\square$, respectively. The square lattice can also be regarded as the Cartesian product of two infinity paths.

THEOREM 5: (1) $\lambda_{0,1}(\Gamma_\square) = 3$, (2) $\lambda_{1,1}(\Gamma_\square) = 4$, and (3) $\lambda_{1,2}(\Gamma_\square) = 7$.

PROOF: (1) We will calculate the labeling number $\lambda_{0,1}(\Gamma_\square)$. Since $\Gamma_\square$ contains a $K_{1,4}$ as an induced subgraph, it is easy to see that $\lambda_{0,1}(K_{1,4}) = 3$, $\lambda_{0,1}(\Gamma_\square) \geq 3$. On the other hand, define a labeling f on $V(\Gamma_\square)$ by $f(i, j) \equiv 2\lfloor i/2 \rfloor + j \pmod 4$, where $f(i, j)$ stands for $f((i, j))$. The maximum label we use is 3.

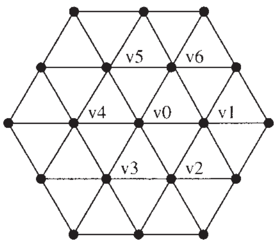

Figure 2. The subgraph $B(v_0)$.

If vertices (i_1, j_1) and (i_2, j_2) are at distance two apart, then $|i_1 - i_2| = 1 = |j_1 - j_2|$, $|i_1 - i_2| = 2$ and $j_1 = j_2$, or $i_1 = i_2$ and $|j_1 - j_2| = 2$. In each case $|f(i_1, j_1) - f(i_2, j_2)|$ is either 1 or 3. Therefore, f is an $L(0, 1)$-labeling with the maximum label 3. Thus, $\lambda_{0,1}(\Gamma_\square) \leq 3$. The result then follows.

(2) We will consider the labeling number $\lambda_{1,1}(\Gamma_\square)$. In this case, we see that $4 = \lambda_{1,1}(K_{1,4}) \leq \lambda_{1,1}(\Gamma_\square)$. The upper bound 4 on $\lambda_{1,1}(\Gamma_\square)$ can be attained by defining a labeling f as $f(i, j) \equiv 3i + j \pmod 5$. It is easy to verify that f is an $L(1, 1)$-labeling with the largest label 4.

(3) We will consider the labeling number $\lambda_{1,2}(\Gamma_\square)$. Define a labeling f by $f(i, j) \equiv 5i + j \pmod 8$. Then we can easily show that f is an $L(1, 2)$-labeling with the largest label 7 on $\Gamma_\square$. The upper bound on $\lambda_{1,2}(\Gamma_\square)$ is then attained. On the other hand, it is trivial that we can assume some vertex, say v, is labeled by 0. Thus, by the definition of $L(1, 2)$-labeling, one of the neighbor of v must have a label at least 7. So $\lambda_{1,2}(\Gamma_\square) \geq 7$, we then obtain the equality. □

THEOREM 6: (1) $\lambda_{0,1}(\Gamma_\Delta) = 3$, (2) $\lambda_{1,1}(\Gamma_\Delta) = 6$, and (3) $\lambda_{1,2}(\Gamma_\Delta) = 9$.

PROOF: For any vertex v, let $N_1(v)$ be the set of its neighbors and $N_2(v)$ be the set of these vertices at distance two from v. Then, for $v = (i, j)$ in Γ_Δ, $N_1((i, j)) = \{(i \pm 1, j), (i, j \pm 1), (i \pm 1, j \mp 1))\}$ and $N_2((i, j)) = \{(i \pm 2, j), (i, j \pm 2), (i \pm 1, j \mp 2), (i \pm 2, j \mp 2), (i \pm 1, j \pm 1), (i \pm 2, j \mp 1)\}$. (Fig. 3 shows a subgraph $B(v_0)$ induced by $\{v_0\} \cup N_1(v_0) \cup N_2(v_0)$.)

Let v_0 be any vertex in Γ_Δ. Then v_0 and $N_1(v_0)$ induce a subgraph, which is a W_6 (see Fig. 2). Thus, the λ-number of Γ_Δ is greater than or equal to the λ-number of W_6 in each case we discuss below.

(1) We evaluate the value of $\lambda_{0,1}(\Gamma_\Delta)$. Let

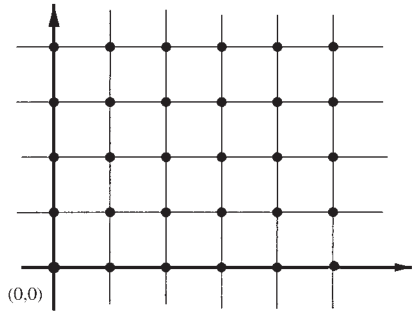

Figure 3. The square lattice $\Gamma_\square$.

$$A = \begin{pmatrix} 3 & 3 & 0 & 1 & 1 & 2 \\ 2 & 3 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 \\ 0 & 0 & 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 3 & 0 & 1 \\ 1 & 2 & 2 & 3 & 0 & 0 \end{pmatrix}.$$

Label (i, j) by $A(6 - j, i + 1)$, for $0 \le i, j \le 5$. For $i, j \ge 6$, label (i, j) by the label of $(i(\mathrm{mod}\ 6), j(\mathrm{mod}\ 6))$. This will be an $L(0, 1)$-labeling. Thus, $\lambda_{0,1}(\Gamma_\Delta) \le 3$.

Consider any $L(0, 1)$-labeling on Γ_Δ. Let v_0 be a vertex labeled by 0. It is easy to check that it is impossible to get an $L(0, 1)$-labeling on the subgraph $B(v_0)$ (Fig. 2) using 0, 1, 2. Therefore, $\lambda_{0,1}(\Gamma_\Delta) = 3$.

(2) We now evaluate the value of $\lambda_{1,1}(\Gamma_\Delta)$. Since Γ_Δ contains a W_6 as an induced subgraph, we have $6 = \lambda_{1,1}(W_6) \le \lambda_{1,1}(\Gamma_\Delta)$, by Theorem 4. In order to obtain an upper bound, we define a labeling f on $V(\Gamma_\Delta)$ by $f(i, j) \equiv i + 5j \pmod 7$. For a vertex (i, j), we get $|f(i, j) - f(i_1, j_1)| \not\equiv 0 \pmod 7$, where $(i_1, j_1) \in N_1((i, j) \cup N_2((i, j))$.

Therefore, we have f an $L(1, 1)$-labeling with the maximum label 6. Thus, $\lambda_{1,1}(\Gamma_\Delta) \le 6$. The result then follows.

(3) We evaluate the value of $\lambda_{1,2}(\Gamma_\Delta)$. Define g on $V(\Gamma_\Delta)$ by $g(i, j) \equiv i + 4j \pmod{10}$. Let (i, j) be any vertex. We get $|g(i, j) - g(i_1, j_1)| \not\equiv 0 \pmod{10}$ for $(i_1, j_1) \in N_1((i, j))$ and $|g(i, j) - g(i_1, j_1)| \not\equiv 0, 1, 9 \pmod{10}$ for $(i_1, j_1) \in N_2((i, j))$.

Thus, g is an $L(1, 2)$-labeling and, hence, $\lambda_{1,2}(\Gamma_\Delta) \le 9$.

Suppose we have an $L(1, 2)$-labeling f of Γ_Δ, with $f(v_0) = 0$ for some vertex v_0. Let $v_1, v_2, \ldots, v_6$ be its six neighbors as in Figure 2. Denote $f_i = f(v_i)$, for $i = 1, \ldots, 6$. Since f is an $L(1, 2)$-labeling, f_i's are all distinct. Further, $|f_i - f_j| \ge 2$ whenever v_i and v_j are not adjacent. If one of the f_i's is greater than 9, then we are done. Suppose $1 \le f_i \le 8$, for $i = 1, \ldots, 6$. We then list all possible choices of $f_1, f_2, \ldots, f_6$. This would not be difficult. In each case, we then consider possible labelings on neighbors of $v_1, \ldots, v_6$, i.e., these vertices at distance two away from v_0. We found that the label 9 is necessary in each labeling. Therefore, $\lambda_{1,2}(\Gamma_\Delta) \ge 9$. The result then follows. □

We would like to note that the result $\lambda_{1,1}(\Gamma_\Delta) = 6$ is also obtained in references [3] and [18], independently.

4. UPPER BOUNDS ON $\lambda_{1,2}$

In this section, we will find upper bounds on $\lambda_{1,2}(G)$ in terms of the maximum degree Δ of G. This is motivated by the chromatic number, which has an upper bound $\Delta + 1$ for all graphs with the maximum degree Δ.

A trivial upper bound on $\lambda_{1,2}$ is $2\Delta^2$ because $\lambda_{1,2} \le \lambda_{2,2} = 2\lambda_{1,1} \le 2\Delta^2$ by Proposition 1 and the previous result on $\lambda_{1,1}$ (cf. [16]).

Recall that for any fixed positive integer i, an *i-independent set* of a graph G is a subset S of $V(G)$ such that every two distinct vertices in S are at distance greater than i. Note that the 1-independent set is the usual independent set.

THEOREM 7: Let G be a graph with the maximum degree Δ. Then $\lambda_{1,2}(G) \le 2\Delta^2 - \Delta$.

PROOF: Let us consider the following labeling scheme. Initially, every vertex is unlabeled. Let $X_{-1} = \varnothing$. When Y_{i-1} is determined and not all vertices in G are labeled, let

$$Y_i = \{u \in V(G) : u \text{ is unlabeled and } d(u, v) \neq 2 \text{ for all } v \in X_{i-1}\}.$$

After, Y_i has been determined, $i \geq 0$, we pick a *maximal* 2-independent subset of Y_i to be X_i, i.e., X_i is a 2-independent subset of Y_i, but X_i is not a proper subset of any 2-independent subset of Y_i. Notice that in case $Y_i = \emptyset$, i.e., for any unlabeled vertex u, there exists some vertex $v \in X_{i-1}$ such that $d(u, v) = 2$, $X_i = \emptyset$. In any case, label vertices in X_i by i. Then increase i by 1 and continue the process (determining Y_i and X_i) until all vertices are labeled. Assume k is the largest label used, and choose a vertex x whose label is k. Let

$$I_1 = \{i : 0 \leq i \leq k-1 \quad \text{and} \quad d(x, y) = 1 \quad \text{for some } y \in X_i\},$$

$$I_2 = \{i : 0 \leq i \leq k-1 \quad \text{and} \quad d(x, y) = 2 \quad \text{for some } y \in X_i\},$$

$$I_3 = \{i : 0 \leq i \leq k-1 \quad \text{and} \quad d(x, y) \geq 3 \quad \text{for all } y \in X_i\}.$$

It is clear that $k \geq |I_1| + |I_2| + |I_3|$. Also, $|I_1| \leq \Delta$ and $|I_2| \leq \Delta^2 - \Delta$. For any $i \in I_3$, $x \notin Y_i$, otherwise $X_i \cup \{x\}$ is a 2-independent subset of Y_i, which contradicts the choice of X_i. That is, $d(x, y) = 2$ for some vertex y in X_{i-1}, i.e., $i - 1 \in I_2$. So, $|I_3| \leq |I_2|$. Then,

$$\lambda_{1,2}(G) \leq k \leq |I_1| + |I_2| + |I_3| \leq |I_1| + |I_2| + |I_2| \leq \Delta + \Delta^2 - \Delta + \Delta^2 - \Delta = 2\Delta^2 - \Delta.$$

□

It is unknown whether the upper bound above is the best possible. However, we do not have an example showing that there is a class of graphs having $L(1, 2)$-numbers close to the bound. We can easily get the upper bound $\lambda_{0,1}(G) \leq \Delta^2 - \Delta$, which is attained by this example. This example also shows that the first and the third inequalities in Proposition 1(3) are tight.

Given a projective plane $\Pi(n)$ of order $n > 1$, define a bipartite graph $G = (A, B, E)$ such that (1) each vertex in A corresponds to a point in $\Pi(n)$ and each vertex in B corresponds to a line in $\Pi(n)$, and (2) a vertex in A is adjacent to a vertex in B if and only if the corresponding point is in the corresponding line.

By the definition of $\Pi(n)$, we know that (1) $|A| = |B| = n^2 + n + 1$, (2) G is $(n + 1)$-regular, (3) for any two vertices in A (or in B), their distance is 2, and (4) for every $x \in A$, $y \in B$ such that they are not adjacent then the distance between x and y is 3. This graph is called an *incidence graph* of the projective plane $\Pi(n)$ (cf. [12]). Then we have the following theorem.

THEOREM 8: If $G = (A, B, E)$ is an incidence graph of a projective plane of order $n \geq 2$, with maximum degree $\Delta = n + 1$, then (1) $\lambda_{0,1}(G) = \Delta^2 - \Delta$, (2) $\lambda_{1,1}(G) = \Delta^2 - \Delta$ [16], and (3) $\lambda_{0,1}(G) = \Delta^2 - \Delta$. Hence, $\lambda_{0,1}(G) = \lambda_{1,1}(G)$ and $\lambda_{1,2}(G) = 2\lambda_{1,1}(G)$.

PROOF: (1) We will evaluate $\lambda_{0,1}(G)$. We have $\lambda_{0,1}(G) \leq \lambda_{1,1}(G) = \Delta^2 - \Delta$. Observe that any pair of vertices in A (or in B) are at distance two apart; hence, they will receive distinct integer labels by the condition. Since $|A| = n^2 + n + 1 = \Delta^2 - \Delta + 1$, we get $\lambda_{0,1}(G) \geq \Delta^2 - \Delta$.

(3) We will evaluate $\lambda_{1,2}(G)$. We have $\lambda_{0,1}(G) = \lambda_{1,1}(G) = \Delta^2 - \Delta$ and $2\lambda_{0,1}(G) = \lambda_{0,2}(G) \leq \lambda_{1,2}(G) \leq \lambda_{2,2}(G) = 2\lambda_{1,1}(G)$. Hence, $\lambda_{1,2}(G) = 2\lambda_{1,1}(G) = 2(\Delta^2 - \Delta)$. □

It is known that $\lambda_{2,1}(G) = \Delta^2 - \Delta$, for G as defined in Theorem 4.2 (cf. [12]). Thus, $\lambda_{2,1} < \lambda_{1,2}$ in this case. However, we also have $\lambda_{2,1}(K_n) = 2(n - 1) > \lambda_{1,2}(K_n) = n - 1$ for $n \geq 2$ and $\lambda_{2,1}(P_n) = 4 > \lambda_{1,2}(P_n) = 3$, for $n \geq 5$. Therefore, it would be interesting to investigate the relation between $\lambda_{2,1}$ and $\lambda_{1,2}$.

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