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the interval with Jacobi weights

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LOCALIZED POLYNOMIAL FRAMES ON THE INTERVAL WITH JACOBI WEIGHTS

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ABSTRACT. As is well known the kernel of the orthogonal projector onto the polynomials of degree n in $L^2(w_{\alpha,\beta}, [-1, 1])$ with $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ can be written in terms of Jacobi polynomials. It is shown that if the coefficients in this kernel are smoothed out by sampling a C^∞ function then the resulting function has nearly exponential (faster than any polynomial) rate of decay away from the main diagonal. This result is used for the construction of tight polynomial frames for $L^2(w_{\alpha,\beta})$ with elements having almost exponential localization.

1. INTRODUCTION

A basic technique in Harmonic analysis is to represent functions or distributions as linear combinations of functions of a particularly simple nature (building blocks), which form bases or frames. Meyer's wavelets [6] and the φ-transform of Frazier and Jawerth [3] provide such building blocks on $\mathbb{R}^d$. A distinctive feature of Meyer's wavelets and the elements of Frazier-Jawerth is their almost exponential space localization and simple structure on the frequency side. This makes them an universal tool for decomposition of spaces of functions and distributions on $\mathbb{R}^d$.

Our primary goal in this article is to develop similar building blocks for decomposition of weighted spaces on $[-1, 1]$ with weight

$$w_{\alpha,\beta}(t) = (1 - t)^\alpha(1 + t)^\beta,$$

and in particular, $L^p(w_{\alpha,\beta})$, Hardy spaces $H^p(w_{\alpha,\beta})$, Besov spaces, and more general Triebel-Lizorkin spaces. The structure of the weighted spaces on $[-1, 1]$ is different and more complicated than the structure of the spaces on $\mathbb{R}$ due to the fact that there are no simple translation or dilation operators in these spaces. This creates a great deal of complications.

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^\infty$ provide a basic vehicle for representation and analysis of functions or distributions in the weighted spaces on $[-1, 1]$ with weight $w_{\alpha,\beta}$. We let $c_{\alpha,\beta}$ denote the normalization constant of $w_{\alpha,\beta}$, i.e. $c_{\alpha,\beta}^{-1} := \int_{-1}^1 w_{\alpha,\beta}(t)dt$. The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^\infty$ are orthogonal with respect to $w_{\alpha,\beta}$, namely [8],

$$c_{\alpha,\beta} \int_{-1}^1 P_n^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) dt = \delta_{n,m} h_n^{(\alpha,\beta)},$$

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where
$$h_n^{(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

For $f \in L^1(w_{\alpha,\beta})$ the Fourier expansion in Jacobi polynomials is

$$f(t) \sim \sum_{n=0}^{\infty} d_n(f)(h_n^{(\alpha,\beta)})^{-1} P_n^{(\alpha,\beta)}(t), \quad d_n(f) = c_{\alpha,\beta} \int_{-1}^{1} f(t) P_n^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) dt.$$

The nth partial sum of the expansion can be written in terms of the reproducing kernel $K_n^{(\alpha,\beta)}(x,y)$ as

$$S_n(f;x) = \sum_{j=0}^{n} d_j(f)(h_j^{(\alpha,\beta)})^{-1} P_j^{(\alpha,\beta)}(x) = c_{\alpha,\beta} \int_{-1}^{1} f(t) K_n^{(\alpha,\beta)}(x,t) w_{\alpha,\beta}(t) dt,$$

where the kernel is given by

$$(1.1) \qquad K_n^{(\alpha,\beta)}(x,y) = \sum_{j=0}^{n} \left(h_j^{(\alpha,\beta)} \right)^{-1} P_j^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y).$$

One of our main results in this article asserts that any polynomial in two variables of the form

$$(1.2) \qquad L_n^{\alpha,\beta}(x,y) = \sum_{j=0}^{\infty} \widehat{a}\left(\frac{j}{n}\right) \left(h_j^{(\alpha,\beta)} \right)^{-1} P_j^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y),$$

where $\widehat{a} \in C^{\infty}[0,\infty)$ with supp $\widehat{a} \subset [1/2, 2]$ (or more generally supp $\widehat{a} \subset [c,d]$, $c > 0$) has almost exponential localization around the main diagonal $y = x$ of $[-1,1]^2$. To be more precise, let

$$w_{\alpha,\beta}(n;x) := (1 - x + n^{-2})^{\alpha+1/2}(1 + x + n^{-2})^{\beta+1/2}.$$

Then for $\alpha, \beta > -1/2$ and any $k \geq 1$, there is a constant $c_k > 0$ such that

$$(1.3) \qquad |L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq c_k \frac{n}{\sqrt{w_{\alpha,\beta}(n;\cos\theta)}\sqrt{w_{\alpha,\beta}(n;\cos\phi)}(1 + n|\theta - \phi|)^k}$$

for $0 \leq \theta, \phi \leq \pi$. This result is a far reaching extension of the recent discovery in [7] that $L_n^{\lambda,\lambda}(x,1)$ (with λ a half integer) has almost exponential localization around $x = 1$, which is utilized in [7] for the construction of frames on the n dimensional sphere. The above result shows that if one smooths out the coefficients of the kernel in (1.1) by sampling a C^{∞} function then the resulting function decays away from the main diagonal at nearly exponential (faster than any polynomial) rate. This fact, which is well known for the trigonometric system, has apparently been overlooked. We believe that it will play an important role in various problems for weighted spaces.

The polynomials $L_n^{\alpha,\beta}$ give us a handy tool for constructing tight frames in $L^2(w_{\alpha,\beta})$ and other weighted spaces (with weight $w_{\alpha,\beta}$) on $[-1,1]$. Our construction of frames utilizes a semi-discrete Calderon type decomposition coupled with discretization using the Gaussian quadrature formula (see §3). A similar scheme is used in [7] for the construction of frames on the sphere.

Let us denote by ψ_ξ, $\xi \in \mathcal{X}$, the constructed frame elements (§3), where $\mathcal{X} = \cup_{j=0}^{\infty} \mathcal{X}_j$ is a multilevel index set consisting of the localization points (poles) of the ψ_ξ's. Then our frame is defined by

$$\Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}.$$

We show that every function $f \in L^2(w_{\alpha,\beta})$ has the representation

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{and} \quad \|f\|_{L^2(w_{\alpha,\beta})} = \Big(\sum_{\xi \in \mathcal{X}} |\langle f, \psi_\xi \rangle|^2 \Big)^{1/2},$$

i.e. Ψ is a tight frame in $L^2(w_{\alpha,\beta})$.

A distinctive property of the jth level frame elements ψ_ξ ($\xi \in \mathcal{X}_j$) is their localization:

$$(1.4) \qquad |\psi_\xi(\cos\theta)| \le c_k \frac{2^{j/2}}{\sqrt{w_{\alpha,\beta}(2^j; \cos\theta)}(1 + 2^j|\theta - \arccos\xi|)^k} \qquad \forall k,$$

which is analogous to the localization of Meyer's wavelets. It is worthwhile to point out that the factor $\sqrt{w_{\alpha,\beta}(2^j; \cos\theta)}$ in the denominator in (1.4) as well as the corresponding terms in (1.3) reflect the expected "effect" at the end points of $[-1, 1]$ and play a critical role in our development. Notice that this term is present even in the case when $w_{\alpha,\beta}(t) = 1$ ($\alpha = \beta = 0$).

The superb localization of the frame elements $\{\psi_\xi\}$ prompted us to term them **needlets**. This along with the semi-orthogonal structure of Ψ and the increasing number of vanishing moments of the ψ_ξ's enables one to utilize the needlet system Ψ for decomposition of spaces other than $L^2(w_{\alpha,\beta})$ such as Besov and Triebel-Lizorkin spaces. We shall report on results of this kind in a follow-up paper.

We do not discuss here the literature on polynomial bases and frames on the interval because to the best of our knowledge the elements of the existing bases or frames do not have the localization of the needlets. It is an open problem to construct bases in $L^2(w_{\alpha,\beta})$ with basis elements having localization similar to (1.4).

This article is organized as follows. In §2, we establish the localization (1.3) of the polynomials $L_n^{\alpha,\beta}$ from (1.2), where in §2.1 we consider the the particular case $\phi = 0$ ($y = 1$) , while in §2.2 we prove (1.3) in general. In §3, we construct our polynomial frames and establish their main properties.

Throughout the article positive constants are denoted by $c, c_1, \ldots$; unless specified, their values may vary at every occurrence. The notation $A \sim B$ means $c_1 A \le B \le c_2 A$, and $A := B$ or $B =: A$ stands for "A is by definition equal to B".

2. Localized polynomials in terms of Jacobi polynomials

In this section we establish the localization properties of $L_n^{\alpha,\beta}(x, y)$ from (1.2) depending on the smoothness of $\widehat{a}$. We first prove (1.3) when $\phi = 0$ ($y = 1$) and then consider the general case.

2.1. The localization of $L_n^{\alpha,\beta}(x, y)$ in the case $y = 1$.

Theorem 2.1. *Let $\widehat{a} \in C^k[0, \infty)$ with $k \ge 1$ and $\operatorname{supp}\widehat{a} \subset [1/2, 2]$. Assume that $\alpha \ge \beta > -1/2$ and define*

$$(2.1) \qquad L_n^{\alpha,\beta}(\cos\theta) = \sum_{j=0}^{\infty} \widehat{a}\left(\frac{j}{n}\right)\left(h_j^{(\alpha,\beta)}\right)^{-1} P_j^{(\alpha,\beta)}(1)P_j^{(\alpha,\beta)}(\cos\theta).$$

Then there exists a constant $c_k > 0$ depending only on k, α, β, and $\widehat{a}$ such that

$$(2.2) \qquad |L_n^{\alpha,\beta}(\cos\theta)| \le c_k \frac{n^{2\alpha+2}}{(1+n\theta)^{k+\alpha-\beta}}, \quad 0 \le \theta \le \pi.$$

The dependence of c_k on $\widehat{a}$ is of the form $c_k = c(\alpha, \beta, k)\max_{1 \le \nu \le k}\|\widehat{a}^{(\nu)}\|_{L^1}$.

Proof. Since $P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n! = \Gamma(n+\alpha+1)/[\Gamma(\alpha+1)\Gamma(n+1)]$ [8, (4.1.1)], it is easy to verify that

$$(2.3) \qquad L_n^{\alpha,\beta}(\cos\theta) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

$$\times \sum_{j=0}^{\infty} \widehat{a}\left(\frac{j}{n}\right) \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha,\beta)}(\cos\theta).$$

Notice that since $\operatorname{supp}\widehat{a} \subset [1/2, 2]$, the sum in (2.1) (and (2.3)) is finite; in fact it is over $n/2 < j < 2n$.

We first prove (2.2) for $0 \le \theta \le \pi/n$. Using the fact that [8, Theorem 7.32.1] $\|P_j^{(\alpha,\beta)}\|_{L^\infty[-1,1]} \sim cj^\alpha$ and $\Gamma(j+a)/\Gamma(j+1) \sim j^{a-1}$ as $j \to \infty$, it follows that

$$|L_n^{\alpha,\beta}(\cos\theta)| \le c\sum_{j=0}^{2n} j^{2\alpha+1} \le cn^{2\alpha+2},$$

which yields (2.2).

Let $\pi/n \le \theta \le \pi$. We will use the identity [4]:

$$\frac{P_m^{(\alpha,\beta)}(\cos\theta)}{P_m^{(\alpha,\beta)}(1)} = d_{\alpha,\beta}(1-\cos\theta)^{-\alpha} \int_0^\theta \cos\left[m+(\alpha+\beta+1)/2\right]\phi$$

$$\times \frac{(\cos\phi - \cos\theta)^{\alpha-1/2}}{(1+\cos\phi)^{(\alpha+\beta)/2}} \, {}_2F_1\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\theta - \cos\phi}{1-\cos\theta}\right) d\phi,$$

for $0 < \theta < \pi$ and $\alpha > -1/2$, where ${}_2F_1$ denotes the hypergeometric function and $d_{\alpha,\beta} = 2^{(\alpha+\beta+1)/2}\Gamma(\alpha+1)/(\sqrt{\pi}\Gamma(\alpha+1/2))$. Recall the identity [8, (4.1.3)] $P_m^{(\alpha,\beta)}(x) = (-1)^m P_m^{(\beta,\alpha)}(-x)$. Now changing first the variables $\phi \mapsto \pi - \phi$ in the integral, and then setting $\theta \mapsto \pi - \theta$ and interchanging the places of α and β, we obtain

$$(2.4) \qquad \frac{P_m^{(\alpha,\beta)}(\cos\theta)}{P_m^{(\beta,\alpha)}(1)} = d_{\beta,\alpha}(1+\cos\theta)^{-\beta} \int_\theta^\pi \cos\left[m\phi - (\alpha+\beta+1)(\pi-\phi)/2\right]$$

$$\times \frac{(\cos\theta - \cos\phi)^{\beta-1/2}}{(1-\cos\phi)^{(\alpha+\beta)/2}} \, {}_2F_1\left(\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}; \beta+\frac{1}{2}; \frac{\cos\phi - \cos\theta}{1-\cos\theta}\right) d\phi.$$

Since $P_j^{(\beta,\alpha)}(1) = \Gamma(j+\beta+1)/[\Gamma(\beta+1)\Gamma(j+1)]$, we have

$$P_j^{(\beta,\alpha)}(1)\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} = \frac{1}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(j+1)}.$$

Using this and combining (2.3) with (2.4), we get

$$L_n^{\alpha,\beta}(\cos\theta) = d_{\alpha,\beta}^*(1+\cos\theta)^{-\beta} \int_\theta^\pi \left[\cos(\lambda\phi - \lambda\pi)A_n^{\cos}(\phi) - \sin(\lambda\phi - \lambda\pi)A_n^{\sin}(\phi)\right]$$

$$(2.5)$$

$$\times \frac{(\cos\theta - \cos\phi)^{\beta-1/2}}{(1-\cos\phi)^{(\alpha+\beta)/2}} \, {}_2F_1\left(\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}; \beta+\frac{1}{2}; \frac{\cos\phi - \cos\theta}{1-\cos\theta}\right) d\phi,$$

where $d_{\alpha,\beta}^* := 2d_{\beta,\alpha}/\Gamma(\alpha+\beta+1)$ and

$$A_n^{\cos}(\theta) := \sum_{j=0}^{\infty} \widehat{a}\left(\frac{j}{n}\right) \frac{(j+\lambda)\Gamma(j+2\lambda)}{\Gamma(j+1)} \cos j\theta,$$

$$A_n^{\sin}(\theta) := \sum_{j=0}^{\infty} \widehat{a}\left(\frac{j}{n}\right) \frac{(j+\lambda)\Gamma(j+2\lambda)}{\Gamma(j+1)} \sin j\theta$$

with $\lambda := (\alpha + \beta + 1)/2$.

The idea of the proof is to derive (2.2) from the analogous localization of the trigonometric polynomials $A_n^{\cos}$ and $A_n^{\sin}$. This in turn will follow by employing the fact that any trigonometric polynomial $A_n(\theta) = \sum_{j=1}^{\infty} a_j e^{ij\theta}$ has an excellent localization around zero whenever the coefficients a_j come from sampling of a smooth compactly supported function.

Define

$$G(t) := \frac{(t+\lambda)\Gamma(t+2\lambda)}{\Gamma(t+1)} \quad \text{if } t \geq 0.$$

Notice that $2\lambda = \alpha + \beta + 1 > 0$, since $\alpha \geq \beta > -1/2$.

With the next lemma we establish the smoothness and decay as $t \to \infty$ of G and its derivatives.

Lemma 2.2. *The function G is analytic on $[0, \infty)$ and for any $k \geq 0$*

$$(2.6) \qquad\qquad |G^{(k)}(t)| \leq c_k t^{2\lambda - k}, \quad t \geq 1,$$

where c_k depends only on k and λ.

Proof. We first show that for $b > a > 0$, $\sigma := a + 1 - b$, and $t > 0$,

$$(2.7) \qquad \frac{d^k}{dt^k} \frac{\Gamma(t+a)}{\Gamma(t+b)} = (-1)^k B_0^{\sigma}(a) \frac{\Gamma(b-a+k)}{\Gamma(b-a)} \frac{1}{t^{b-a+k}} (1 + \mathcal{O}(t^{-1})) \quad \text{as } t \to \infty,$$

where $B_n^{\sigma}(u)$ (here $n = 0$) are the generalized Bernoulli polynomials defined in [1, (C.4.2)]. We will slightly modify the argument from [1].

The relation between Gamma and Beta functions gives

$$\frac{\Gamma(t+a)}{\Gamma(t+b)} = \frac{1}{\Gamma(b-a)} \int_0^1 u^{t+a-1}(1-u)^{b-a-1} du$$

$$= \frac{1}{\Gamma(b-a)} \int_0^{\infty} e^{-s(t+a)}(1-e^{-s})^{b-a-1} ds.$$

It is readily seen that the last integral is an analytic function for $t \geq 0$ and it can be differentiated with respect to t as follows

$$\frac{d^k}{dt^k} \frac{\Gamma(t+a)}{\Gamma(t+b)} = \frac{(-1)^k}{\Gamma(b-a)} \int_0^{\infty} e^{-st} s^k e^{-sa}(1-e^{-s})^{b-a-1} ds.$$

Now exactly as in [1, C.4, p. 615],

$$\frac{s^k e^{-as}}{(1-e^{-s})^{\sigma}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n^{\sigma}(a) s^{n-\sigma+k}, \quad |t| < 2\pi,$$

and by Watson's lemma, we obtain the asymptotic expansion

$$\frac{d^k}{dt^k} \frac{\Gamma(t+a)}{\Gamma(t+b)} \sim \sum_{n=0}^{\infty} \frac{(-1)^{k+n}}{n!} B_n^{\sigma}(a) \frac{\Gamma(b-a+k+n)}{\Gamma(b-a)} \frac{1}{t^{b-a+k+n}} \quad \text{as } t \to \infty,$$

which implies (2.7).

Let $a := 2\lambda - \lfloor 2\lambda \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than x. Then $0 < a \le 1$. Using that $\Gamma(1+x) = x\Gamma(x)$, we have

$$
\begin{aligned}
G(t) &= \frac{(t+1)(t+\lambda)\Gamma(t+2\lambda)}{\Gamma(t+2)} \\
&= (t+1)(t+\lambda)(t+2\lambda-1)\dots(t+a)\frac{\Gamma(t+a)}{\Gamma(t+2)} = p(t)\frac{\Gamma(t+a)}{\Gamma(t+2)},
\end{aligned}
$$

where $p(t)$ is a polynomial of degree $\lfloor 2\lambda \rfloor + 2$. Now by (2.7),

$$
\left| \frac{d^\nu}{dt^\nu}\frac{\Gamma(t+a)}{\Gamma(t+2)} \right| \le \frac{c}{t^{2-a+\nu}}, \quad t \ge 1, \quad \nu = 0, 1, \dots,
$$

and hence

$$
\begin{aligned}
|G^{(k)}(t)| &\le \sum_{\nu=0}^{k} \binom{k}{\nu} |p^{(\nu)}(t)| \left| \frac{d^{k-\nu}}{dt^{k-\nu}}\frac{\Gamma(t+a)}{\Gamma(t+2)} \right| \\
&\le c\sum_{\nu=0}^{k} \binom{k}{\nu} \frac{t^{\lfloor 2\lambda \rfloor + 2 - \nu}}{t^{2-a+k-\nu}} \le ct^{2\lambda-k}.
\end{aligned}
$$

Thus (2.6) is established. $\square$

Now relying on the smoothness and localization properties of $\widehat{a}$ and G, we derive the desired localization of $A_n^{\cos}$ and $A_n^{\sin}$.

Lemma 2.3. *If $A_n = A_n^{\cos}$ or $A_n = A_n^{\sin}$, then*

$$
(2.8) \qquad |A_n(\theta)| \le c_k \frac{n^{2\lambda+1}}{(1+n|\theta|)^k}, \quad |\theta| \le \pi,
$$

where c_k is of the form $c_k = c(\alpha, \beta, k) \max_{1 \le \nu \le k} \|\widehat{a}^{(\nu)}\|_{L^1}$ with $c(\alpha, \beta, k)$ depending only on α, β, and k.

Proof. Let $n \ge 2$ and set

$$
\Lambda_n(\theta) := \sum_{j\in\mathbb{Z}} \widehat{a}(|j|/n)s(j)G(|j|)e^{ij\theta},
$$

where $s(t) = 1$ or $s(t) = \operatorname{sign} t$. Evidently, $\Lambda_n(\theta) = 2A_n^{\cos}(\theta)$ if $s(t) = 1$ and $\Lambda_n(\theta) = 2iA_n^{\sin}(\theta)$ if $s(t) = \operatorname{sign} t$.

Set $\widehat{\Phi}_n(t) := \widehat{a}(|t|/n)s(t)G(|t|)$ and define $\Phi_n(t) := \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_n(\xi)e^{it\xi}d\xi$. Clearly, $\widehat{\Phi}_n(t) \in C^k(\mathbb{R})$ and $\operatorname{supp}\widehat{\Phi}_n \subset [-2n, -n/2] \cup [n/2, 2n]$. Consequently,

$$
t^k\Phi_n(t) = \frac{i^k}{2\pi}\int_{\mathbb{R}} \frac{d^k}{d\xi^k}\widehat{\Phi}_n(\xi)e^{it\xi}d\xi
$$

and using (2.6),

$$|t^k \Phi_n(t)| \le \int_{\mathbb{R}} \left| \frac{d^k}{d\xi^k} [\widehat{a}(\xi/n) G(\xi)] \right| d\xi$$

$$\le \sum_{j=0}^{k} \binom{k}{j} \frac{1}{n^j} \int_{\mathbb{R}} |\widehat{a}^{(j)}(\xi/n) G^{(k-j)}(\xi)| d\xi$$

$$= 2 \sum_{j=0}^{k} \binom{k}{j} \frac{n}{n^j} \int_{1/2}^{2} |\widehat{a}^{(j)}(u) G^{(k-j)}(nu)| du$$

$$\le c n^{2\lambda-k+1} \sum_{j=0}^{k} \binom{k}{j} \int_{1/2}^{2} |\widehat{a}^{(j)}(u)| u^{2\lambda+j-k} du$$

$$\le c n^{2\lambda-k+1} \max_{0 \le j \le k} \|\widehat{a}^{(j)}\|_{L^1} = c_k n^{2\lambda-k+1}.$$

Therefore, $|\Phi_n(t)| \le c_k n^{2\lambda-k+1} |t|^{-k}$. As above (with $k=0$) but easier, we obtain $|\Phi_n(t)| \le c n^{2\lambda+1}$. Combining these, we get

$$(2.9) \qquad |\Phi_n(t)| \le c_k \frac{n^{2\lambda+1}}{(1+n|t|)^k}, \quad t \in \mathbb{R}.$$

Applying the Poisson summation formula

$$2\pi \sum_{j \in \mathbb{Z}} f(2\pi j) = \sum_{j \in \mathbb{Z}} \widehat{f}(j)$$

to the function $f(t) := \Phi_n(t+\theta)$ (then $\widehat{f}(t) = \widehat{\Phi}_n(t) e^{it\theta}$), we get

$$\Lambda_n(\theta) = \sum_{j \in \mathbb{Z}} \widehat{\Phi}_n(j) e^{ij\theta} = 2\pi \sum_{j \in \mathbb{Z}} \Phi_n(2\pi j + \theta).$$

Now using (2.9), we obtain for $|\theta| \le \pi$,

$$|\Lambda_n(\theta)| = 2\pi \left| \sum_{j \in \mathbb{Z}} \Phi_n(2\pi j + \theta) \right| \le c \sum_{j \in \mathbb{Z}} \frac{n^{2\lambda+1}}{(1+n|2\pi j + \theta|)^k}$$

$$\le c \frac{n^{2\lambda+1}}{(1+n|\theta|)^k} + c \sum_{j=1}^{\infty} \frac{n^{2\lambda+1}}{(1+n(2j-1)\pi)^k}$$

$$\le c \frac{n^{2\lambda+1}}{(1+n|\theta|)^k} + c \frac{n^{2\lambda+1}}{(1+n)^k} \sum_{j=1}^{\infty} j^{-k}$$

$$\le c \frac{n^{2\lambda+1}}{(1+n|\theta|)^k}$$

and estimate (2.8) follows. $\qquad \square$

We are now in a position to complete the proof of estimate (2.2) in the case $\pi/n \le \theta \le \pi$. Since $0 \le (\cos\theta - \cos\phi)/(1 - \cos\theta) \le 1$ for $\theta \le \phi \le \pi$, the absolute value of the $_2F_1$ term under the integral in (2.5) is bounded by a constant. Hence, using (2.5) and Lemma 2.3, we infer

$$(2.10) \qquad |L_n^{\alpha,\beta}(\cos\theta)| \le c(1+\cos\theta)^{-\beta} \int_{\theta}^{\pi} \frac{n^{\alpha+\beta+2}}{(1+n\phi)^k} \frac{(\cos\theta - \cos\phi)^{\beta-1/2}}{(1-\cos\phi)^{(\alpha+\beta)/2}} d\phi.$$

In the following we will use the identities $\cos\theta - \cos\phi = 2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$, $1-\cos\phi = 2\sin^2\frac{\phi}{2}$, and $1+\cos\phi = 2\cos^2\frac{\phi}{2}$.

Consider first the case $\pi/n \le \theta \le \pi/2$. Then for $\theta \le \phi \le \pi$,

$$\frac{2\sqrt{2}}{3\pi}(\theta+\phi) \le \sin\frac{\theta+\phi}{2} \le \frac{\theta+\phi}{2} \quad \text{and} \quad \frac{\phi-\theta}{\pi} \le \sin\frac{\phi-\theta}{2} \le \frac{\phi-\theta}{2}.$$

Using these in (2.10), we get

$$\left|L_n^{\alpha,\beta}(\cos\theta)\right| \le cn^{\alpha+\beta+2}\int_\theta^\pi \frac{(\phi^2-\theta^2)^{\beta-1/2}}{(1+n\phi)^k\phi^{\alpha+\beta}}d\phi$$

$$\le cn^{\alpha+\beta+2}\frac{\theta^{2\beta}}{(n\theta)^k\theta^{\alpha+\beta}}\int_1^{\frac{\pi}{\theta}} \frac{(u^2-1)^{\beta-1/2}}{u^{k+\alpha+\beta}}du$$

$$\le c\frac{n^{2\alpha+2}}{(n\theta)^{k+\alpha-\beta}}\int_1^\infty \frac{(u^2-1)^{\beta-1/2}}{u^{k+\alpha+\beta}}du \le c\frac{n^{2\alpha+2}}{(1+n\theta)^{k+\alpha-\beta}}.$$

In the case $\pi/2 \le \theta \le \pi$, estimate (2.10) evidently gives

$$\left|L_n^{\alpha,\beta}(\cos\theta)\right| \le c\frac{n^{\alpha+\beta+2}}{(1+n\theta)^k}(1+\cos\theta)^{-\beta}\int_\theta^\pi \frac{(\cos\theta-\cos\phi)^{\beta-1/2}}{(1-\cos\phi)^{\alpha+\beta}}d\phi.$$

Let $\theta' = \pi - \theta$. Then $0 \le \theta' \le \pi/2$. Hence, for $0 \le \phi \le \theta'$, $(\theta'\pm\phi)/\pi \le \sin[(\theta'\pm\phi)/2] \le (\theta'\pm\phi)/2$. Consequently, substituting $\phi \mapsto \pi - \phi$ gives

$$\int_\theta^\pi \frac{(\cos\theta-\cos\phi)^{\beta-1/2}}{(1-\cos\phi)^{\alpha+\beta}}d\phi = \int_0^{\theta'} \frac{(\cos\phi-\cos\theta')^{\beta-1/2}}{(1+\cos\phi)^{\alpha+\beta}}d\phi$$

$$\le c\int_0^{\theta'} (\theta'^2-\phi^2)^{\beta-1/2}d\phi = c\theta'^{2\beta}.$$

Since $1+\cos\theta = 1-\cos\theta' = 2\sin^2(\theta'/2) \sim (\theta')^2$, this shows that

$$\left|L_n^{\alpha,\beta}(\cos\theta)\right| \le c\frac{n^{\alpha+\beta+2}}{(1+n\theta)^k} \le c\frac{n^{2\alpha+2}}{(1+n\theta)^{k+\alpha-\beta}}$$

as $\pi/2 \le \theta \le \pi$. The proof of Theorem 2.1 is complete. $\qquad\square$

2.2. The localization of $L_n^{\alpha,\beta}(x,y)$ in the general case.

Recall the definition of $L_n^{\alpha,\beta}(x,y)$ from (1.2):

$$L_n^{\alpha,\beta}(x,y) := \sum_{j=0}^\infty \widehat{a}\left(\frac{j}{n}\right)\left(h_j^{(\alpha,\beta)}\right)^{-1} P_j^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(y).$$

In this subsection we estimate the localization of $L_n^{\alpha,\beta}(x,y)$ around the main diagonal $y = x$ of $[-1,1]^2$, which depends on the smoothness of $\widehat{a}$. To this end we will need the quantity

$$(2.11) \quad w_{\alpha,\beta}(n;x) := (1-x+n^{-2})^{\alpha+1/2}(1+x+n^{-2})^{\beta+1/2}, \qquad -1 \le x \le 1.$$

Notice that since $\sin\frac{\theta}{2} \sim \sin\theta$ on $[0, 2\pi/3]$ and $\cos\frac{\theta}{2} \sim \sin\theta$ on $[\pi/3, \pi]$, then

$$(2.12) \quad w_{\alpha,\beta}(n;\cos\theta) \sim (\sin\theta + n^{-1})^{2\alpha+1}, \quad 0 \le \theta \le 2\pi/3,$$

and

$$(2.13) \quad w_{\alpha,\beta}(n;\cos\theta) \sim (\sin\theta + n^{-1})^{2\beta+1}, \quad \pi/3 \le \theta \le \pi.$$

Theorem 2.4. *Let* $\alpha, \beta > -1/2$ *and let* $\widehat{a} \in C^k[0, \infty)$ *with* $k \geq 2\alpha + 2\beta + 3$ *and* $\operatorname{supp} \widehat{a} \subset [1/2, 2]$. *Then there is a constant* $c_k > 0$ *depending only on* k, α, β, *and* $\widehat{a}$ *such that for* $0 \leq \theta, \phi \leq \pi$

$$(2.14) \quad |L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq c_k \frac{n}{\sqrt{w_{\alpha,\beta}(n; \cos\theta)}\sqrt{w_{\alpha,\beta}(n; \cos\phi)}(1 + n|\theta - \phi|)^\sigma},$$

where $\sigma = k - 2\alpha - 2\beta - 3$. *Here the dependence of* c_k *on* $\widehat{a}$ *is of the form* $c_k = c(\alpha, \beta, k) \max_{1 \leq \nu \leq k} \|\widehat{a}^{(\nu)}\|_{L^1}$.

Proof. We need the product formula of Jacobi polynomials [5]: For $\alpha > \beta > -1/2$,

$$\frac{P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} = c_{\alpha,\beta} \int_0^\pi \int_0^1 P_n^{(\alpha,\beta)}(t(x, y, r, \psi)) dm(r, \psi),$$

where

$$t(x, y, r, \psi) = \tfrac{1}{2}(1 + x)(1 + y) + \tfrac{1}{2}(1 - x)(1 - y)r^2 + r\sqrt{1 - x^2}\sqrt{1 - y^2}\cos\psi - 1,$$

the integral is against

$$dm(r, \psi) = (1 - r^2)^{\alpha - \beta - 1} r^{2\beta + 1}(\sin\psi)^{2\beta} dr d\psi,$$

and the constant $c_{\alpha,\beta}$ is selected so that

$$c_{\alpha,\beta} \int_0^\pi \int_0^1 1 \, dm(r, \psi) = 1.$$

Once (2.14) is established for $\alpha > \beta > -1/2$, the case $\beta > \alpha > -1/2$ will follow from the relation $P_n^{(\alpha,\beta)}(\cos\theta) = (-1)^n P_n^{(\beta,\alpha)}(\cos(\pi - \theta))$. In the case $\alpha = \beta = \lambda - 1/2$ we use the product formula of Gegenbauer polynomials [2, Vol I, Sec. 3.15.1, (20)]

$$\frac{C_j^\lambda(\cos\theta) C_j^\lambda(\cos\phi)}{C_j^\lambda(1)} = c_\lambda \int_{-1}^1 C_j^\lambda(\cos\theta \cos\phi + u \sin\theta \sin\phi)(1 - u^2)^{\lambda - 1} du,$$

where c_λ is the normalization constant of the weight function $(1 - u^2)^{\lambda - 1}$. Since the case $\alpha = \beta$ is much easier, we shall only give the proof when $\alpha > \beta$.

The product formula allows us to write $L_n^{\alpha,\beta}(\cos\theta, \cos\phi)$ in terms of $L_n^{\alpha,\beta}(t)$ defined in Theorem 2.1, namely,

$$(2.15) \qquad L_n^{\alpha,\beta}(x, y) = c_{\alpha,\beta} \int_0^\pi \int_0^1 L_n^{\alpha,\beta}(t(x, y, r, \psi)) dm(r, \psi).$$

For $t = \cos\theta$ with $0 \leq \theta \leq \pi$, we have $\theta \sim \sin\theta/2 \sim \sqrt{1 - t}$, and consequently the estimate of $|L_n^{\alpha,\beta}(t)|$ from Theorem 2.1 can be rewritten as

$$(2.16) \qquad \left| L_n^{\alpha,\beta}(t) \right| \leq c_k \frac{n^{2\alpha + 2}}{(1 + n\sqrt{1 - t})^{k + \alpha - \beta}}, \quad -1 \leq t \leq 1.$$

We will apply this estimate with $t = t(x, y, r, \psi)$. Let $x = \cos\theta$ and $y = \cos\phi$, $0 \leq \theta, \phi \leq \pi$. Then

$$
\begin{aligned}
1 - t(x, y, r, \psi) =\ & 1 - \cos\theta \cos\phi - \sin\theta \sin\phi \\
& + \tfrac{1}{2}(1 - \cos\theta)(1 - \cos\phi)(1 - r^2) + \sin\theta \sin\phi(1 - r\cos\psi) \\
=\ & 2\sin^2 \frac{\theta - \phi}{2} + 2\sin^2 \frac{\theta}{2}\sin^2 \frac{\phi}{2}(1 - r^2) + \sin\theta \sin\phi(1 - r\cos\psi),
\end{aligned}
$$

(2.17)

yielding

$$1 - t(x, y, r, \psi) \geq 2\sin^2 \frac{\theta - \phi}{2} \sim (\theta - \phi)^2.$$

Then by (2.15)-(2.16), we get

$$(2.18) \qquad |L_n^{\alpha,\beta}(\cos\theta,\cos\phi)| \leq \frac{cn^{2\alpha+2}}{(1+n|\theta-\phi|)^{k+\alpha-\beta}}, \quad 0\leq\theta,\phi\leq\pi.$$

Our next step is to show that this estimate can be substantially improved away from the end points of the interval.

Case (a): $0\leq\theta\leq 2\pi/3$ and $0\leq\phi\leq\pi/2$. By (2.17), we have

$$1 - t(x,y,r,\psi) \geq c(|\theta-\phi|^2 + \sin\theta\sin\phi(1 - r\cos\psi)).$$

Then by (2.15)-(2.16),

$$|L_n^{\alpha,\beta}(\cos\theta,\cos\phi)| \leq cn^{2\alpha+2}\int_0^\pi\int_0^1 \frac{(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}drd\psi}{\left(1+n[|\theta-\phi|^2+\sin\theta\sin\phi(1-r\cos\psi)]^{1/2}\right)^\sigma},$$

where $\sigma = k + \alpha - \beta$. Set $F(u) := 1/(1+n(|\theta-\phi|^2+\sin\theta\sin\phi(1-u))^{1/2})^\sigma$. In the following we evaluate the double integral by changing the variables several times:

$$\int_0^\pi\int_0^1 F(r\cos\psi)dm(r,\psi) = \int_0^1\int_{-1}^1 F(rs)(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(1-s^2)^{\beta-1/2}dsdr$$

$$= \int_0^1\int_{-r}^r F(u)r(1-r^2)^{\alpha-\beta-1}(r^2-u^2)^{\beta-1/2}dudr$$

$$= \int_{-1}^1 F(u)\int_{|u|}^1 r(1-r^2)^{\alpha-\beta-1}(r^2-u^2)^{\beta-1/2}drdu$$

$$= c\int_{-1}^1 F(u)(1-u^2)^{\alpha-1/2}du,$$

where $c = (1/2)\int_0^1 v^{\beta-1/2}(1-v)^{\alpha-\beta-1}dv$ (here $v = (r^2-u^2)/(1-u^2)$ is the last substitution). This gives

$$|L_n^{\alpha,\beta}(\cos\theta,\cos\phi)| \leq cn^{2\alpha+2}\int_{-1}^1 \frac{(1-u^2)^{\alpha-1/2}du}{(1+n[|\theta-\phi|^2+\sin\theta\sin\phi(1-u)]^{1/2})^\sigma}.$$

We now split up the last integral into two integrals: one over $[-1,0]$ and the other over $[0,1]$. For the first integral, $1 - u \geq 1$ and hence

$$\int_{-1}^0 F(u)(1-u^2)^{\alpha-1/2}du \leq \frac{c}{(1+n[|\theta-\phi|^2+\sin\theta\sin\phi]^{1/2})^\sigma}$$

$$\leq \frac{c}{(n^2\sin\theta\sin\phi)^{\alpha+1/2}(1+n|\theta-\phi|)^{k-\alpha-\beta-1}}.$$

To estimate the integral over $[0,1]$ we apply the substitution $t = 1 - u$ and obtain

$$\int_0^1 F(u)(1-u^2)^{\alpha-1/2}du \leq c\int_0^1 \frac{(1-u)^{\alpha-1/2}du}{(1+n[|\theta-\phi|^2+\sin\theta\sin\phi(1-u)]^{1/2})^\sigma}$$

$$\leq \frac{c}{(n^2\sin\theta\sin\phi)^{\alpha+1/2}}\int_0^{n^2\sin\theta\sin\phi} \frac{t^{\alpha-1/2}dt}{(1+[n^2(\theta-\phi)^2+t]^{1/2})^\sigma}$$

$$\leq \frac{c}{(n^2\sin\theta\sin\phi)^{\alpha+1/2}(1+n|\theta-\phi|)^{\sigma-2\alpha-2}}\int_0^\infty \frac{t^{\alpha-1/2}dt}{(1+t)^{\alpha+1}}$$

$$\leq \frac{c}{(n^2\sin\theta\sin\phi)^{\alpha+1/2}(1+n|\theta-\phi|)^{k-\alpha-\beta-2}}.$$

Putting these estimates together, we get

$$(2.19) \qquad |L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq \frac{cn}{(\sin\theta\sin\phi)^{\alpha+1/2}(1+n|\theta-\phi|)^{k-\alpha-\beta-2}}.$$

Our last step here is to show that estimates (2.18)-(2.19) imply (2.14). Observe first that since $0 \leq \theta, \phi \leq 2\pi/3$, we have by (2.12),

$$(2.20) \qquad w_{\alpha,\beta}(n; \cos\theta)w_{\alpha,\beta}(n; \cos\phi) \sim (\sin\theta + n^{-1})^{2\alpha+1}(\sin\phi + n^{-1})^{2\alpha+1}$$

$$\sim (\theta + n^{-1})^{2\alpha+1}(\phi + n^{-1})^{2\alpha+1}.$$

We need the following simple inequality

$$(2.21) \qquad (\theta + n^{-1})(\phi + n^{-1}) \leq 3(\theta\phi + n^{-2})(1 + n|\theta - \phi|), \quad \theta, \phi \geq 0.$$

To prove this we assume that $\phi \geq \theta$ and define $\gamma \geq 1$ from $\phi = \gamma\theta$. Suppose $\gamma \geq 3$. Then (2.21) will follow if we show that $(\theta + \phi)n^{-1} \leq 3n^{-2}n|\theta - \phi|$, which is equivalent to $\gamma + 1 \leq 3(\gamma - 1)$. But this holds since $\gamma \geq 3$.

Let now $1 \leq \gamma < 3$. Then it suffices to show that $(\theta + \phi)n^{-1} \leq 2(\theta\phi + n^{-2})$. In turn this inequality holds if $4\theta n^{-1} \leq 2(\theta^2 + n^{-2})$, which is apparently true. Thus (2.21) is established.

It is now readily seen that (2.18)-(2.19) yield (2.14) by using (2.20)-(2.21).

Case (b): $\pi/3 \leq \theta \leq \pi$ and $\pi/2 \leq \phi \leq \pi$. In this case, we have by (2.17)

$$(2.22) \qquad 1 - t(x, y, r, \psi) \geq c(|\theta - \phi|^2 + (1 - r) + r(1 - \cos\psi)\sin\theta\sin\phi)$$

upon writing $1 - r\cos\psi = 1 - r + r(1 - \cos\psi)$. In particular,

$$1 - t(x, y, r, \psi) \geq c(|\theta - \phi|^2 + (1 - r)),$$

which shows, using (2.15)-(2.16) , that

$$|L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq cn^{2\alpha+2} \int_0^1 \frac{(1 - r^2)^{\alpha-\beta-1}r^{2\beta+1}dr}{\left(1 + n[|\theta - \phi|^2 + (1 - r)]^{1/2}\right)^\sigma},$$

with $\sigma := k + \alpha - \beta$ as before. Applying the substitution $n^2(1 - r) = s$, we get

$$(2.23) \quad |L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq cn^{2\alpha+2} \int_0^1 \frac{(1 - r)^{\alpha-\beta-1}dr}{\left(1 + n(|\theta - \phi|^2 + (1 - r))^{1/2}\right)^\sigma}$$

$$\leq cn^{2\beta+2} \int_0^{n^2} \frac{s^{\alpha-\beta-1}ds}{\left(1 + (n^2|\theta - \phi| + s)^{1/2}\right)^\sigma}$$

$$\leq c\frac{n^{2\beta+2}}{(1 + n|\theta - \phi|)^{\sigma-2(\alpha-\beta)-1/2}} \int_0^\infty \frac{s^{\alpha-\beta-1}ds}{(1 + s)^{\alpha-\beta+1/4}}$$

$$\leq c\frac{n^{2\beta+2}}{(1 + n|\theta - \phi|)^{k-\alpha+\beta-1/2}}.$$

Furthermore, using (2.15)-(2.16) and (2.22), we obtain

$$|L_n^{\alpha,\beta}(\cos\theta, \cos\phi)|$$

$$\leq cn^{2\alpha+2} \int_0^1 \int_0^\pi \frac{(1 - r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}d\psi dr}{\left(1 + n[|\theta - \phi|^2 + (1 - r) + r(1 - \cos\psi)\sin\theta\sin\phi]^{1/2}\right)^\sigma}.$$

The inner integral can be estimated by changing the variables. Set $A := |\theta - \phi|^2 + 1 - r$. Then

$$
\int_0^{\pi/2} = \int_0^1 \frac{(1-s)^{\beta-1/2}ds}{(1 + n[A + r(1-s)\sin\theta\sin\phi]^{1/2})^\sigma}
$$

$$
= \frac{1}{(n^2 r \sin\theta\sin\phi)^{\beta+1/2}} \int_0^{n^2 r \sin\theta\sin\phi} \frac{t^{\beta-1/2}dt}{(1 + (n^2 A + t)^{1/2})^\sigma}
$$

$$
\leq \frac{1}{(n^2 r \sin\theta\sin\phi)^{\beta+1/2}(1 + nA^{1/2})^{\sigma-2\beta-2}} \int_0^\infty \frac{t^{\beta-1/2}dt}{(1+t)^{\beta+1}}
$$

$$
\leq \frac{c}{(n^2 r \sin\theta\sin\phi)^{\beta+1/2}(1 + nA^{1/2})^{k+\alpha-3\beta-2}},
$$

and since $1 - s \geq 1$ for $-1 \leq s \leq 0$,

$$
\int_{\pi/2}^\pi = \int_{-1}^0 \frac{(1-s)^{\beta-1/2}ds}{(1 + n(A + r(1-s)\sin\theta\sin\phi)^\sigma} \leq \frac{c}{(1 + n(A + r\sin\theta\sin\phi)^{1/2})^\sigma}
$$

$$
\leq \frac{c}{(n^2 r \sin\theta\sin\phi)^{\beta+1/2}(1 + nA^{1/2})^{k+\alpha-3\beta-2}}.
$$

Combining these, we get

$$
|L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq c \frac{n^{2\alpha+2}}{(n^2 \sin\theta\sin\phi)^{\beta+1/2}} \int_0^1 \frac{(1-r^2)^{\alpha-\beta-1}r^{\beta+1/2}dr}{\left(1 + n(|\theta-\phi|^2 + (1-r))^{1/2}\right)^\tau},
$$

where $\tau := k + \alpha - 3\beta - 2$. The last integral (with σ in place of τ) has already been estimated in (2.23). Consequently,

$$
|L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq \frac{cn^{2\beta+2}}{(n^2 \sin\theta\sin\phi)^{\beta+1/2}(1 + n|\theta-\phi|)^{k-\alpha-\beta-5/2}}.
$$

As in Case (a), this estimate along with (2.23) implies (2.14), using (2.13) and (2.21).

Case (c): Either $0 \leq \theta \leq \pi/3$ and $\pi/2 \leq \phi \leq \pi$ or $2\pi/3 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi/2$. In this case, we have $|\theta - \phi| \geq \pi/6$, so that

$$
1 - t(x, y, r, \psi) \geq c|\theta - \phi|^2 \geq c > 0.
$$

Hence, by (2.15)-(2.16),

$$
|L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq \frac{cn^{2\alpha+2}}{(1+n)^{k+\alpha-\beta}} \leq \frac{cn}{(1+n)^{k-\alpha-\beta-1}},
$$

which yields (2.14) since $w_{\alpha,\beta}(n; \cos\theta)w_{\alpha,\beta}(n; \cos\phi) \leq c$. The proof of Theorem 2.4 is complete. $\square$

The estimate of $|L_n^{\alpha,\beta}(x, y)|$ from Theorem 2.4 allows us to control its L^p integral.

Proposition 2.5. *Let $k \geq 3\alpha + 3\beta + 5$. Then for $1 \leq p < \infty$,*

$$
(2.24) \qquad \int_{-1}^1 |L_n^{\alpha,\beta}(x, y)|^p w_{\alpha,\beta}(y)dy \leq c \frac{n^{p-1}}{(w_{\alpha,\beta}(n; x))^{p-1}}, \quad -1 \leq x \leq 1.
$$

Proof. We will prove (2.24) only in the case $0 \leq x \leq 1$, since the case $-1 \leq x < 0$ is the same. Let $x =: \cos\theta$ and denote $I_{[a,b]} := \int_a^b |L_n^{\alpha,\beta}(x,y)|^p w_{\alpha,\beta}(y) dy$. We first estimate $I_{[-1/2,1]}$. Using Theorem 2.4 and (2.12) and substituting $y = \cos\phi$, we get

$$I_{[-1/2,1]} \leq c \frac{n^p}{(w_{\alpha,\beta}(n;x))^{p/2}} \int_0^{2\pi/3} \frac{(\sin\phi)^{2\alpha+1} d\phi}{(\sin\phi + n^{-1})^{(\alpha+1/2)p}(1+n|\theta-\phi|)^{\sigma p}}$$

with $\sigma = k - 2\alpha - 2\beta - 3$. Denoting the last integral by J, we use the relation $\sin\phi \sim \phi$ for $0 \leq \phi \leq 2\pi/3$ and apply the substitution $u = n\theta$ to obtain

$$J \leq c \frac{n^{(\alpha+1/2)p}}{n^{2\alpha+1}} \int_0^{2\pi/3} \frac{d\phi}{(1+n\phi)^{(\alpha+1/2)(p-2)}(1+n|\theta-\phi|)^{\sigma p}}$$

$$\leq c \frac{n^{(\alpha+1/2)p}}{n^{2\alpha+2}} \int_0^{2n\pi/3} \frac{du}{(1+u)^{(\alpha+1/2)(p-2)}(1+|u-n\theta|)^{\sigma p}}.$$

If $p \geq 2$, then we split the last integral into two integrals: one over $[0, n\theta/2]$ and the other over $[n\theta/2, 2n\pi/3]$. We have

$$\int_0^{n\theta/2} \leq \frac{c}{(1+n\theta)^{\sigma p}} \int_0^{n\theta/2} \frac{du}{(1+u)^{(\alpha+1/2)(p-2)}} \leq \frac{c}{(1+n\theta)^{\sigma p - 1}}$$

and

$$\int_{n\theta/2}^{2n\pi/3} \leq \frac{c}{(1+n\theta)^{(\alpha+1/2)(p-2)}} \int_{n\theta/2}^{2n\pi/3} \frac{du}{(1+|u-n\theta|)^{\sigma p}} \leq \frac{c}{(1+n\theta)^{(\alpha+1/2)(p-2)}},$$

using that the last integral is bounded by $\int_{-\infty}^{\infty}(1+|u|)^{-\sigma p} du \leq c$.

The assumption $k \geq 3\alpha + 3\beta + 5$ yields $\sigma p - 1 \geq \alpha(p-2)$ and hence

$$J \leq c \frac{n^{(\alpha+1/2)p}}{n^{2\alpha+2}(1+n\theta)^{(\alpha+1/2)(p-2)}} \leq \frac{c}{n(\theta+n^{-1})^{(\alpha+1/2)(p-2)}}$$

$$\leq \frac{c}{n(w_{\alpha,\beta}(n;\cos\theta))^{p/2-1}}.$$

If $p < 2$, then we have

$$J \leq c \frac{n^{(\alpha+1/2)p}}{n^{2\alpha+2}} \int_0^{2n\pi/3} \frac{(1+u)^{(\alpha+1/2)(2-p)} du}{(1+|u-n\theta|)^{\sigma p}}$$

$$= c \frac{n^{(\alpha+1/2)p}}{n^{2\alpha+2}} \int_{-n\theta}^{n(2\pi/3-\theta)} \frac{(1+v+n\theta)^{(\alpha+1/2)(2-p)} dv}{(1+|v|)^{\sigma p}}.$$

Using the inequality $(A+B)^a \leq 2^a(A^a + B^a)$ we conclude that

$$J \leq c \frac{n^{(\alpha+1/2)p}}{n^{2\alpha+2}} \left[\int_{-\infty}^{\infty} \frac{dv}{(1+|v|)^{\sigma p - (\alpha+1/2)(2-p)}} + (n\theta)^{(\alpha+1/2)(2-p)} \int_{-\infty}^{\infty} \frac{dv}{(1+|v|)^{\sigma p}} \right]$$

$$\leq c \frac{n^{(\alpha+1/2)p}}{n^{2\alpha+2}} (1+n\theta)^{(\alpha+1/2)(2-p)} \leq \frac{c}{n(w_{\alpha,\beta}(n;\cos\theta))^{p/2-1}},$$

which is the same as in the case of $p \geq 2$. Here we have used that $k \geq 3\alpha + 3\beta + 5$, which yields $\sigma p - (\alpha + 1/2)(2-p) > 1$.

Putting these estimates together, we conclude that

$$(2.25) \qquad I_{[-1/2,1]} \leq c \frac{n^p J}{(w_{\alpha,\beta}(n;x))^{p/2}} \leq c \frac{n^{p-1}}{(w_{\alpha,\beta}(n;x))^{p-1}}.$$

To estimate $I_{[-1,-1/2]}$ we again apply the substitution $y = \cos\phi$. Notice that since $0 \le \theta \le \pi/2$ and $2\pi/3 \le \phi \le \pi$, we have $|\theta - \phi| \ge \pi/6$. Employing Theorem 2.4 and (2.13), we obtain as above

$$I_{[-1,-1/2]} \le c\frac{n^p}{(w_{\alpha,\beta}(n;x))^{p/2}}\frac{n^{(\beta+1/2)p}}{n^{\sigma p}}\int_{2\pi/3}^{\pi}\frac{(\sin\phi)^{2\beta+1}}{(1+n\sin\phi)^{(\beta+1/2)p}}d\phi$$

$$\le c\frac{n^p}{(w_{\alpha,\beta}(n;x))^{p/2}}\frac{n^{(\beta+1/2)p}}{n^{\sigma p}}\frac{1}{n^{(\beta+1/2)(p-1)}} \le c\frac{n^{p-1}}{(w_{\alpha,\beta}(n;x))^{p-1}},$$

using the trivial inequalities $n^{-(2\alpha+1)} \le w_{\alpha,\beta}(n;\cos\theta) \le c$ for $0 \le \theta \le \pi/2$ and the fact that $k \ge 3\alpha + 3\beta + 5$. The last estimate coupled with (2.25) yields (2.24). The proof of Proposition 2.5 is complete.   □

3. Tight polynomial frames in $L^2(w_{\alpha,\beta})$

We now turn to the construction of polynomial frames. In addition to the localized polynomials from §2, we shall need the Gaussian quadrature formula. Let $\xi_j = \cos\theta_j$, $1 \le j \le n$, denote the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$, ordered so that

$$0 =: \theta_0 < \theta_1 < \cdots < \theta_n < \theta_{n+1} := \pi.$$

Let $\lambda_n(t)$ be the Christoffel function and $b_\nu = \lambda_n(\xi_\nu)$. It is known that

$$\theta_{\nu+1} - \theta_\nu \sim n^{-1} \text{ and hence } \theta_\nu \sim \nu n^{-1} \ (1 \le \nu \le n);$$

and also

$$b_\nu \sim n^{-1}w_{\alpha,\beta}(\xi_\nu)(1-\xi_\nu^2)^{1/2} \sim n^{-1}w_{\alpha,\beta}(n;\xi_\nu).$$

Here the constants of equivalence depend only on α, β (cf. [9, p. 282]). These quantities appear in the well-known Gaussian quadrature:

Proposition 3.1. *For each $n \ge 1$, the quadrature*

$$(3.1) \qquad c_{\alpha,\beta}\int_{-1}^{1}f(t)w_{\alpha,\beta}(t)dt \sim \sum_{\nu=1}^{n}b_\nu f(\xi_\nu)$$

is exact for all polynomials of degree $2n-1$.

We shall utilize Proposition 3.1 with $n = 2^j$. Let us denote by $\xi_{j,\nu}$ and $b_{j,\nu}$ ($\nu = 1, 2, \ldots, 2^j$) the knots and the coefficients of the Gaussian quadrature when $n = 2^j$, and set

$$\mathcal{X}_j := \{\xi_{j,\nu} : \nu = 1, 2, \ldots, 2^j\}, \quad \text{and} \quad b_\xi := b_{j,\nu} \quad \text{if } \xi = \xi_{j,\nu}.$$

Then the Gaussian quadrature

$$(3.2) \qquad c_{\alpha,\beta}\int_{-1}^{1}f(t)w_{\alpha,\beta}(t)dt \sim \sum_{\xi\in\mathcal{X}_j}b_\xi f(\xi)$$

is exact for all polynomials of degree at most $2^{j+1} - 1$.

Let $\widehat{a}$ satisfy the conditions:

$$(3.3) \qquad \widehat{a} \in C^\infty(\mathbb{R}), \quad \widehat{a} \ge 0, \quad \operatorname{supp}\widehat{a} \subset [1/2, 2],$$

$$(3.4) \qquad \widehat{a}(t) > c > 0, \qquad \text{if } t \in [3/5, 5/3],$$

$$(3.5) \qquad \widehat{a}^2(t) + \widehat{a}^2(2t) = 1, \qquad \text{if } t \in [1/2, 1].$$

It follows from these conditions that

$$(3.6) \qquad \sum_{\nu=0}^{\infty} \widehat{a}^2(2^{-\nu}t) = 1, \qquad t \in [1, \infty).$$

It is easy to construct functions that satisfy properties (3.3)-(3.5). Indeed, a standard construction (e.g. in wavelets) shows that there exists a function g with the properties: $g \in C^{\infty}(\mathbb{R})$, $\operatorname{supp} g = [-1, 1]$, $g(t) > 0$ on $(-1, 1)$, $g(-t) = g(t)$, $g(0) = 1$ and $|g(t)|^2 + |g(t+1)|^2 = 1$ on $[-1, 0]$. Then the function $\widehat{a}(t) := g(\log_2 t)$ has the required properties.

We introduce the orthonormal Jacobi polynomials $\mathsf{P}_n^{(\alpha, \beta)}$, which are given by

$$\mathsf{P}_n^{(\alpha, \beta)}(x) := (h_n^{(\alpha, \beta)})^{-1/2} P_n^{(\alpha, \beta)}(x).$$

The sequence $\{\mathsf{P}_\nu^{(\alpha, \beta)}\}_{\nu=0}^{\infty}$ is an orthonormal basis in $L^2(w_{\alpha, \beta}, [-1, 1])$ with inner product

$$\langle f, g \rangle := c_{\alpha, \beta} \int_{-1}^{1} f(t)\overline{g(t)} w_{\alpha, \beta}(t) dt.$$

Assuming that $\widehat{a}$ satisfies conditions (3.3)-(3.5), we write $L_0^{\alpha, \beta} := 1$ and

$$(3.7) \qquad L_j^{\alpha, \beta}(x, y) := \sum_{\nu=0}^{\infty} \widehat{a}\left(\frac{\nu}{2^{j-1}}\right) \mathsf{P}_\nu^{(\alpha, \beta)}(x) \mathsf{P}_\nu^{(\alpha, \beta)}(y), \quad j = 1, 2, \dots.$$

We now define our frame elements (**needlets**) by

$$(3.8) \qquad \psi_\xi(x) := \sqrt{b_\xi} \cdot L_j^{\alpha, \beta}(x, \xi) \quad \text{for} \quad \xi \in \mathcal{X}_j, \ j = 0, 1, \dots,$$

where $\mathcal{X}_j$ is the set of knots and the b_ξ's are the weights in the Gaussian quadrature (3.2).

Denote $\mathcal{X} := \cup_{j=0}^{\infty} \mathcal{X}_j$, where every two points $\xi, \eta \in \mathcal{X}$ from different levels $\mathcal{X}_j \neq \mathcal{X}_k$ are considered to be different elements of $\mathcal{X}$ even if they coincide. We use $\mathcal{X}$ as an index set in the definition of the needlet system

$$\Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}},$$

which as will be shown in Theorem 3.3 below is a tight frame in $L^2(w_{\alpha, \beta})$.

We next show that each ψ_ξ has nearly exponential (faster than any polynomial) rate of decay away from ξ. This property of needlets is critical for their implementation to decomposition of spaces other than $L^2(w_{\alpha, \beta})$.

Theorem 3.2. *For any $k \geq 1$ there exists a constant c_k depending only on k, α, β, and $\widehat{a}$ such that for $\xi \in X_j$, $j = 1, 2, \dots$,*

$$(3.9) \qquad |\psi_\xi(\cos \theta)| \leq c_k \frac{2^{j/2}}{\sqrt{w_{\alpha, \beta}(2^j; \cos \theta)}(1 + 2^j|\theta - \arccos \xi|)^k}, \qquad \theta \in [0, \pi],$$

and, consequently, for $\nu = 1, 2, \dots, 2^j$,

$$(3.10) \qquad |\psi_{\xi_{j, \nu}}(\cos \theta)| \leq c_k \frac{2^{j/2}}{\sqrt{w_{\alpha, \beta}(2^j; \cos \theta)}(1 + 2^j|\theta - \nu 2^{-j}|)^k}, \qquad \theta \in [0, \pi].$$

Proof. Estimates (3.9)-(3.10) follow immediately from the estimate in Theorem 2.4, taking into account the simple fact that $1 + 2^j|\theta \pm c2^{-j}| \sim 1 + 2^j\theta$ if $0 \leq \theta \leq \pi$ and the fact that $b_\xi \sim 2^{-j} w_{\alpha, \beta}(2^{-j}; \xi)$. Since by assumption $\widehat{a}$ is in C^{∞}, estimates (3.9)-(3.10) holds for every $k \geq 1$. $\qquad \square$

Remark. It is worthwhile to mention that the estimate (3.9) or (3.10) (also Proposition 2.5 with $p = 2$) immediately yields $\|\psi_{\xi_{j,\nu}}\|_{L^2(w_{\alpha,\beta})} \leq c$.

Our next theorem shows that Ψ is a tight frame in $L^2(w_{\alpha,\beta})$.

Theorem 3.3. *If $f \in L^2(w_{\alpha,\beta})$, then*

$$(3.11) \qquad f = \sum_{j=0}^{\infty} \sum_{\nu=1}^{2^j} \langle f, \psi_{\xi_{j,\nu}} \rangle \psi_{\xi_{j,\nu}} = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad in \ L^2(w_{\alpha,\beta})$$

and

$$(3.12) \qquad \|f\|_{L^2(w_{\alpha,\beta})} = \Big(\sum_{\xi \in \mathcal{X}} |\langle f, \psi_\xi \rangle|^2 \Big)^{1/2}.$$

Proof. Suppose $f \in L^2(w_{\alpha,\beta})$. Then $f = \sum_{\nu=0}^{\infty} d_\nu(f) \mathsf{P}_\nu^{(\alpha,\beta)}$, where $d_\nu(f) := \langle f, \mathsf{P}_\nu^{(\alpha,\beta)} \rangle$. Here and elsewhere in the following the convergence is in $L^2(w_{\alpha,\beta})$. Denote briefly

$$(L_j * f)(x) := c_{\alpha,\beta} \int_{-1}^{1} L_j^{\alpha,\beta}(x,t) f(t) w_{\alpha,\beta}(t) dt,$$

where $L_j^{\alpha,\beta}$ are defined in (3.7). One can regard this as a nonstandard convolution which is evidently associative but not commutative. We next show that the following Calderon type decomposition holds

$$(3.13) \qquad f = \sum_{j=0}^{\infty} L_j^{\alpha,\beta} * L_j^{\alpha,\beta} * f.$$

Indeed, clearly for $j \geq 1$,

$$L_j^{\alpha,\beta} * f = \sum_{\nu=1}^{2^j} \widehat{a}\Big(\frac{\nu}{2^{j-1}} \Big) d_\nu(f) \mathsf{P}_\nu^{(\alpha,\beta)}$$

and then

$$L_j^{\alpha,\beta} * L_j^{\alpha,\beta} * f = \sum_{\nu=1}^{2^j} \widehat{a}^2\Big(\frac{\nu}{2^{j-1}} \Big) d_\nu(f) \mathsf{P}_\nu^{(\alpha,\beta)}.$$

Consequently,

$$\sum_{j=0}^{\infty} L_j^{\alpha,\beta} * L_j^{\alpha,\beta} * f = d_0(f) \mathsf{P}_0^{(\alpha,\beta)} + \sum_{j=1}^{\infty} \sum_{\nu=1}^{2^j} \widehat{a}^2\Big(\frac{\nu}{2^{j-1}} \Big) d_\nu(f) \mathsf{P}_\nu^{(\alpha,\beta)}$$

$$= d_0(f) \mathsf{P}_0^{(\alpha,\beta)} + \sum_{\nu=1}^{\infty} \sum_{j=1}^{\infty} \widehat{a}^2\Big(\frac{\nu}{2^{j-1}} \Big) d_\nu(f) \mathsf{P}_\nu^{(\alpha,\beta)}$$

$$= \sum_{\nu=0}^{\infty} d_\nu(f) \mathsf{P}_\nu^{(\alpha,\beta)} = f,$$

where we used (3.6). Thus (3.13) is established.

As we already mentioned $L_j^{\alpha,\beta} * L_j^{\alpha,\beta} * f = (L_j^{\alpha,\beta} * L_j^{\alpha,\beta}) * f$. Since $L_j^{\alpha,\beta}(x,t) L_j^{\alpha,\beta}(t,y)$ is a polynomial of degree at most $2^{j+1} - 1$ in t and the quadrature formula (3.2) is

exact for such polynomials, we have

$$
\begin{aligned}
(L_j^{\alpha,\beta} * L_j^{\alpha,\beta})(x,y) &= c_{\alpha,\beta} \int_{-1}^{1} L_j^{\alpha,\beta}(x,t) L_j^{\alpha,\beta}(t,y) w_{\alpha,\beta}(t) dt \\
&= \sum_{\xi \in \mathcal{X}_j} \sqrt{b_\xi} \cdot L_j^{\alpha,\beta}(x,\xi) \sqrt{b_\xi} \cdot L_j^{\alpha,\beta}(\xi,y) \\
&= \sum_{\xi \in \mathcal{X}_j} \psi_\xi(x) \psi_\xi(y).
\end{aligned}
$$

Here we also used that $L_j^{\alpha,\beta}(x,y) = L_j^{\alpha,\beta}(y,x)$ and the definition of the frame elements in (3.8). Consequently,

$$
L_j^{\alpha,\beta} * L_j^{\alpha,\beta} * f = \sum_{\xi \in \mathcal{X}_j} \langle f, \psi_\xi \rangle \psi_\xi,
$$

which along with (3.13) yields (3.11).

For the proof of (3.12) we denote $F_J f := \sum_{j=0}^{J} \sum_{\xi \in \mathcal{X}_j} \langle f, \psi_\xi \rangle \psi_\xi$. Evidently,

$$
\langle f, F_J f \rangle = \sum_{j=0}^{J} \sum_{\xi \in \mathcal{X}_j} |\langle f, \psi_\xi \rangle|^2
$$

and passing to the limit as $J \to \infty$ we arrive at (3.12).   □

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