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Finite volume methods in general surfaces

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FINITE VOLUME METHODS ON GENERAL SURFACES

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Abstract. In this paper, we study the finite volume method for numerical solution of a set of model partial differential equations defined on a smooth surface. The discretization is defined via a surface mesh consisting of piecewise planar triangles and piecewise polygons. We prove the optimal error estimates of the approximate solution in both H^1 norm and L^2 norm that are of first order and second order respectively under mesh regularity assumptions.

1. Introduction

Numerical solutions of partial differential equations on arbitrary surfaces or two dimensional Riemannian manifolds are needed in diverse applications such as fluid dynamics, weather forecast and climate modelling, chemical coating, cell membrane modelling and image processing [3, 7, 13, 20, 24, 26, 27, 34, 36]. Many discretization techniques developed for these type of problems are based on finite element methods or finite difference methods, including direct discretizations on surface meshes [2, 19] or discretizations via level set techniques for implicitly defined surfaces [3, 32]. On the other hand, finite volume methods for the numerical solution of partial differential equations have also been gaining popularity in recent decades due to their discrete conservation properties, see for instance, [4, 6, 10, 11, 21, 23, 24, 26, 27, 28, 29, 30, 31, 33, 34, 35]. The application of finite volume methods to solve PDEs on general surfaces is the subject studied here.

In this paper, we analyze a finite volume method for the numerical solution of some linear second order elliptic equations defined on smooth surfaces. We choose to work directly with a surface discretization, in the form of a piecewise linear complex representation, rather than using an implicitly defined surface approach. The latter often avoids the difficulty of dealing with complex (and perhaps evolving) surfaces at the expense of solving equations in a higher space dimension. The former approach, on the other hand, relies its success more on a good geometric representation of the underlying surface. Naturally, another alternative is to use the surface parameterization to map the problem to a planar domain entirely and then make it treatable via conventional discretization methods in $\mathbb{R}^2$. A comprehensive discussion on the pros and cons of these different approaches is beyond the scope of this paper. The focus here is rather on some theoretical issues related to the discrete approximations, in the situation where a good piecewise (locally defined) representation of the surface is available. The main objective of this paper is to present some rigorous analysis of a finite volume method based on primal-dual surface meshes. In particular, since there has not been any rigorous error estimate

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in the literature for the finite volume methods on surfaces, we hereby prove some optimal error estimates of the approximate solutions. By carefully analyzing both the errors of the discrete mesh approximation of the surface and the finite volume discretization of the differential equation, we show that the errors of our finite volume approximation in the discrete H^1 norm and the L^2 norm are of first and second order respectively in the mesh parameter under some mesh regularity assumptions, similar to the results established for planar problems. In addition, we also discuss how to efficiently construct and optimize the meshes for general surfaces so that the mesh regularity assumptions may be satisfied.

The paper is organized as follows: we first introduce the model equation on general surfaces in section 2. Then in section 3, we present the finite volume discretization schemes. A short summary of some notations used in the paper is given in the beginning of section 3 as references. In section 4, the existence of the discrete solution and stability estimates are discussed. The rigorous H^1 and L^2 error estimates are given in sections 5 and 6 respectively. Finally, discussions on the surface mesh regularity and concluding remarks are given in section 7.

2. Model Problem and Weak solution

Let $\mathbf{S}$ be a compact $C^{k,\alpha}$-hypersurface [22, 19] in $\mathbb{R}^3$ ($k \in \mathbb{N} \cup \{0\}$ and $0 \le \alpha < 1$), represented globally by some oriented distance function (level set function) d defined on some open subset Ω of $\mathbb{R}^3$ such as $\mathbf{S} = \{\mathbf{x} \in \Omega \mid d(\mathbf{x}) = 0\}$ where $d \in C^{k,\alpha}$ and $\nabla d \neq 0$. Then the unit outward normal to $\mathbf{S}$ (with increasing d) at $\mathbf{x}$ is given by

$$\vec{\mathbf{n}}(\mathbf{x}) = \big(n_1(\mathbf{x}), n_2(\mathbf{x}), n_3(\mathbf{x})\big) = \frac{\nabla d(\mathbf{x})}{\|\nabla d(\mathbf{x})\|}$$

where $\|\cdot\|$ denotes the Euclidean norm and ∇ denotes the standard gradient operator in $\mathbb{R}^3$. Without loss of generality, we assume that $\|\nabla d\| \equiv 1$.

Let ∇_s be the tangential (surface) gradient operator [22] on $\mathbf{S}$ defined by

$$\nabla_s = (\nabla_{s,1}, \nabla_{s,2}, \nabla_{s,3})u = \nabla - \vec{\mathbf{n}}(\vec{\mathbf{n}} \cdot \nabla)\,,$$

and we use the standard notation for Sobolev spaces $L^p(\mathbf{S})$, $W^{m,p}(\mathbf{S})$, and $H^m(\mathbf{S}) = W^{m,2}(\mathbf{S})$ on $\mathbf{S}$. To make the space $H^m(\mathbf{S})$ well defined, we need $k + \alpha \ge 1$ and $k + \alpha \ge m$, see [22]. To avoid technical complexities, we assume that $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth (say, of class C^3) for the rest of the paper unless stated otherwise.

We are interested in the following model equation on $\mathbf{S}$:

$$-\nabla_s \cdot \big(a(\mathbf{x})\nabla_s u(\mathbf{x})\big) + b(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \mathbf{S}\,. \tag{2.1}$$

where the coefficients satisfy the following assumption:

Assumption 1. $a \in W^{1,\infty}(\mathbf{S})$, $b \in L^\infty(\mathbf{S}^2)$, $f \in L^2(\mathbf{S})$, $a(\mathbf{x}) \ge \alpha_1 > 0$, and $b(\mathbf{x}) \ge \alpha_2$ where $\alpha_2 \ge 0$ if $\partial\mathbf{S} \neq \emptyset$ and $\alpha_2 > 0$ if $\partial\mathbf{S} = \emptyset$.

We note that our discussion here can be extended to the case with the coefficient $a = a(\mathbf{x})$ being a symmetric positive definite tensor. Note also that there are diverse application for the above elliptic problem on general surfaces including texture synthesis and the images inpainting on surfaces [7].

For any $u, v \in H^1(\mathbf{S})$, define the bilinear functional $\mathcal{A}$ such that

$$\mathcal{A}(u,v) = \int_{\mathbf{S}} a(\mathbf{x})\big(\nabla_s u(\mathbf{x}) \cdot \nabla_s v(\mathbf{x})\big)\, ds + \int_{\mathbf{S}} b(\mathbf{x})u(\mathbf{x})v(\mathbf{x})\, ds, \tag{2.2}$$

then we have (for some constants $c > 0$ and $\alpha_0 > 0$)

$$\mathcal{A}(u,v) \leq c\|u\|_{H^1(\mathbf{S})}\|v\|_{H^1(\mathbf{S})}\ , \tag{2.3}$$

$$\mathcal{A}(u,u) \geq \alpha_0\|u\|^2_{H^1(\mathbf{S})}\ . \tag{2.4}$$

We say that $u \in H^1(\mathbf{S})$ is a weak solution of the equation (2.1) if and only if

$$\mathcal{A}(u,v) = (f,v), \quad \forall\, v \in H^1(\mathbf{S}) \tag{2.5}$$

where

$$(f,v) = \int_{\mathbf{S}} f(\mathbf{x})v(\mathbf{x})\, ds.$$

Since $\mathbf{S}$ is compact, we have the following classical results.

Theorem 1. Assume that Assumption 1 is satisfied. Then,
a) $\partial\mathbf{S} \neq \emptyset$. For every $f \in L^2(\mathbf{S})$, there exists a unique weak solution $u \in H^1_0(\mathbf{S})$ of (2.1), and consequently, u satisfies the estimate: for some constant $C > 0$,

$$\|u\|_{H^2(\mathbf{S})} \leq C\|f\|_{L^2(\mathbf{S})}\ . \tag{2.6}$$

b) $\partial\mathbf{S} = \emptyset$. For any $f \in L^2(\mathbf{S})$, there exists a unique weak solution $u \in H^1(\mathbf{S})$ of (2.1), and consequently u also satisfies the estimate (2.6).

3. Finite Volume Discretization

In this section, a finite volume discretization is presented for the equation (2.1). The discrete solution is determined by the equation (3.4) given later, but first, to make it easier for the readers to follow the discussion, let us briefly summarize some of the notations to be used later. For example, $\mathcal{T} = \{T_i\}_1^n$ and $\mathcal{T}^h = \{T_i^h\}_1^n$ are used to denote the curved and planar triangulations of the surface $\mathbf{S}$ and its piecewise polygonal approximation $\mathbf{S}^h$, these triangulations are related to each other by the lift map $\mathcal{L}$ from $\mathbf{S}^h$ to $\mathbf{S}$ as defined in (5.1); $\mathcal{K}$ and $\mathcal{K}^h$ are corresponding dual tessellations of $\mathbf{S}$ and $\mathbf{S}^h$; $\mathcal{U}$ and $\mathcal{V}$ denote piecewise linear and piecewise constant function spaces defined on the triangulation $\mathcal{K}^h$ of $\mathbf{S}^h$; Π_u and Π_v are interpolation operators into $\mathcal{U}$ and $\mathcal{V}$, while π_u and π_v, defined by (5.3) are the counterparts onto the pair of spaces induced by $\mathcal{U}$ and $\mathcal{V}$ on $\mathbf{S}$ through the lift $\mathcal{L}$; $\mathbf{P}_h$ and $\mathbf{P}$ are projection operators defined by (5.2); $\mathcal{A}$, $\mathcal{A}^h_G$, $\mathcal{A}^h_*$ and $\mathcal{A}_G$ are bilinear forms defined by (2.2), (3.3), (3.6) and (5.5) respectively (the subscript G refers to the use of the Green's formula in the definition).

We now present detailed discussions. For the smooth surface $\mathbf{S}$, we may assume that there is a strip (*band*)

$$\mathbf{U} = \{\mathbf{x} \in \Omega \ \mid\ dist(\mathbf{x},\mathbf{S}) < \delta\}, \quad \text{for some } \ \delta > 0$$

around $\mathbf{S}$ such that there is a unique decomposition for any $\mathbf{x} \in \mathbf{U}$

$$\mathbf{x} = \mathbf{p}(\mathbf{x}) + d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{x})$$

where $\mathbf{p}(\mathbf{x}) \in \mathbf{S}$, $d(\mathbf{x})$ is the signed distance to $\mathbf{S}$, and $\vec{\mathbf{n}}(\mathbf{x})$ denotes the unit outward normal of $\mathbf{S}$ at $\mathbf{p}(\mathbf{x})$. The parameter δ can be determined by the surface curvatures if $\mathbf{S}$ is sufficiently smooth. Then, a function u defined on $\mathbf{S}$ can be extended uniquely in the strip by

$$U(\mathbf{x}) = u(\mathbf{p}(\mathbf{x})) = u(\mathbf{x} - d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathbf{U}.$$

Let $\mathbf{S}$ be approximated by a continuous piecewise linear complex $\mathbf{S}^h \subset \mathbf{U}$ which consists of a regular triangulation $\mathcal{T}^h = \{T_i^h\}_{i=1}^m$ with vertices $\{\mathbf{x}_i\}_{i=1}^n$ on $\mathbf{S}$ (i.e.,

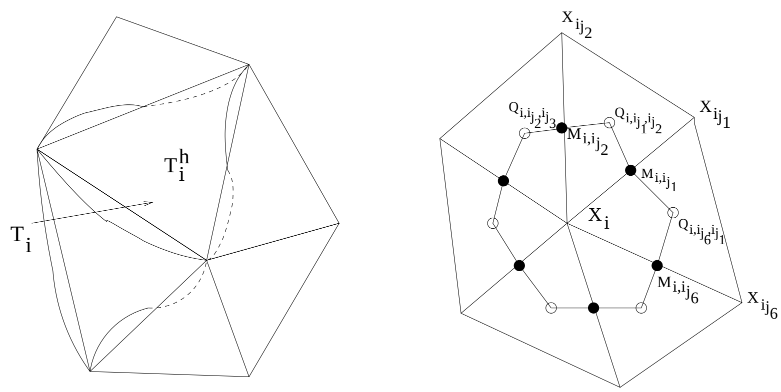

FIGURE 1. Approximate mesh surface and the control volume.

$\{\mathbf{x}_i\}_{i=1}^n \in \mathbf{S} \cap \mathbf{S}^h$), see Fig. 1 (left). Clearly, $\mathbf{S}^h$ is globally of class $C^{0,1}$. Let $m(\cdot)$ denote the area for planar regions or the length for arcs and segments.

We assume that $\mathcal{T}^h$ satisfies the following mesh regularity condition:

$$c_1 h^2 \le m(T_i^h) \le c_2 h^2 \tag{3.1}$$

where h is the mesh parameter (size) for $\mathcal{T}^h$, c_1 and c_2 are positive constants independent of h. Comments on meshes satisfying such regularity conditions are to be given later.

By the uniqueness of the vector decomposition discussed above, we define $T_i = \{\mathbf{p}(\mathbf{x}) \in \mathbf{S} \mid \mathbf{x} \in T_i^h\}$ and let $\mathcal{T} = \{T_i\}_{i=1}^m$, then $\mathbf{S} = \cup_{i=1}^n T_i$. Note that this implies in particular that $\mathbf{p}(\partial \mathbf{S}^h) = \partial \mathbf{S}$.

Let the tangential gradient operator ∇_{s_h} on $\mathbf{S}^h$ be given by:

$$\nabla_{s_h} = (\nabla_{s_h,1}, \nabla_{s_h,2}, \nabla_{s_h,3}) = \nabla - \vec{\mathbf{n}}_h(\vec{\mathbf{n}}_h \cdot \nabla u)$$

where $\vec{\mathbf{n}}_h(\mathbf{x}) = \big(n_{h1}(\mathbf{x}), n_{h2}(\mathbf{x}), n_{h3}(\mathbf{x})\big)$ is the unit outward normal to $\mathbf{S}^h$. Since $\vec{\mathbf{n}}_h$ is constant on each triangle T_i^h, ∇_{s_h} only needs to be locally defined as a two dimensional gradient operator on the plane formed by T_i^h, and the Sobolev space $W^{m,p}(\mathbf{S}^h)$ is well-defined for $m \le 1$.

We take the similar strategy adopted in [19] to numerically solve the equation on $\mathbf{S}^h$ instead of $\mathbf{S}$, but a finite volume method [6, 29] is used instead of their finite element methods there. For simplicity, we only consider the case of $\partial \mathbf{S} \neq \emptyset$ in this paper.

We now discuss the discretization scheme. First, we project the coefficients and the data a, b and f in (2.1) from $\mathbf{S}$ onto $\mathbf{S}^h$ such that for any $\mathbf{x} \in \mathbf{S}^h$, $A(\mathbf{x}) = a(\mathbf{p}(\mathbf{x}))$, $B(\mathbf{x}) = b(\mathbf{p}(\mathbf{x}))$, and $F(\mathbf{x}) = f(\mathbf{p}(\mathbf{x}))$.

Denote by $\mathcal{U}$ the space of continuous piecewise linear polynomials on S^h with respect to $\mathcal{T}^h$, that is,

$$\mathcal{U} = \{U^h \in C^0(S^h) \;\mid\; U^h|_{\partial \mathbf{S}^h} = 0,\ U^h|_{T_i^h} \in \mathbb{P}_1(T_i^h)\}\ , \tag{3.2}$$

where $\mathbb{P}_k(D)$ denote the space of polynomials of degree no larger than k on any planar domain D. It is easy to see that $U^h \in H_0^1(\mathbf{S}^h)$ and $\nabla_{s_h} U^h$ is constant on each triangle $T_i^h \in \mathcal{T}^h$.

We now construct the dual tessellation of $\mathcal{T}^h$ on $\mathbf{S}^h$, see Fig. 1 (right). For each vertex $\mathbf{x}_i$, let $\chi_i = \{i_s\}_{s=1}^{m_i}$ be the set of indices of its neighbors, $Q_{i,i_j,i_{j+1}}$ (where

$i_{s+1} = i_1$ if $s = m_i$) be the centroid of the triangle $T^h_{i_j} = \triangle \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}}$ and M_{i,i_j} be the midpoint of $\overline{\mathbf{x}_i \mathbf{x}_{i_j}}$ for $i_j \in \chi_i$. Let $K^h_i = \cup_{i_j \in \chi_i} \Omega_{i,i_j,i_{j+1}}$ where $\Omega_{i,i_j,i_{j+1}}$ denotes the polygonal region bounded by $\mathbf{x}_i$, M_{i,i_j}, $Q_{i,i_j,i_{j+1}}$ and $M_{i,i_{j+1}}$. K^h_i is in general only piecewise planar and we define its projection on $\mathbf{S}$ by $K_i = \{\mathbf{p}(\mathbf{x}) \in \mathbf{S} \mid \mathbf{x} \in K^h_i\}$.

Now, denote by σ the set of indices of the interior vertices of $\mathcal{T}^h$, then, $\mathcal{K} = \{K_i\}_{i\in\sigma}$ and $\mathcal{K}^h = \{K^h_i\}_{i\in\sigma}$ may be viewed as dual tessellations of $\mathbf{S} = \cup^n_{i=1} T_i$ and $\mathbf{S}^h = \cup^n_{i=1} T^h_i$. Denote by $\mathcal{V}$ the space of grid functions on S^h with respect to $\mathcal{K}^h$:

$$\mathcal{V} = \{V^h \;\mid\; V^h|_{\partial \mathbf{S}^h} = 0,\ V^h|_{K^h_i} \in \mathbb{P}_0(K^h_i)\}\ .$$

A set of basis functions $\{\Psi_i\}_{i\in\sigma}$ of $\mathcal{V}$ is given by

$$\Psi^h_i(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in K^h_i \\ 0, & \mathbf{x} \in \mathbf{S} - K^h_i\ . \end{cases}$$

For any $U \in H^1(\mathbf{S}^h)$ and $V^h \in \mathcal{V}$, define the bilinear functionals $\mathcal{A}^h_G$ such that

$$\mathcal{A}^h_G(U, V^h) = \sum_{i\in\sigma} V^h_i \mathcal{A}^h(U, \Psi^h_i)\ , \tag{3.3}$$

where $V^h_i = V^h(\mathbf{x}_i)$ and

$$\begin{aligned} \mathcal{A}^h_G(U, \Psi^h_i) &= -\int_{\partial K^h_i} A(\mathbf{x}) \nabla_{s_h} U(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K^h_i}\, d\gamma_h + \int_{K^h_i} B(\mathbf{x}) U(\mathbf{x})\, ds_h \\ &= -\sum_{i_j \in \chi_i} \int_{\Gamma_{i,i_j,i_{j+1}}} A(\mathbf{x}) \nabla_{s_h} U(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K^h_i}\, d\gamma_h + \int_{K^h_i} B(\mathbf{x}) U(\mathbf{x})\, ds_h \end{aligned}$$

with $\Gamma_{i,i_j,i_{j+1}} = \partial K^h_i \cap \triangle \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}} = \overline{M_{i,i_j} Q_{i,i_j,i_{j+1}} M_{i,i_{j+1}}}$ and $\vec{\mathbf{n}}_{K^h_i}$ the outward unit normal of ∂K^h_i.

For any $V^h \in \mathcal{V}$, define

$$(F, V^h)_{s_h} = \int_{\mathbf{S}^h} F(\mathbf{x}) V^h(\mathbf{x})\, ds_h.$$

Then the *discrete finite volume method* is given by: find $U^h \in \mathcal{U}$ such that

$$\mathcal{A}^h_G(U^h, V^h) = (F, V^h)_{s_h}, \quad \forall\, V^h \in \mathcal{V}\ . \tag{3.4}$$

In practical implementation, noticing that U^h is piecewise linear on $\mathbf{S}^h$ with respect to $\mathcal{T}^h$, $\nabla_{s_h} U^h$ is constant on each triangle $T^h_{i_j} = \triangle \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}}$, and defining

$$B_i = \frac{1}{m(K^h_i)} \int_{K^h_i} B(\mathbf{x})\, ds_h\ , \quad F_i = \frac{1}{m(K^h_i)} \int_{K^h_i} F(\mathbf{x})\, ds_h$$

as averages over K^h_i, we could use the approximations:

$$(F, V^h)_{s_h} = \sum_{i\in\sigma} \int_{K^h_i} F(\mathbf{x}) V^h(\mathbf{x}_i)\, ds_h = \sum_{i\in\sigma} m(K^h_i) V^h_i F_i\ , \tag{3.5}$$

$$\mathcal{A}^h_*(U^h, V^h) = \sum_{i\in\sigma} V^h_i \mathcal{A}^h_*(U^h, \Psi^h_i)\ . \tag{3.6}$$

Here,

$$\begin{aligned}\mathcal{A}_*^h(U^h,\Psi_i^h) &= -\sum_{i_j\in\chi_i} A_{i,i_j,i_{j+1}}\Big[q^1_{i,i_j,i_{j+1}}(U^h_{i_j}-U^h_i)+q^2_{i,i_j,i_{j+1}}(U^h_{i_{j+1}}-U^h_i)\Big]\\ &\quad +m(K_i^h)B_iU_i^h\\ &= -\sum_{i_j\in\chi_i} p_{i,i_j}(U^h_{i_j}-U^h_i)+m(K_i^h)B_iU_i^h \end{aligned}\tag{3.7}$$

and

$$\begin{aligned} &U_i^h = U^h(\mathbf{x}_i)\,, \qquad A_{i,i_j,i_{j+1}} = A(Q_{i,i_j,i_{j+1}})\,,\\ &p_{i,i_j} = A_{i,i_j,i_{j+1}}q^1_{i,i_j,i_{j+1}} + A_{i,i_{j-1},i_j}q^2_{i,i_{j-1},i_j},\\ &q^k_{i,i_j,i_{j+1}} = \frac{1}{8m(\triangle\mathbf{x}_i\mathbf{x}_{i_j}\mathbf{x}_{i_{j+1}})}\Big((-1)^{k-1}\|\mathbf{x}_{i_{j+1}}-\mathbf{x}_i\|^2\\ &\qquad\qquad +(-1)^k\|\mathbf{x}_{i_j}-\mathbf{x}_i\|^2+\|\mathbf{x}_{i_j}-\mathbf{x}_{i_{j+1}}\|^2\Big), \qquad k=1,2. \end{aligned}$$

With numerical integration, we may transform (3.4) to the following problem in the practical implementation: find $U^h\in\mathcal{U}$ such that

$$\mathcal{A}_*^h(U^h,V^h) = (F,v^h)_{s_h}, \quad \forall\, V^h\in\mathcal{V}\,. \tag{3.8}$$

Rewriting (3.8) in a form of a discrete linear system, we get:

$$-\frac{1}{m(K_i^h)}\sum_{i_j\in\chi_i} p_{i,i_j}(U^h_{i_j}-U^h_i)+B_iU_i^h = F_i, \quad \text{for } \ i\in\sigma. \tag{3.9}$$

Remark 1. It is clear that the above system (3.9) satisfies the discrete conservation law since

$$\sum_{i\in\sigma} -\frac{1}{m(K_i^h)}\sum_{i_j\in\chi_i} p_{i,i_j}(U^h_{i_j}-U^h_i) = 0. \tag{3.10}$$

Remark 2. Although a global triangulation for $\mathbf{S}$ is provided for the description of the algorithm, we note that the finite volume discretization may be constructed locally using the geometry of a locally defined triangular meshes and the corresponding dual cells as seen from the equation (3.9).

In this paper, we only analyze the error of the finite volume approximation (3.4). The bilinear form $\mathcal{A}_*^h$ given above turns out to be useful in the derivation of the coercivity of $\mathcal{A}_G^h$. The analysis can be generalized to (3.9) but more stringent regularity assumptions on the data and the exact solution would be required.

4. Existence and Stability Estimates

The analysis below takes the similar framework used in [19, 29] and also [6, 11]. For given functions $U^h\in\mathcal{U}$, $V^h\in\mathcal{U}$ or $\mathcal{V}$, we define, similar to [4, 21, 11, 29], the following discrete inner products and norms associated with $\mathcal{T}^h$ and a particular triangle $T_i^h = \triangle\mathbf{x}_{i_1}\mathbf{x}_{i_2}\mathbf{x}_{i_3}$:

$$\begin{cases} (U^h,V^h)_{T_i^h} = \dfrac{1}{3}m(T_i^h)\Big(\displaystyle\sum_{j=1}^{3}U^h(\mathbf{x}_{i_j})V^h(\mathbf{x}_{i_j})\Big),\\ \|U^h\|^2_{0,T_i^h} = (U^h,U^h)_{T_i^h}\,, \qquad |U^h|^2_{1,T_i^h} = m(T_i^h)\|\nabla_{s_h}U^h\|^2_{T_i^h}. \end{cases}$$

and $\|U^h\|^2_{0,\mathcal{T}^h} = (U^h, U^h)_{\mathcal{T}^h}$, $\|U^h\|^2_{1,\mathcal{T}^h} = \|U^h\|^2_{0,\mathcal{T}^h} + |U^h|^2_{1,\mathcal{T}^h}$ where

$$\begin{cases} (U^h, V^h)_{\mathcal{T}^h} = \sum_{T_i^h \in \mathcal{T}^h} (U^h, V^h)_{T_i^h}\,, \\ \\ |U^h|^2_{1,\mathcal{T}^h} = \sum_{T_i^h \in \mathcal{T}^h} |U^h|^2_{1,T_i^h}\,, \end{cases}$$

As the norms are defined locally with piecewise planar triangles, the following technical lemma is a trivial generalization of the same result given in [29].

Lemma 1. There exist some constants $c_1, c_2 > 0$ such that for any $U_h \in \mathcal{U}$,

$$\begin{array}{c} c_1\|U^h\|_{0,\mathcal{T}^h} \le \|U^h\|_{L^2(\mathbf{S}^h)}\| \le c_2\|U^h\|_{0,\mathcal{T}^h}\,, \\ c_1\|U^h\|_{1,\mathcal{T}^h} \le \|U^h\|_{H^1(\mathbf{S}^h)}\| \le c_2\|U^h\|_{1,\mathcal{T}^h}\,. \end{array} \tag{4.1}$$

Similarly, for any $U \in C^0(\mathbf{S}^h)$, denote by $\Pi_u(U)$ the interpolant of U onto $\mathcal{U}$ and by $\Pi_v(U)$ the interpolant onto $\mathcal{V}$, then we have the following classical approximation results:

Lemma 2. If $U \in H^2(T_i^h)$ for $T_i^h \in \mathcal{T}^h$, then there exist some $c_1, c_2 > 0$ such that

$$\begin{cases} \|U - \Pi_u(U)\|_{L^2(T_i^h)} + h\|U - \Pi_u(U)\|_{H^1(T_i^h)} \le c_1 h^2 \|U\|_{H^2(T_i^h)}, \\ \|U - \Pi_v(U)\|_{L^2(T_i^h)} \le c_2 h \|U\|_{H^1(T_i^h)}\,. \end{cases} \tag{4.2}$$

We then have the coercivity of the operator $\mathcal{A}_G^h$.

Proposition 1. There exists a constant $c > 0$ such that

$$\mathcal{A}_G^h(U^h, \Pi_v(U^h)) \ge c\|U^h\|^2_{H^1(\mathbf{S}^h)} \tag{4.3}$$

for any $U^h \in \mathcal{U}$.

Proof. First we have

$$\begin{aligned} \mathcal{A}_G^h(U^h, \Pi_v(U^h)) \quad &= \quad \left[\mathcal{A}_G^h(U^h, \Pi_v(U^h)) - \mathcal{A}_*^h(U^h, \Pi_v(U^h))\right] \\ & \quad\quad + \mathcal{A}_*^h(U^h, \Pi_v(U^h)) \end{aligned} \tag{4.4}$$

From (3.4), we get

$$\begin{aligned} &\mathcal{A}_*^h(U^h, \Pi_v(U^h)) = \sum_{i\in\sigma} U_i^h \mathcal{A}_*^h(U^h, \Psi_i^h) \\ &= \sum_{i\in\sigma} \Big(- \sum_{i_j\in\chi_i} A_{i,i_j,i_{j+1}} U_i^h \int_{\Gamma_{i,i_j,i_{j+1}}} \nabla_{s_h} U^h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + m(K_i^h) B_i (U_i^h)^2 \Big) \\ &\ge - \sum_{i\in\sigma} \sum_{i_j\in\chi_i} A(Q_{i,i_j,i_{j+1}}) U_i^h \int_{\Gamma_{i,i_j,i_{j+1}}} \nabla_{s_h} U^h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h \end{aligned}$$

Let $Q_i = Q_{i_1,i_2,i_3}$ be the centroid of $T_i^h = \triangle \mathbf{x}_{i_1}\mathbf{x}_{i_2}\mathbf{x}_{i_3} \in \mathcal{T}^h$, by Lemma 1 and some simple calculations, we have

$$\begin{aligned} \mathcal{A}_*^h(U^h, \Pi_v(U^h)) \quad &= \quad \sum_{T_i^h\in\mathcal{T}^h} A(Q_i) \Big(- \sum_{j=1}^{3} U_{i_j}^h \int_{\partial K_{i_j}^h \cap T_i^h} \nabla_{s_h} U^h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_{i_j}^h}\, d\gamma_h \Big) \\ &= \quad \sum_{T_i^h\in\mathcal{T}^h} A(Q_i) m(T_i^h) \|\nabla_s U^h\|^2_{T_i^h} \\ &\ge \quad \sum_{T_i^h\in\mathcal{T}^h} \alpha_1 |U^h|^2_{1,T_i^h} \ge \alpha_1 |U^h|^2_{H^1(\mathbf{S}^h)}\,. \end{aligned} \tag{4.5}$$

On the other hand, we have

$$\left|\mathcal{A}_G^h(U^h,\Pi_v(U^h)) - \mathcal{A}_*^h(U^h,\Pi_v(U^h))\right| \le I_1 + I_2$$

where

$$I_1 = \Big|\sum_{T_i^h\in\mathcal{T}^h}\Big(-\sum_{j=1}^3 U_{i_j}^h \int_{\partial K_{i_j}^h\cap T_i^h}(A(\mathbf{x})-A(Q_i))\nabla_{s_h}U^h(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_{i_j}^h}\,d\gamma_h\Big)\Big|,$$

$$I_2 = \Big|\sum_{i\in\sigma}\int_{k_i^h} B(\mathbf{x})(U^h(\mathbf{x})-U_i^h)U_i^h\,ds_h\Big|.$$

Rearranging I_1, we get (let $j = j\ mod\ 3$)

$$\begin{aligned} I_1 = \Big|\sum_{T_i^h\in\mathcal{T}^h}&\Big(\sum_{j=1}^3(U_{i_{j+2}}^h - U_{i_{j+1}}^h)\\ &\cdot\int_{M_{i_{j+1},i_{j+2}}Q_i}(A(\mathbf{x})-A(Q_i))\nabla_{s_h}U^h(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_{i_{j+1}}^h}\,d\gamma_h\Big|\end{aligned}$$

Since in each triangle T_i^h, we have

$$\begin{aligned}|U_{i_{j+2}}^h - U_{i_{j+1}}^h| &\le h\|\nabla_{s_h}U^h\|_{T_j^h},\\ |A(\mathbf{x})-A(Q_i)| &< ch\|A\|_{W^{1,\infty}(\mathbf{S}^h)},\\ |\nabla_{s_h}U^h(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_{i_{j+1}}^h}| &\le \|\nabla_{s_h}U^h|_{T_i^h}.\|\end{aligned}$$

With the mesh regularity assumption and Lemma 1, we get

$$\begin{aligned} I_1 &\le \sum_{T_i^h\in\mathcal{T}^h} ch^3\|\nabla_{s_h}U^h\|_{T_i^h}^2 \\ &\le ch\sum_{T_i^h\in\mathcal{T}^h}\|\nabla_{s_h}U^h\|_{T_j^h}^2 m(T_i^h) \le ch\|U^h\|_{H^1(\mathbf{S}^h)}^2. \end{aligned} \tag{4.6}$$

As for I_2, with Lemma 2, we have

$$\begin{aligned} I_2 &\le c\|B\|_{L^\infty(\mathbf{S}^h)}\|U^h-\Pi_v(U^h)\|_{L^2(\mathbf{S}^h)}\|\Pi_v(U^h)\|_{L^2(\mathbf{S}^h)}\\ &\le ch\|U^h\|_{H^1(\mathbf{S}^h)}\|\Pi_v(U^h)\|_{L^2(\mathbf{S}^h)}. \end{aligned}$$

Since Q_i is the centroid of T_i^h, it is easy to find that for any $U^h\in\mathcal{U}$

$$\begin{aligned}\|\Pi_v(U^h)\|_{L^2(\mathbf{S}^h)} &= \Big(\sum_{T_i^h\in\mathcal{T}^h}\sum_{j=1}^3(U_{i_j}^h)^2 m(K_{i_j}^h\cap T_i^h)\Big)^{1/2}\\ &= \Big(\sum_{T_i^h\in\mathcal{T}^h}\frac{1}{3}(\sum_{j=1}^3(U_{i_j}^h)^2)m(T_i^h))\Big)^{1/2} = \|U^h\|_{0,\mathcal{T}^h}\ .\end{aligned}$$

So we get

$$I_2 \le ch\|U^h\|_{L^2(\mathbf{S}^h)}\|U^h\|_{H^1(\mathbf{S}^h)}\ . \tag{4.7}$$

Combining (4.6) and (4.7), we know

$$\left|\mathcal{A}_G^h(U^h,\Pi_v(U^h)) - \mathcal{A}_*^h(U^h,\Pi_v(U^h))\right| \le ch\|U^h\|_{H^1(\mathbf{S}^h)}^2\ . \tag{4.8}$$

Using (4.4), (4.5), (4.8), and the Poincare inequality in $H_0^1(\mathbf{S}^h)$, we finally obtain (4.3). □

It is also easy to see

$$\begin{aligned} \left|(F,\Pi_v(U^h))_{S^h}\right| &\leq \|F\|_{L^2(\mathbf{S}^h)}\|\Pi_v(U^h)\|_{L^2(\mathbf{S}^h)} \\ &= c\|F\|_{L^2(\mathbf{S}^h)}\|U^h\|_{0,\mathcal{T}^h} \end{aligned} \tag{4.9}$$

By Proposition 1 and (4.9), we have the following stability results:

Theorem 2. The discrete problem (3.4) have an unique solution $U^h \in \mathcal{U}$, and the U^h satisfies the stability estimate:

$$\|U^h\|_{H^1(\mathbf{S}^h)} \leq c\|F\|_{L^2(\mathbf{S}^h)} \tag{4.10}$$

for some constant $c > 0$.

5. H^1 Error Estimate

When h is small enough, it is easy to find

$$|d(\mathbf{x})| \leq ch^2, \qquad \forall \mathbf{x} \in \mathbf{S}^h,$$

see [19]. To compare the discrete solution on $\mathbf{S}^h$ with the continuous solution on $\mathbf{S}$, we lift a function U defined from $\mathbf{S}^h$ onto $\mathbf{S}$ by

$$\mathcal{L} : U \to u = \mathcal{L}(U) \quad \text{where} \quad u(\mathbf{y}) = U(\mathbf{p}^{-1}(\mathbf{y})), \quad \forall\, \mathbf{y} \in \mathbf{S}\ , \tag{5.1}$$

that is, $U(\mathbf{x}) = u(\mathbf{p}(\mathbf{x})) = u(\mathbf{x} - d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{x})$ for $\mathbf{x} \in \mathbf{S}^h$. Let $\mathbf{y} = \mathbf{p}(\mathbf{x})$ and

$$\mu_h(\mathbf{x}) = \frac{ds(\mathbf{x})}{ds_h(\mathbf{p}(\mathbf{x}))}\ , \quad \xi_h(\mathbf{x}) = \frac{d\gamma(\mathbf{x})}{d\gamma_h(\mathbf{p}(\mathbf{x}))}.$$

Since $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth, we have

$$|1-\mu_h(\mathbf{x})| \leq ch^2, \quad |1-\xi_h(\mathbf{x})| \leq ch^2, \quad \|\vec{\mathbf{n}}(\mathbf{y}) - \vec{\mathbf{n}}_h(\mathbf{x}))\| < ch.$$

For the relations between ∇_s and ∇_{s_h}, we have

$$\begin{gathered} \nabla_{s_h} U(\mathbf{x}) = \mathbf{P}_h \nabla U(\mathbf{x}), \quad \nabla_s u(\mathbf{y}) = \mathbf{P}\nabla u(\mathbf{y}), \\ \nabla U(\mathbf{x}) = (\mathbf{P} - d\mathbf{H})\nabla u(\mathbf{y}) \end{gathered}$$

where

$$\begin{aligned} &\mathbf{P}_h = (\delta_{i,j} - n_{hi}n_{hj}), \ \ \mathbf{P} = (\delta_{i,j} - n_i n_j), \\ &\mathbf{H} = (d_{x_i,x_j}) = ((n_i)_{x_j}) = ((n_j)_{x_i}). \end{aligned} \tag{5.2}$$

Since $\mathbf{P}$ is in fact a projection, we can easily find that

$$\mathbf{PP} = \mathbf{P}, \qquad \mathbf{PH} = \mathbf{HP} = \mathbf{H},$$

and consequently

$$\nabla_{s_h} U(\mathbf{x}) = \mathbf{P}_h(\mathbf{I} - d\mathbf{H})\nabla_s u(\mathbf{y})\ .$$

The following results were proved in [19]:

Lemma 3. There exists some constants $c_1, c_2, c_3, c_4, c > 0$ such that

$$\left\{ \begin{array}{l} c_1\|U\|_{L^2(T_i^h)} \leq \|u\|_{L^2(T_i)} \leq c_2\|U\|_{L^2(T_i^h)}, \\ c_3\|U\|_{H^1(T_i^h)} \leq \|u\|_{H^1(T_i)} \leq c_4\|U\|_{H^1(T_i^h)}, \\ |U|_{H^2(T_i^h)} \leq c\big[|u|_{H^2(T_i)} + h|u|_{H^1(T_i)}\big] \end{array} \right. .$$

For any $u \in C^0(\mathbf{S})$, we define the interpolants $\pi_u(u)$ and $\pi_v(u)$ by

$$\pi_u(u) = \mathcal{L}(\Pi_u(\mathcal{L}^{-1}(u)), \quad \pi_v(u) = \mathcal{L}(\Pi_v(\mathcal{L}^{-1}(u)). \tag{5.3}$$

Then we have the following results (see [19]):

Lemma 4. If $u \in H^2(\mathbf{S})$, then there exist some $c_1, c_2 > 0$ such that

$$\left\{ \begin{array}{l} \|u - \pi_u(u)\|_{L^2(\mathbf{S})} + h\|u - \pi_u(u)\|_{H^1(\mathbf{S})} \le c_1 h^2 \|u\|_{H^2(\mathbf{S})}, \\ \|u - \pi_v(u)\|_{L^2(\mathbf{S})} \le c_2 h \|u\|_{H^1(\mathbf{S})} \ . \end{array} \right. \tag{5.4}$$

For any $U^h \in \mathcal{U}$ and $V^h \in \mathcal{V}$, lift them onto $\mathbf{S}$ by $u^h = \mathcal{L}(U^h)$ and $v^h = \mathcal{L}(V^h)$, and let

$$\psi_i^h(\mathbf{x}) = \left\{ \begin{array}{ll} 1, & \mathbf{x} \in K_i, \\ 0, & \mathbf{x} \in \mathbf{S} - K_i. \end{array} \right.$$

Let $\vec{\mathbf{n}}_{K_i}$ denote the outward normal of ∂K_i. For any $u \in H^1(\mathbf{S})$ and $v^h \in \mathcal{L}(\mathcal{V})$, we then define the bilinear functional $\mathcal{A}_G$ such as

$$\mathcal{A}_G(u, v^h) = \sum_{i \in \sigma} v_i^h \mathcal{A}_G(u, \psi_i^h) \ , \tag{5.5}$$

where $v_i^h = v^h(\mathbf{x}_i)$ and

$$\mathcal{A}_G(u, \psi_i^h) \ = \ -\int_{\partial K_i} a(\mathbf{x}) \nabla_s u(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i} \, d\gamma + \int_{K_i} b(\mathbf{x}) u(\mathbf{x}) \, ds \ .$$

To avoid excessively long formulae, we assume $a(\mathbf{x}) \equiv 1$, so that $A(\mathbf{x}) \equiv 1$ in the remaining parts of this paper. We note that the results hold in fact for general coefficients.

Lemma 5. For any $u \in H^2(\mathbf{S})$ and $W^h \in \mathcal{U}$, there exists a constant $c > 0$ such that

$$\left| \mathcal{A}_G^h(U, \Pi_v(W^h)) - \mathcal{A}_G^h(\Pi_u(U), \Pi_v(W^h)) \right| \le ch \|u\|_{H^2(\mathbf{S})} \|W^h\|_{H^1(\mathbf{S}^h)} \tag{5.6}$$

where $U = \mathcal{L}^{-1}(u)$ and $w^h = \mathcal{L}(W^h)$.

Proof. It is easy to see that $U \in H^2(T_i^h)$ and $w^h \in H^1(\mathbf{S})$. We know

$$\mathcal{A}_G^h(U, \Pi_v(W^h)) - \mathcal{A}_G^h(\Pi_u(U), \Pi_v(W^h)) = I_1 + I_2$$

where

$$I_1 = \sum_{i \in \sigma} -W^h(\mathbf{x}_i) \int_{\partial K_i^h} \nabla_{s_h}(U - \Pi_u(U)) \cdot \vec{\mathbf{n}}_{K_i^h} \, d\gamma_h,$$

$$I_2 = \sum_{i \in \sigma} W^h(\mathbf{x}_i) \int_{K_i^h} B(U - U(\mathbf{x}_i)) \, ds_h \ .$$

Let $W_i^h = W^h(\mathbf{x}_i)$ and $T_i^h = \triangle \mathbf{x}_{i_1} \mathbf{x}_{i_2} \mathbf{x}_{i_3}$, then we get

$$\begin{aligned} I_1 \ &= \ \sum_{T_i^h \in \mathcal{T}^h} \Big(-\sum_{j=1}^{3} W_{i_j}^h \int_{\partial K_{i_j}^h \cap T_i^h} \nabla_{s_h}(U(\mathbf{x}) - \Pi_u(U)) \cdot \vec{\mathbf{n}}_{K_{i_j}^h} \, d\gamma_h \Big) \\ &= \ \sum_{T_i^h \in \mathcal{T}^h} \Big(\sum_{j=1}^{3} (W_{i_{j+2}}^h - W_{i_{j+1}}^h) \\ &\qquad \cdot \int_{\overline{M_{i_{j+1}, i_{j+2}} Q_i}} \nabla_{s_h}(U - \Pi_u(U)) \cdot \vec{\mathbf{n}}_{K_{i_{j+1}}^h} \, d\gamma_h \Big) \ . \end{aligned}$$

In each triangle T_i^h, by the mesh regularity assumption, we have

$$|W_{i_{j+2}}^h - W_{i_{j+1}}^h| \le h \|\nabla_{s_h} W^h\|_{T_j^h} \le c \|W^h\|_{1, T_i^h} \ .$$

Using trace theorem on each $K^h_{i_j} \cap T^h_i$ and the mesh regularity assumption again, we get

$$
\begin{aligned}
&|\int_{\overline{M_{i_{j+1},i_{j+2}}Q_i}} \nabla_{s_h}(U - \Pi_u(U)) \cdot \vec{\mathbf{n}}_{K^h_{i_{j+1}}} \, d\gamma_h| \\
&\quad \le ch^{1/2}\Big(\int_{\overline{M_{i_{j+1},i_{j+2}}Q_i}} \|\nabla_{s_h}(U - \Pi_u(U))\|^2 \, d\gamma_h\Big)^{1/2} \\
&\quad \le ch\|U\|_{H^2(T^h_i)}.
\end{aligned}
$$

By Lemma 1 and 3, we then obtain

$$
\begin{aligned}
|I_1| &\le \sum_{T^h_i \in \mathcal{T}^h} ch\|U\|_{H^2(T^h_i)}\|W^h\|_{1,T^h_i} \le ch \sum_{T^h_i \in \mathcal{T}^h} \|u\|_{H^2(T_i)}\|w^h\|_{H^1(T^h_i)} \\
&\le ch\|u\|_{H^2(\mathbf{S})}\|W^h\|_{H^1(\mathbf{S}^h)} \ . \qquad (5.7)
\end{aligned}
$$

Also by Lemma 2 and 3, we achieve

$$
\begin{aligned}
|I_2| &= |\sum_{i\in\sigma} \int_{K^h_i} B\Pi_v(W^h)(U - \Pi_v(U)) \, ds_h| \\
&\le \|B\|_{L^\infty(\mathbf{S}^h)} \int_{\mathbf{S}^h} |\Pi_v(W)| \cdot |U - \Pi_v(U)| \, ds_h \\
&\le c\|b\|_{L^\infty(\mathbf{S})}\|\Pi_v(W^h)\|_{L^2(\mathbf{S}^h)}\|U - \Pi_v(U)\|_{L^2(\mathbf{S}^h)} \\
&\le c\|W^h\|_{0,\mathcal{T}^h}\|U\|_{H^1(\mathbf{S}^h)} \le ch\|u\|_{H^1(\mathbf{S})}\|W^h\|_{L^2(\mathbf{S}^h)}. \qquad (5.8)
\end{aligned}
$$

Combining (5.7) and (5.8), we get (5.6). □

Lemma 6. For any $u \in H^2(\mathbf{S})$ and $W^h \in \mathcal{U}$, there exists a constant $c > 0$ such that

$$
\big|\mathcal{A}_G(u, \pi_v(w^h)) - \mathcal{A}^h_G(U, \Pi_v(W^h))\big| \le ch^2\|u\|_{H^2(\mathbf{S})}\|W^h\|_{H^1(\mathbf{S}^h)}. \qquad (5.9)
$$

where $U = \mathcal{L}^{-1}(u)$ and $w^h = \mathcal{L}(W^h)$.

Proof. We know

$$
\mathcal{A}_G(u, \pi_v(w^h)) - \mathcal{A}^h_G(U, \Pi_v(W^h)) = I_1 + I_2 + I_3
$$

where

$$
\begin{aligned}
I_1 &= \sum_{i\in\sigma} -W^h_i\Big(\int_{\partial K_i} \nabla_s u(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i}(\mathbf{x}) \, d\gamma - \int_{\partial K^h_i} \nabla_{s_h} U(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x})) \, d\gamma_h\Big), \\
I_2 &= \sum_{i\in\sigma} -W^h_i\Big(\int_{\partial K^h_i} \nabla_{s_h} U(\mathbf{x}) \cdot (\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x})) - \vec{\mathbf{n}}_{K^h_i}(\mathbf{x})) \, d\gamma_h\Big), \\
I_3 &= \sum_{i\in\sigma} W^h_i\Big(\int_{K_i} b(\mathbf{x})u(\mathbf{x}) \, ds - \int_{K^h_i} B(\mathbf{x})U(\mathbf{x}) \, ds_h\Big).
\end{aligned}
$$

As for I_1, we have

$$\begin{aligned}
I_1 &= \sum_{i\in\sigma} -W_i^h\Big(\int_{\partial K_i^h} \nabla_s u(\mathbf{p}(\mathbf{x}))\cdot\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x}))\xi_h\,d\gamma_h - \\
&\qquad \int_{\partial K_i^h} \nabla_{s_h} U(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x}))\,d\gamma_h\Big) \\
&= \sum_{i\in\sigma} -W_i^h \int_{\partial K_i^h} (\xi_h\nabla_s u(\mathbf{p}(\mathbf{x})) - \nabla_{s_h} U(\mathbf{x}))\cdot\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x}))\,d\gamma_h \\
&= \sum_{T_i\in\mathcal{T}} \Big(-\sum_{j=1}^{3} W_{i_j}^h \int_{\partial K_i^h\cap T_i^h} (\xi_h\nabla_s u(\mathbf{p}(\mathbf{x})) - \nabla_{s_h} U(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x}))\,d\gamma_h\Big) \\
&= \sum_{T_i^h\in\mathcal{T}^h} \Big(\sum_{j=1}^{3}(W_{i_{j+2}}^h - W_{i_{j+1}}^h) \\
&\qquad \cdot\int_{\overline{M_{i_{j+1},i_{j+2}}Q_i}} (\xi_h\nabla_s u(\mathbf{p}(\mathbf{x})) - \nabla_{s_h} U(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_{i_{j+1}}^h}\,d\gamma_h\Big).
\end{aligned}$$

We observe that

$$\begin{aligned}
\xi_h\nabla_s u(\mathbf{p}(\mathbf{x})) - \nabla_{s_h} U(\mathbf{x}) &= \big(\xi_h\mathbf{I} - \mathbf{P}_h(\mathbf{I} - d\mathbf{H})\big)\nabla_s u(\mathbf{p}(\mathbf{x})) \\
&= \xi_h\mathbf{P}\big(\mathbf{I} - \frac{1}{\xi_h}\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}\big)\nabla_s u(\mathbf{p}(\mathbf{x})).
\end{aligned}$$

Since $|1-\xi_h| < ch^2$, we have

$$\begin{aligned}
|\xi_h\mathbf{P}(\mathbf{I} - \frac{1}{\xi_h}\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P})| &\le |\mathbf{P} - \mathbf{P}\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P})| + ch^2 \\
&\le |\mathbf{P} - \mathbf{P}\mathbf{P}_h\mathbf{P}| + ch^2 \\
&\le c\|\vec{\mathbf{n}}\times\vec{\mathbf{n}}_h\|^2 + ch^2 \le ch^2.
\end{aligned}$$

So we know

$$\|\xi_h\nabla_s u(\mathbf{p}(\mathbf{x})) - \nabla_{s_h} U(\mathbf{x})\| \le ch^2\|\nabla_s u(\mathbf{p}(\mathbf{x}))\| \le ch^2\|\nabla_{s_h} U(\mathbf{x})\|.$$

Then using the similar analysis for I_1, we could find

$$|I_1| \quad\le\quad ch^2\|u\|_{H^2(\mathbf{S})}\|W^h\|_{H^1(\mathbf{S}^h)}. \tag{5.10}$$

As for I_2, since

$$\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x})) = (\mathbf{P} - d\mathbf{H})\vec{\mathbf{n}}_{K_i^h}(\mathbf{x}).$$

and $\mathcal{P}_h$ is a projection, we have

$$\begin{aligned}
\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x})) - \vec{\mathbf{n}}_{K_i^h}(\mathbf{x}) &= (\mathbf{P} - d\mathbf{H} - \mathbf{I})\vec{\mathbf{n}}_{K_i^h}(\mathbf{x}) \\
&= \mathbf{P}_h(\mathbf{P} - d\mathbf{H} - \mathbf{I})\mathbf{P}_h\vec{\mathbf{n}}_{K_i^h}(\mathbf{x}),
\end{aligned}$$

and again we get

$$\begin{aligned}
|\mathbf{P}_h(\mathbf{P} - d\mathbf{H} - \mathbf{I})\mathbf{P}_h| &\le |\mathbf{P}_h\mathbf{P}\mathbf{P}_h - \mathbf{P}_h| + ch^2 \\
&\le c\|\vec{\mathbf{n}}_h\times\vec{\mathbf{n}}|^2 + ch^2 \le ch^2.
\end{aligned}$$

By using a similar analysis as I_1, we easily obtain

$$|I_2| \quad\le\quad ch^2\|u\|_{H^2(\mathbf{S})}\|W^h\|_{H^1(\mathbf{S}^h)}. \tag{5.11}$$

As for I_3, we have

$$\begin{aligned} I_3 &= \sum_{i\in\sigma} W_i^h\Big(\int_{K_i^h} BU\mu_h\, ds_h - \int_{K_i^h} B(\mathbf{x})U(\mathbf{x})\, ds_h\Big) \\ &= \int_{\mathbf{S}^h}(1-\mu_h)BU\Pi_v(W^h)ds_h \end{aligned}$$

which deduces

$$\begin{aligned} |I_3| &\leq ch^2\|b\|_{L^\infty(\mathbf{S})}\|U\|_{L^2(\mathbf{S}^h)}\|W^h\|_{0,\mathcal{T}^h} \\ &\leq ch^2\|u\|_{L^2(\mathbf{S})}\|W^h\|_{L^2(\mathbf{S}^h)}. \end{aligned} \tag{5.12}$$

Combining (5.10), (5.11) and (5.12), we get (5.9). □

Theorem 3. Suppose that u is the weak solution the problem (2.1) with $u|_{\partial\mathbf{S}} = 0$, $U^h \in \mathcal{U}$ is the solution of discrete problem (3.4) and $u^h = L(U^h)$. If $u \in H^2(\mathbf{S})$, then we have that for some $c > 0$,

$$\|u - u^h\|_{H^1(\mathbf{S})} \leq ch\|u\|_{H^2(\mathbf{S})}\ . \tag{5.13}$$

Proof. Let us extend u onto $\mathbf{S}^h$ by $U = \mathcal{L}^{-1}(u)$. By Proposition 1, we have

$$\|U^h - \Pi_u(U)\|^2_{H^1(\mathbf{S}^h)} \leq c\mathcal{A}_G^h(U^h - \Pi_u(U), \Pi_v(U^h - \Pi_u(U))\ . \tag{5.14}$$

For any $W^h \in \mathcal{U}$, let $w^h = \mathcal{L}(W^h)$, then we get

$$\begin{aligned} \mathcal{A}_G^h(U^h - \Pi_u(U), \Pi_v(W^h)) &= \big[\mathcal{A}_G^h(U^h, \Pi_v(W^h)) - \mathcal{A}_G(u, \pi_v(w^h))\big] \\ &\quad + \mathcal{A}_G^h(U - \Pi_u(U), \Pi_v(W^h)) \\ &\quad + \big[\mathcal{A}_G(u, \pi_v(w^h)) - \mathcal{A}_G^h(U, \Pi_v(W^h))\big]\ . \end{aligned} \tag{5.15}$$

According to Stokes theorem and (3.4), we have

$$\mathcal{A}_G^h(U^h, \Pi_v(W^h)) = (F, \Pi_v(W^h))_{s_h}, \quad \mathcal{A}_G(u, \pi_v(W^h)) = (f, \pi_v(W^h)),$$

So by Lemma 1-3 and Theorem 1, we get

$$\begin{aligned} &\big|\mathcal{A}_G^h(U^h, \Pi_v(w^h)) - \mathcal{A}_G(u, \pi_v(w^h))\big| \\ &= |(F, \Pi_v(W^h))_{s_h} - (f, \pi_v(w^h))| \\ &= \Big|\int_{\mathbf{S}^h} F\Pi_v(W^h)\, ds_h - \int_{\mathbf{S}} f\pi_v(w^h)\, ds\Big| \\ &= \Big|\int_{\mathbf{S}^h} F\Pi_v(W^h)\, ds_h - \int_{\mathbf{S}^h} F\Pi_v(W^h)\mu_h\, ds_h\Big| \\ &= \Big|\int_{\mathbf{S}^h}(1-\mu_h)F\Pi_v(W^h)\, ds_h\Big| \\ &\leq ch^2\|F\|_{L^2(\mathbf{S}^h)}\|\Pi_v(W^h)\|_{L^2(\mathbf{S}^h)} \\ &\leq ch^2\|f\|_{L^2(\mathbf{S})}\|W^h\|_{L^2(\mathbf{S}^h)} \leq ch^2\|u\|_{H^2(\mathbf{S})}\|W^h\|_{L^2(\mathbf{S}^h)}. \end{aligned} \tag{5.16}$$

By Lemma 5, we have

$$\big|\mathcal{A}_G^h(U - \Pi_u(U), \Pi_v(W^h))\big| \leq ch\|u\|_{H^2(\mathbf{S})}\|W^h\|_{H^1(\mathbf{S}^h)}. \tag{5.17}$$

By Lemma 6, we get

$$\big|\mathcal{A}_G(u, \pi_v(w^h)) - \mathcal{A}_G^h(U, \Pi_v(W^h))\big| \leq ch\|u\|_{H^2(\mathbf{S})}\|W^h\|_{H^1(\mathbf{S}^h)}. \tag{5.18}$$

Using (5.14)–(5.18) and setting $W^h = U^h - \Pi_u(U)$, we then obtain

$$\|U^h - \Pi_u(U)\|^2_{H^1(\mathbf{S}^h)} \le ch^2 \|u\|_{H^2(\mathbf{S})} \|U^h - \Pi_u(U)\|_{H^1(\mathbf{S}^h)},$$

that is,

$$\|U^h - \Pi_u(U)\|_{H^1(\mathbf{S}^h)} \le ch\|u\|_{H^2(\mathbf{S})}. \tag{5.19}$$

Additionally, by Lemma 4, we have

$$\|U - \Pi_u(U)\|_{H^1(\mathbf{S}^h)} \quad \le \quad \|u - \pi_u(u)\|_{H^1(\mathbf{S})} \le ch\|u\|_{H^2(\mathbf{S})}. \tag{5.20}$$

Combining (5.19) and (5.20), we finally have

$$\begin{aligned}
\|u - u^h\|_{H^1(\mathbf{S})} &\le c\|U - U^h\|_{H^1(\mathbf{S}^h)} \\
&\le c\big(\|U^h - \Pi_u(U)\|_{H^1(\mathbf{S}^h)} + \|U - \Pi_u(U)\|_{H^1(\mathbf{S}^h)}\big) \\
&\le ch\|u\|_{H^2(\mathbf{S})}.
\end{aligned}$$

□

The optimal error estimate presented in Theorem 3 is similar to that obtained by the finite element method, see [19].

6. L^2 Error Estimate

Before presenting the main results for L^2 error estimate, let us first prove additional estimates on the bilinear forms.

Lemma 7. Suppose that u is the weak solution the problem (2.1) with $u|_{\partial\mathbf{S}} = 0$, and $U^h \in \mathcal{U}$ is the solution of discrete problem (3.4). For any $w \in H^2(\mathbf{S})$, there exists a constant $c > 0$ such that

$$\left|\mathcal{A}^h_G(U^h, \Pi_v(W)) - \mathcal{A}_G(u^h, \pi_v(w))\right| \le ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}. \tag{6.1}$$

where $u^h = \mathcal{L}(U^h)$ and $W = \mathcal{L}^{-1}(w)$.

Proof. We know

$$\mathcal{A}^h_G(U^h, \Pi_v(W) - \mathcal{A}_G(u^h, \pi_v(w)) = I_1 + I_2 + I_3$$

where

$$\begin{aligned}
I_1 &= \sum_{i\in\sigma} -W_i\Big(\int_{\partial K_i} \nabla_s u^h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i}(\mathbf{x})\, d\gamma - \int_{\partial K_i^h} \nabla_{s_h} U^h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x})\, d\gamma_h\Big), \\
I_2 &= \sum_{i\in\sigma} -W_i\Big(\int_{\partial K_i^h} \nabla_{s_h} U^h(\mathbf{x}) \cdot (\vec{\mathbf{n}}_{K_i}(\mathbf{p}(\mathbf{x})) - \vec{\mathbf{n}}_{K_i^h}(\mathbf{x}))\, d\gamma_h\Big), \\
I_3 &= \sum_{i\in\sigma} W_i\Big(\int_{K_i} b(\mathbf{x}) u^h\, ds - \int_{K_i^h} B(\mathbf{x}) U^h(\mathbf{x})\, ds_h\Big).
\end{aligned}$$

with $W_i = W(\mathbf{x}_i)$.

Since

$$\begin{aligned}
I_1 \quad = \quad & \sum_{T_i^h \in \mathcal{T}^h} \Big(\sum_{j=1}^{3} (W_{i_{j+2}} - W_{i_{j+1}}) \\
& \cdot \int_{\overline{M_{i_{j+1},i_{j+2}} Q_i}} (\xi_h \nabla_s U^h(\mathbf{p}(\mathbf{x})) - \nabla_{s_h} u^h(\mathbf{x})) \cdot \vec{\mathbf{n}}_{K^h_{i_{j+1}}}\, d\gamma_h\Big),
\end{aligned}$$

by the fact that U^h is piecewise linear, using similar analysis of (4.6) and (5.10), Theorem 3 and Lemma 4, we have

$$
\begin{aligned}
|I_1| &\le ch^2\|U^h\|_{H^1(\mathbf{S}^h)}\|\Pi_u(W)\|_{H^1(\mathbf{S}^h)} \\
&\le ch^2\|u^h\|_{H^1(\mathbf{S})}\|\pi_u(w)\|_{H^1(\mathbf{S})} \\
&\le ch^2(\|u\|_{H^1(\mathbf{S})}+\|u^h-u\|_{H^1(\mathbf{S})})(\|w\|_{H^1(\mathbf{S})}+\|w-\pi_u(w)\|_{H^2(\mathbf{S}^h)}) \\
&\le ch^2\|u\|_{H^2(\mathbf{S}^h)}\|w\|_{H^2(\mathbf{S})}. \qquad (6.2)
\end{aligned}
$$

By similar analysis of (4.6), (5.10) and (5.11), we

$$
\begin{aligned}
|I_2| &\le ch^2\|U^h\|_{H^1(\mathbf{S}^h)}\|\Pi_u(W)\|_{H^1(\mathbf{S}^h)} \\
&\le ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})})\,. \qquad (6.3)
\end{aligned}
$$

As for I_3, we also can get

$$
\begin{aligned}
|I_3| &\le ch^2\|U^h\|_{L^2(\mathbf{S}^h)}\|\Pi_u(W)\|_{H^1(\mathbf{S}^h)} \\
&\le ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}\,. \qquad (6.4)
\end{aligned}
$$

Combining (6.2), (6.3) and (6.4), we get (6.1). $\square$

Lemma 8. Suppose that u is the weak solution the problem (2.1) with $u|_{\partial\mathbf{S}} = 0$, and $U^h \in \mathcal{U}$ is the solution of discrete problem (3.4). If $u \in H^3(\mathbf{S})$, then for any $w \in H^2(\mathbf{S})$, there exists a constant $c > 0$ such that

$$
\left|\mathcal{A}(u-u^h,\pi_u(w)) - \mathcal{A}_G(u-u^h,\pi_v(w))\right| \le ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}. \qquad (6.5)
$$

where $u^h = \mathcal{L}(U^h)$.

Proof. We first have

$$
\begin{aligned}
&\mathcal{A}(u-u^h,\pi_u(w)) \\
&= \int_{\mathbf{S}} \nabla_s(u-u^h)\cdot\nabla_s\pi_u(w\ ds + \int_{\mathbf{S}} b(u-u^h)\pi_u(w)\ ds \\
&= \sum_{T_i\in\mathcal{T}}\int_{T_i} \nabla_s(u-u^h)\cdot\nabla_s\pi_u(w)\ ds + \int_{\mathbf{S}} b(u-u^h)\pi_u(w)\ ds \\
&= \sum_{T_i\in\mathcal{T}}\Big(\int_{T_i} -\triangle_s(u-u^h)\ \pi_u(w)\ ds + \int_{\partial T_i}(\nabla_s(u-u^h)\cdot\vec{\mathbf{n}}_{T_i})\pi_u(w)\ d\gamma\Big) \\
&\quad + \int_{\mathbf{S}} b(u-u^h)\pi_u(w)\ ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{A}_G(u-u^h,\Pi_v(w)) \\
&= \sum_{i\in\sigma}\Big(\int_{\partial K_i} -(\nabla_s(u-u^h)\cdot\vec{\mathbf{n}}_{K_i})\pi_v(w)\ d\gamma + \int_{K_i} b(u-u^h)\Pi_v(w)\ ds\Big) \\
&= \sum_{T_i\in\mathcal{T}}\sum_{j=1}^{3}\int_{K_{i_j}\cap T_i} -(\nabla_s(u-u^h)\cdot\vec{\mathbf{n}}_{K_i})\pi_v(w)\ d\gamma + \int_{\mathbf{S}} b(u-u^h)\pi_v(w)\ ds \\
&= \sum_{T_i\in\mathcal{T}}\Big(\int_{T_i} -\triangle_s(u-u^h)\ \pi_v(w)\ ds + \int_{\partial T_i}(\nabla_s(u-u^h)\cdot\vec{\mathbf{n}}_{T_i})\pi_v(w)\ d\gamma\Big) \\
&\quad + \int_{\mathbf{S}} b(u-u^h)\pi_v(w)\ ds.
\end{aligned}
$$

So we obtain

$$\mathcal{A}(u-u^h,\pi_u(w)) - \mathcal{A}_G(u-u^h,\pi_v(w)) = I_1+I_2+I_3+I_4$$

where

$$I_1 = \sum_{T_i\in\mathcal{T}} \int_{T_i} -\triangle_s u\ (\pi_u(w)-\pi_v(w))\ ds,$$

$$I_2 = \sum_{T_i\in\mathcal{T}} \int_{T_i} -\triangle_s u^h\ (\pi_u(w)-\pi_v(w))\ ds,$$

$$I_3 = \sum_{T_i\in\mathcal{T}} \int_{\partial T_i} (\nabla_s(u-u^h)\cdot\vec{\mathbf{n}}_{T_i})(\pi_u(w)-\pi_v(w))\ d\gamma,$$

$$I_4 = \int_{\mathbf{S}} b(u-u^h)(\pi_u(w)-\pi_v(w)\ ds.$$

Consider I_1, we have

$$I_1 = J_1+J_2$$

where

$$J_1 = \sum_{T_i\in\mathcal{T}} \int_{T_i} (\triangle_s u - \triangle_s u(\mathbf{p}(Q_i))\ (\pi_u(w)-\pi_v(w))\ ds,$$

$$J_2 = \sum_{T_i\in\mathcal{T}} \triangle_s u(\mathbf{p}(Q_i)) \int_{T_i} (\pi_u(w)-\pi_v(w))\ ds.$$

Clearly,

$$\begin{aligned} |J_1| &\leq ch^2\|u\|_{H^3(\mathbf{S})}\|\pi_u(w)-\pi_v(w)\|_{L^2(\mathbf{S})} \\ &\leq ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{H^2(\mathbf{S})} \end{aligned}$$

Let $W=\mathcal{L}^{-1}(w)$, since Q_i is the centroid of T_i^h, we have

$$\int_{T_i^h} \Pi_u(W)-\Pi_v(W)\ ds_h = 0\ .$$

Then it is easy to find

$$\begin{aligned} \Big|\int_{T_i} \pi_u(w)-\pi_v(w)\ ds\Big| &= \Big|\int_{T_i} \pi_u(w)-\pi_v(w)\ ds - \int_{T_i^h} \Pi_u(W)-\Pi_v(W)\ ds_h\Big| \\ &= \Big|\int_{T_i} (1-\mu_h)(\Pi_u(W)-\Pi_v(W))\ ds_h\Big| \\ &\leq ch^2 \int_{T_i} |\Pi_u(W)-\Pi_v(W)|\ ds_h \\ &\leq ch^2 \int_{T_i} |\pi_u(w)-\pi_v(w)|\ ds, \end{aligned}$$

then we have

$$\begin{aligned} |J_2| &= ch^2\|\pi_v(\triangle_s u)\|_{L^2(\mathbf{S})}\|\pi_u(w)-\pi_v(w)\|_{L^2(\mathbf{S})} \\ &\leq ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}. \end{aligned}$$

So we get

$$|I_1| \leq |J_1|+|J_2| \leq ch^2\|u\|_{H^3(\mathbf{S})}\|u-u^h\|_{L^2(\mathbf{S})}. \tag{6.6}$$

As for I_2, by Theorem 3 and Lemma 4, we have

$$\begin{aligned}
|I_2| &\leq \sum_{T_i\in\mathcal{T}}\int_{T_i} |\triangle_s u^h\ (\pi_u(w)-\pi_v(w))|\ ds \\
&\leq ch\sum_{T_i\in\mathcal{T}}\int_{T_i} |\nabla_s u^h|\ |(\pi_u(w)-\pi_v(w))|\ ds \\
&\leq ch^2\|u^h\|_{H^1(\mathbf{S})}\|w\|_{H^2(\mathbf{S})} \\
&\leq ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{L^2(\mathbf{S})}. \qquad (6.7)
\end{aligned}$$

According to the continuity of $\nabla_s u$ on ∂T_i, we have

$$\sum_{T_i\in\mathcal{T}}\int_{\partial T_i}(\nabla_s u\cdot\vec{\mathbf{n}}_{T_i})(\pi_u(w)-\pi_v(w))\ d\gamma = 0.$$

Since $U^h\cdot\vec{\mathbf{n}}_{T_i^h}$ is constant on each edge of the triangle T_i^h, it is also easy to find

$$\sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h}(\nabla_{s_h}U^h\cdot\vec{\mathbf{n}}_{T_i^h})(\Pi_u(W)-\Pi_v(W))\ , d\gamma_h = 0\ ,$$

we than have

$$\begin{aligned}
&\sum_{T_i\in\mathcal{T}}\int_{\partial T_i}(\nabla_s u^h\cdot\vec{\mathbf{n}}_{T_i})(\pi_u(w)-\pi_v(w))\ d\gamma \\
&= \sum_{T_i\in\mathcal{T}^h}\int_{\partial T_i}(\nabla_s u^h\cdot\vec{\mathbf{n}}_{T_i})(\pi_u(w)-\pi_v(w))\ d\gamma \\
&\quad - \sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h}(\nabla_{s_h} u^h\cdot\vec{\mathbf{n}}_{T_i^h})(\Pi_u(W)-\Pi_v(W))\ d\gamma_h \\
&= \sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h}(\xi_h\nabla_s u^h(\mathbf{p}(\mathbf{x}))-\nabla_{s_h}U^h(\mathbf{x}))(\Pi_u(W)-\Pi_v(W))\cdot\vec{\mathbf{n}}_{T_i^h}\ d\gamma_h \\
&\quad + \sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i}\nabla_{s_h}U^h(\mathbf{x})\cdot(\vec{\mathbf{n}}_{T_i}(\mathbf{p}(\mathbf{x})-\vec{\mathbf{n}}_{T_i^h}(\mathbf{x}))(\Pi_u(W)-\Pi_v(W))d\gamma_h.
\end{aligned}$$

Then it is easy to find

$$\begin{aligned}
|I_3| &\leq ch^2\|U^h\|_{H^1(\mathbf{S}^h)}\|\Pi_u(W)\|_{H^1(\mathbf{S}^h)} \\
&\leq ch^2\|u^h\|_{H^1(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}\leq ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}. \qquad (6.8)
\end{aligned}$$

About I_4, we have

$$\begin{aligned}
|I_4| &\leq c\|b\|_{L^\infty(\mathbf{S})}\|u-u^h\|_{L^2(\mathbf{S})}\|\pi_u(w)-\pi_v(w)\|_{L^2(\mathbf{S})} \\
&\leq ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}. \qquad (6.9)
\end{aligned}$$

Combining (6.6)-(6.9), we obtain (6.5).

□

Theorem 4. Suppose that u is the weak solution the problem (2.1) with $u|_{\partial\mathbf{S}}=0$, $U^h\in\mathcal{U}$ is the solution of discrete problem (3.4) and $u^h=\mathcal{L}(U^h)$. If $u\in H^3(\mathbf{S})$, then we have for some $c>0$,

$$\|u-u^h\|_{L^2(\mathbf{S})}\leq ch^2\|u\|_{H^3(\mathbf{S})}\ . \qquad (6.10)$$

Proof. Since $u - u^h \in H^1(\mathbf{S})$, according to Theorem 1, we know that there exists a weak solution $w \in H^2(\mathbf{S})$ satisfying

$$\mathcal{A}(w, v) = (u - u^h, v), \qquad \forall\, v \in H^1(\mathbf{S})$$

Put $v = u - u^h$ in the above equality, then we get

$$\|u - u^h\|_{L^2(\mathbf{S})} = (u - u^h, u - u^h) = \mathcal{A}(w, u - u^h).$$

Furthermore, we know

$$\|w\|_{H^2(\mathbf{S})} \le c\|u - u^h\|_{L^2(\mathbf{S})}. \tag{6.11}$$

Let $W = \mathcal{L}^{-1}(w)$, then we get

$$\begin{aligned}
\|u - u^h\|^2_{L^2(\mathbf{S})} \quad \le \quad & |\mathcal{A}(u - u^h, w - \pi_u(w))| \\
& + |\mathcal{A}_G(u, \pi_v(w)) - \mathcal{A}^h_G(U^h, \Pi_v(W)| \\
& + |\mathcal{A}^h_G(U^h, \Pi_v(W) - \mathcal{A}_G(u^h, \pi_v(w))| \\
& + |\mathcal{A}(u - u^h, \pi_u(w)) - \mathcal{A}_G(u - u^h, \pi_v(w)|.
\end{aligned} \tag{6.12}$$

First by Theorem 3, we have

$$\begin{aligned}
|\mathcal{A}(u - u^h, w - \pi_u(w))| \quad & \le \quad c\|u - u^h\|_{H^1(\mathbf{S})}\|w - \pi_u(w)\|_{H^1(\mathbf{S})} \\
& \le \quad ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})} \\
& \le \quad ch^2\|u\|_{H^2(\mathbf{S})}\|u - u^h\|_{L^2(\mathbf{S})}.
\end{aligned} \tag{6.13}$$

Since

$$\mathcal{A}^h_G(U^h, \Pi_v(W)) = (F, \Pi_v(W))_{s_h}, \quad \mathcal{A}_G(u, \pi_v(w)) = (f, w),$$

using (5.16) and Theorem 1, we get

$$\begin{aligned}
& \left|\mathcal{A}_G(u, \pi_v(w)) - \mathcal{A}^h_G(U^h, \Pi_v(W)\right| \le ch^2\|f\|_{L^2(\mathbf{S})}\|\Pi_v(W)\|_{L^2(\mathbf{S}^h)} \\
& \le ch^2\|f\|_{L^2(\mathbf{S})}(\|W\|_{L^2(\mathbf{S}^h)} + \|W - \Pi_v(W)\|_{L^2(\mathbf{S}^h)}) \\
& \le ch^2\|f\|_{L^2(\mathbf{S})}(\|W\|_{L^2(\mathbf{S}^h)} + ch\|W\|_{H^1(\mathbf{S}^h)}) \\
& \le ch^2\|f\|_{L^2(\mathbf{S})}(\|w\|_{L^2(\mathbf{S})} + ch\|w\|_{H^1(\mathbf{S})}) \\
& \le ch^2\|f\|_{L^2(\mathbf{S})}\|w\|_{H^1(\mathbf{S})} \\
& \le ch^2\|u\|_{H^2(\mathbf{S})}\|u - u^h\|_{L^2(\mathbf{S})}.
\end{aligned} \tag{6.14}$$

By Lemma 7 and (6.11), we get

$$\begin{aligned}
\left|\mathcal{A}^h_G(U^h, \Pi_v(W) - \mathcal{A}_G(u^h, \pi_v(w))\right| \quad & \le \quad ch^2\|u\|_{H^2(\mathbf{S})}\|u - u^h\|_{L^2(\mathbf{S}^h)}) \\
& \le \quad ch^2\|u\|_{H^2(\mathbf{S})}\|u - u^h\|_{L^2(\mathbf{S}^h)}).
\end{aligned} \tag{6.15}$$

By Lemma 8 and (6.11), we get

$$\begin{aligned}
\left|\mathcal{A}(u - u^h, \pi_u(w)) - \mathcal{A}_G(u - u^h, \pi_v(w))\right| \quad & \le \quad ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{H^2(\mathbf{S})} \\
& \le \quad ch^2\|u\|_{H^3(\mathbf{S})}\|u - u^h\|_{L^2(\mathbf{S})}.
\end{aligned} \tag{6.16}$$

Combining (6.12)–(6.16), we finally get

$$\|u - u^h\|^2_{L^2(\mathbf{S})} \le ch^2\|u\|_{H^3(\mathbf{S})}\|u - u^h\|_{L^2(\mathbf{S})}$$

which deduces (6.10) directly. □

Remark 3. All results proved in Theorems 2, 3 and 4 can easily generalized to the case of $\partial\mathbf{S} = \emptyset$ with $b(\mathbf{x}) > \alpha_2 > 0$.

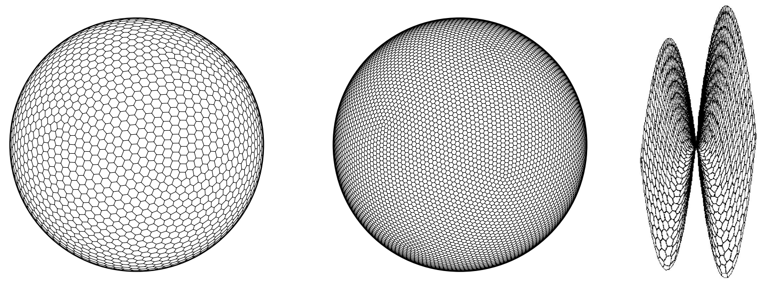

FIGURE 2. CVTs on spheres and a saddle surface.

7. Discussions and Conclusions

In this paper, a finite volume method for solving second order elliptic PDEs on surfaces of arbitrary geometry has been studied using a piecewise linear complex representation of the surface. Optimal order error estimates have been proved under some mesh regularity assumptions. For surface with complex geometry, a natural issue is how to generate a mesh with such regularity.

To address this issues, let us briefly recall the concept of constrained CVTs [9] which are special Voronoi tessellations of the surface with the generators coincide with the constrained centroids of the corresponding Voronoi regions. The concept has been extended to the case constrained to a surface with the standard Euclidean metric [9] and also to the case of a one-sided distance function associated to a Riemannian metric [17], see Figure 2 for some examples of CVT representations of spheres and a saddle. Moreover, these extensions allow us to efficiently generate high quality surface unstructured meshes and triangulations. Applications to full 3d volume mesh generations and optimizations have been explored [14]. Robust and efficient boundary recovery schemes for 3D meshing have also been developed to match given boundary surface specifications [15, 16].

The surface meshes produced using the CVT technology tend to enjoy certain optimality properties. In particular, they are often much more evenly spaced when a uniform density function is used, see Figure 3 for some examples of surface triangulations of a saddle surface, the surface for some connected cubes and balls, and a surface with punched holes. We refer to [18] for a review on the recent progress in this direction. For these surface meshes, the mesh regularity assumption is almost assured to be valid. Thus, they provide excellent surface meshes on which the finite volume methods can be further constructed. An example on the application of such meshes in connection to finite volume methods has been given in [11] where CVT meshes on spherical surfaces have been used. Due to the excellent meshing quality, the finite volume solutions display superconvergent properties. We refer to recent works for extensive numerical experiments and applications [10, 11, 12, 13].

There are additional interesting questions related to the development of finite volume schemes of even higher order accuracy for smooth surfaces and solutions. Some works for the planar cases have been given in the literature, for example, [29]. With singular surfaces and solutions, local mesh refinement can also be considered by generalizing the discussions in earlier works (see for instance [30]). Connections

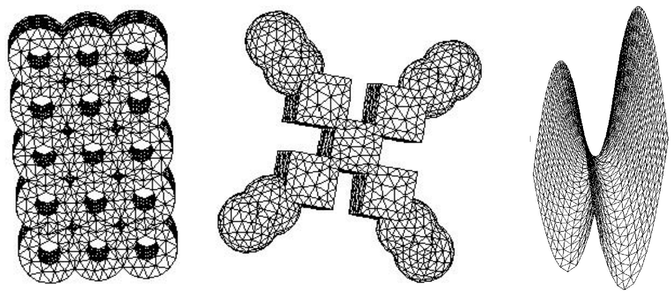

FIGURE 3. High quality surface triangluation of a surface with holes, a surface of connected cubes and balls, and a saddle surface.

with standard and mixed finite element methods [5], non-conforming and discontinuous finite element methods [1, 8]. can also be considered for problems on surfaces. These issues will be explored in our future research.

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