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Abstract

Non-oscillatory schemes are widely used in numerical approximations of nonlinear conservation laws. The Nessyahu-Tadmor (NT) scheme is an example of a second order scheme that is both robust and simple. In this paper, we prove a new stability property of the NT scheme based on the standard minmod reconstruction in the case of a scalar strictly convex conservation law. This property is similar to the One-Sided Lipschitz Condition for first order schemes. Using this new stability, we derive the convergence of the NT scheme to the exact entropy solution without imposing any nonhomogeneous limitations on the method. We also derive an error estimate for monotone initial data.

AMS subject classification: Primary 65M15; Secondary 65M12
Key Words: scalar convex conservation laws, second order non-oscillatory schemes, Minmod limiter.

1 Introduction

We are interested in the scalar hyperbolic conservation law

$$(1) \qquad \begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u^0(x), & x \in \mathbb{R}, \end{cases}$$

where f is a given flux function. In recent years, there has been enormous activity in the development of the mathematical theory and in the construction of numerical methods for (1). Even though the existence-uniqueness theory is complete, there are many numerically efficient methods for which the questions of convergence and error estimates are still open. For example, there are many second or higher order non-oscillatory schemes based on minmod limiters which are numerically robust but

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theoretical results about convergence or error estimates are still missing [18, 6, 7, 22, 8, 9]. Usually second order schemes are constructed to be Total Variation Diminishing (TVD) but that property only guarantees the convergence of such schemes to a weak solution, see [10]. No property was known that implies convergence of such schemes to the entropy solution even in the case of a genuinely nonlinear scalar conservation law. The usual approach is to try to prove a single cell entropy inequality which usually leads to additional nonhomogeneous limitations on a second order scheme in order to fit it into the existing convergence theory. There are few results on convergence of non-oscillatory second order schemes which do not require nonhomogeneous limitations and we are going to mention them here. LeFloch and Liu in [11] consider piecewise smooth data and prove a different entropy inequality in different monotonicity regions of the numerical solution. Their result is valid for a specific second order upwind scheme and it may work for other schemes but the conditions are hard to check, the NT schemes does not fit into their framework, and there is little hope to prove any error estimates with that approach. In [24, 25], Yang reduces the convergence of a special type second order scheme to a convergence of that scheme for a Riemann problem. Again, that type of argument has no potential for any error estimates. Finally, P.L. Lions and P.E. Souganidis develop in [12] a convergence theory for second order schemes for scalar convex conservation laws and Hamilton-Jacobi equations. Unfortunately, their results for conservation laws do not hold for any of the explicit second order schemes used in practice because of the very strong restriction imposed on the CFL condition, see [12]. The main reason for such difficulties is hidden in the fact that besides a TVD property very little was known for non-oscillatory schemes because they use nonlinear limiters such as Minmod. This is in contrast to the theory for first order schemes where in the convex case there are many different approaches. For example, Tadmor's dual approach based on Lip+ stability [19] and the Kruzkov-Kuznetsov argument based on an entropy diminishing property [1, 2, 17]. In our previous work [13], in the case of a linear flux, we derive a new stability result for a generic second order scheme (central or upwind) based on the Minmod limiter. Here, we prove the one-sided analog of this result for the NT schemes in the case of any scalar conservation law with a strictly convex flux. This new property, to the best of our knowledge, is the first one-sided stability result for a second order scheme. We use that result to prove convergence of the NT scheme to the unique entropy solution without imposing any nonhomogeneous limitation on the method. This stability result and our results in [14] imply an error estimate in the case of a monotone initial data. The question of a general error estimate framework based on the new stability will be addressed elsewhere. All results in this paper are also valid for the non-staggered version of the NT scheme based on the Minmod limiter given in [8].

The paper is organized as follows. In section 2, we describe the staggered NT scheme. In section 3, we present our main result: a new one-sided stability property of the NT scheme. Then, we use that property to prove the convergence of the scheme to the entropy solution and derive an error estimate for monotone initial data in section 4. In the appendix, we give proof of the localization argument we need in the proof of our main result.

2 Non-Oscillatory Central Schemes

In this section, we are concerned with second order non-oscillatory central differencing approximations to the scalar conservation law

$$u_t + f(u)_x = 0. \tag{2}$$

The prototype of all such schemes is the staggered Nessyahu-Tadmor (NT) scheme [18]. We limit our attention to the staggered NT scheme but all results in this paper are valid for the corresponding non-staggered version in [8]. We now recall the basic step in the NT scheme [18]. Let $v(x,t)$ be an approximate solution to (2), and assume that the space mesh Δx and the time mesh Δt are uniform. Let $x_j := j\Delta x$, $j \in \mathbb{Z}$, $\lambda := \frac{\Delta t}{\Delta x}$ and

$$v_j(t) := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} v(x,t)\,dx \tag{3}$$

be the average of v at time t over $(x_{j-1/2}, x_{j+1/2})$. Let us assume that $v(\cdot,t)$ is a piecewise linear function, and it is linear on the intervals $(x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$, of the form

$$v(x,t) = L_j(x,t) := v_j(t) + (x - x_j)\frac{1}{\Delta x}v'_j, \quad x_{j-1/2} < x < x_{j+1/2}, \tag{4}$$

where $\frac{1}{\Delta x}v'_j$ is the numerical derivative of v which is yet to be determined. Integration of (2) over the staggered space-time cell $(x_j, x_{j+1}) \times (t, t+\Delta t)$ yields

$$\begin{aligned} v_{j+1/2}(t+\Delta t) = {} & \tfrac{1}{\Delta x}\left(\int_{x_j}^{x_{j+1/2}} L_j(x,t)\,dx + \int_{x_{j+1/2}}^{x_{j+1}} L_{j+1}(x,t)\,dx\right) \\ & -\tfrac{1}{\Delta x}\left(\int_t^{t+\Delta t} f(v(x_{j+1},\tau))\,d\tau - \int_t^{t+\Delta t} f(v(x_j,\tau))\,d\tau\right). \end{aligned} \tag{5}$$

The first two integrals on the right of (5) can be evaluated exactly. Moreover, if the CFL condition

$$\lambda \max_{x_j \le x \le x_{j+1}} |f'(v(x,t))| \le \frac{1}{2}, \quad j \in \mathbb{Z}, \tag{6}$$

is met, then the last two integrants on the right of (5) are smooth functions of τ. Hence, they can be integrated approximately by the midpoint rule with third order local truncation error. Note that, in the case of zero slopes $\frac{1}{\Delta x}v'_j$ and $\frac{1}{\Delta x}v'_{j+1}$, the time integration is exact for any flux f. Thus, following [18], we arrive at

$$\begin{aligned} v_{j+1/2}(t+\Delta t) &= \frac{1}{2}(v_j(t) + v_{j+1}(t)) + \frac{1}{8}(v'_j - v'_{j+1}) \\ &- \lambda\left(f(v(x_{j+1}, t+\Delta t/2)) - f(v(x_j, t+\Delta t/2))\right). \end{aligned} \tag{7}$$

By Taylor expansion and the conservation law (2), we obtain

$$v(x_j, t+\Delta t/2) = v_j(t) - \frac{1}{2}\lambda f'_j, \tag{8}$$

where $\frac{1}{\Delta x} f'_j$ stand for an approximate numerical derivative of the flux $f(v(x = x_j, t))$. The following choices are widely used as approximations of the numerical derivatives (we drop t to simplify the notation)

$$
\begin{aligned}
(9) \qquad v'_j &= \mathrm{m}(v_{j+1} - v_j, v_j - v_{j-1}), \\
(10) \qquad f'_j &= \mathrm{m}(f(v_{j+1}) - f(v_j), f(v_j) - f(v_{j-1})) \quad \text{or} \quad f'_j = f'(v_j^n)\, v'_j,
\end{aligned}
$$

where $\mathrm{m}(a,b)$ stands for the standard minmod limiter

$$
(11) \qquad \mathrm{m}(a,b) \equiv \mathrm{MinMod}(a,b) := \frac{1}{2}(\mathrm{sgn}(a) + \mathrm{sgn}(b)) \cdot \min(|a|, |b|).
$$

Using the approximate slopes (9) and flux derivatives (10), we construct a family of central schemes in the predictor-corrector form

$$
(12) \qquad
\begin{aligned}
v(x_j, t + \Delta t/2) &= v_j(t) - \frac{1}{2}\lambda f'_j, \\
v_{j+1/2}(t + \Delta t) &= \frac{1}{2}(v_j(t) + v_{j+1}(t)) + \frac{1}{8}(v'_j - v'_{j+1}) \\
&- \lambda \left(f(v(x_{j+1}, t + \Delta t/2)) - f(v(x_j, t + \Delta t/2)) \right),
\end{aligned}
$$

where we start with $v_j(0) := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\, dx$. Note that, this is the description of the NT scheme when we compute the staggered averages from the averages on regular grid. The other step, from staggered averages to averages on regular grid, is completely analogous – we have to shift the index j to $j+1/2$ everywhere. Therefore, we compute averages on two staggered uniform partitions of the real line: (i) all intervals $I_j := (x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$ for time $t = 2n\Delta t$, $n = 0, 1, \ldots$; (ii) all intervals $J_j := (x_j, x_{j+1})$, $j \in \mathbb{Z}$ for time $t = (2n+1)\Delta t$, $n = 0, 1, \ldots$

3 One-Sided Stability of the NT Scheme

In this section we will present the main result of the paper: a new stability result for the NT scheme. Let us denote the numerical solution of the NT scheme at time step t_n with v^n, $v^n := v(\cdot, t_n)$, and its cell averages with v_j^n, where the cell averages of v^0 are equal to the cell averages of the initial condition u^0: $v_j^0 := u_j^0$, $j \in \mathbb{Z}$. We define the numerical solution $v(\cdot, t_n)$ to be a linear function on each cell $I_j := (x_{j-1/2}, x_{j+1/2})$

$$
(13) \qquad v^n|_{I_j} = v_j^n + (x - x_j)\frac{1}{\Delta x}\mathrm{m}(v_{j+1}^n - v_j^n, v_j^n - v_{j-1}^n).
$$

Note that the above formula is valid only for an even time index n because for odd n we have a similar piecewise linear reconstruction but on the staggered grid, i.e., we have a minmod-type reconstruction on $I_{j+1/2} := (x_j, x_{j+1})$, $j \in \mathbb{Z}$ and we have to shift the index j to $j + 1/2$ in (13). We will continue the presentation considering only the case of even n with the case of odd n being analogous.

We age going to use the notation: v_j^n for the cell averages at time t_n, $v_{j+1/2}^{n+1}$ for the cell averages at time t_{n+1}, and $a(u) := f'(u)$ for the derivative of the flux. Then, the averages at time t_{n+1} are given by

$$v_{j+1/2}^{n+1} = \frac{1}{2}(v_j^n + v_{j+1}^n) + \frac{1}{8}(v_j' - v_{j+1}') - \lambda \left[f(v_{j+1}^{n+1/2}) - f(v_j^{n+1/2}) \right], \tag{14}$$

where v_j' is given by (9), $v_j^{n+1/2} = v_j^n - \frac{\lambda}{2} f_j'$, and f_j' is one of the flux approximations given in (10).

Denote the *new* jumps at time t_{n+1} with $\delta_j^{n+1} := v_{j-1/2}^{n+1} - v_{j-3/2}^{n+1}$, the *old* jumps (at time t_n) with $\delta_j^n := v_j^n - v_{j-1}^n$, and let $\|a\|_\infty$ be the maximum speed of propagation

$$\|a\|_\infty := \max_{|w| \le \|u^0\|_{L^\infty}} |f'(w)|. \tag{15}$$

With this notation we have the following theorem which is our main result.

Theorem 1. *Let $u^0 \in L^\infty(\mathbb{R})$ and f be strictly convex in the range of u^0. That is, there exist constants $\gamma_1 \le \gamma_2$ such that*

$$0 < \gamma_1 \le f''(w) \le \gamma_2$$

for any $|w| \le \|u^0\|_{L^\infty}$. Then, there exists a constant $\kappa > 0$ which depends only on γ_1,γ_2, $\|a\|_\infty$, and $\|u^0\|_{L^\infty}$ such that under the CFL condition

$$\lambda \|a\|_\infty \le \kappa \tag{16}$$

the l_2 norm of the nonnegative jumps of the NT scheme is non-increasing in time. That is, the NT scheme satisfies the following One-Sided Lipschitz Condition

$$\sum_{j \in \mathbb{Z}} \left(\delta_j^{n+1}\right)_+^2 \le \sum_{j \in \mathbb{Z}} \left(\delta_j^n\right)_+^2, \tag{17}$$

for all $n \ge 0$, where we use the standard $+$ notation: $x_+ = \max(x, 0)$.

Proof. It is enough to prove the result for one time step, assuming that $\sum_{j\in\mathbb{Z}} \left(\delta_j^n\right)_+^2 < \infty$. We will always assume that the CFL condition (16) is satisfied with $\kappa \le 0.32$ because this guarantees the TVD property of the NT scheme. It is enough to prove the theorem for a single nondecreasing sequence. The general result follows from a localization argument similar to the one we used in the case of linear flux in [13] and it is given in the appendix. Hence, we assume that all jumps δ_j^n are nonnegative. There are two different choices for f_j', see (10), and the proof is very similar for either one. We are only going to consider the second one here

$$f_j' = a(v_j^n)\, v_j'. \tag{18}$$

We use (9) and (18) in (14) and derive the formula for the new jumps $\delta_j' := \delta_j^{n+1}$ from the old ones $\delta_j = v_j^n - v_{j-1}^n$. That is

$$\delta_{j+1}' = \frac{1}{2}(\delta_j + \delta_{j+1}) - \frac{1}{8}(y_{j+1} - 2y_j + y_{j-1}) - \lambda \left[f(v_{j+1}^{n+1/2}) - 2f(v_j^{n+1/2}) + f(v_{j-1}^{n+1/2}) \right],$$

where we define $y_j := v'_j$, see (9). Now, we consider the flux difference

$$f(v_{j+1}^{n+1/2}) - f(v_j^{n+1/2}) = \bar{a}_{j+1/2}(v_{j+1}^{n+1/2} - v_j^{n+1/2}),$$

where $\bar{a}_{j+1/2}$ is some averaged velocity. We will use the notation $a_j := a(v_j^n)$ and the standard divided difference notation $\bar{a}_{j+1/2} = f[v_{j+1}^{n+1/2}, v_j^{n+1/2}]$. Note that

$$v_{j+1}^{n+1/2} - v_j^{n+1/2} = \delta_{j+1} - \frac{\lambda}{2}(a_{j+1}y_{j+1} - a_j y_j). \tag{19}$$

Then

$$\begin{aligned}
\delta'_{j+1} &= \frac{1}{2}(\delta_j + \delta_{j+1}) - \frac{1}{8}(\Delta y_{j+1} - \Delta y_j) - \lambda\left[\bar{a}_{j+1/2}\left(\delta_{j+1} - \frac{\lambda}{2}(a_{j+1}y_{j+1} - a_j y_j)\right)\right. \\
&- \left.\bar{a}_{j-1/2}\left(\delta_j - \frac{\lambda}{2}(a_j y_j - a_{j-1}y_{j-1})\right)\right].
\end{aligned}$$

Our goal is to show that the l_2 norm of the jumps decreases in time. That is, $\sum(\delta'_j)^2 \leq \sum(\delta_j)^2$ for the NT scheme with a CFL condition $\lambda\|a\|_\infty \leq \kappa$ with sufficiently small but fixed κ. We are going to follow the steps in our proof in the case of linear flux, see [20]. Unfortunately, in the case of strictly convex flux the formula for $\{\delta'_j\}$ is more complicated and we need to make a sequence of perturbations in order to mimic the linear proof. Let s be a term appearing in the formulation of $\{\delta'_j\}$. If we replace s with another term w, then the sequence $\{\delta'_j\}$ will change to a new sequence $\{\delta''_j\}$. We will use the notation $s \sim w$ if the l_2 norm of the two sequences satisfy

$$|\sum(\delta'_j)^2 - \sum(\delta''_j)^2| \leq C\lambda^2 \sum(\delta_j)^3.$$

In general, we are going to use the equivalence $a \approx b$ if

$$|a - b| \leq C\lambda^2 \sum(\delta_j)^3.$$

The positive constant C could be different at the different appearances but it can depend only on the maximum convexity γ_2, maximum speed $\|a\|_\infty$, and $\|u^0\|_{L^\infty}$. This formal calculus will simplify our presentation. First, we are going to replace the term $a_{j+1}y_{j+1} - a_j y_j$ with $\bar{a}_{j+1/2}\Delta y_{j+1}$, and the term $a_j y_j - a_{j-1}y_{j-1}$ with $\bar{a}_{j-1/2}\Delta y_j$ in the the formula for δ'_{j+1}, where we use the standard finite difference notation $\Delta y_j := y_j - y_{j-1}$. Let

$$\sigma_{j+1} := \bar{a}_{j+1/2}\left((a_{j+1}y_{j+1} - a_j y_j) - \bar{a}_{j+1/2}\Delta y_{j+1}\right)$$

and

$$\begin{aligned}
\delta''_{j+1} &= \frac{1}{2}(\delta_j + \delta_{j+1}) - \frac{1}{8}(\Delta y_{j+1} - \Delta y_j) - \lambda\left[\bar{a}_{j+1/2}(\delta_{j+1} - \frac{\lambda}{2}\bar{a}_{j+1/2}\Delta y_{j+1})\right. \\
&- \left.\bar{a}_{j-1/2}(\delta_j - \frac{\lambda}{2}\bar{a}_{j-1/2}\Delta y_j)\right].
\end{aligned}$$

Then, we have that

$$\delta'_{j+1} = \delta''_{j+1} + \frac{\lambda^2}{2}(\sigma_{j+1} - \sigma_j).$$

Using $|\bar{a}_{j-1/2}| \leq \|a\|_\infty$, we get

$$|\sigma_j| \leq \|a\|_\infty \left| y_j(a_j - \bar{a}_{j-1/2}) - y_{j-1}(a_{j-1} - \bar{a}_{j-1/2}) \right|.$$

Note that

$$a_j - \bar{a}_{j-1/2} = a(v_j^n) - f[v_j^{n+1/2}, v_{j-1}^{n+1/2}] = a(v_j^n) - a(\xi_j^n),$$

where $\xi_j^n \in [v_{j-1}^{n+1/2}, v_j^{n+1/2}]$ because v^n is monotone. Assuming the CFL condition $\lambda \|a\|_\infty \leq 0.5$, we have

$$v_j^{n+1/2} = v_j^n - \frac{\lambda a_j}{2} y_j \leq v_j^n + \frac{\delta_j}{4} =: v_j^+$$

and

$$v_{j-1}^{n+1/2} = v_{j-1}^n - \frac{\lambda a_{j-1}}{2} y_{j-1} \geq v_{j-1}^n - \frac{\delta_{j-1}}{4} =: v_{j-1}^-$$

The above bounds and the Mean Value Theorem give

$$|a_j - \bar{a}_{j-1/2}| \leq \max_{w \in [v_{j-1}^-, v_j^+]} |a'(w)| \max(v_j^n - v_{j-1}^-, v_j^+ - v_j^n) \leq \gamma_2(\delta_j + \delta_{j-1}). \tag{20}$$

Similarly, we obtain

$$|a_{j-1} - \bar{a}_{j-1/2}| \leq \gamma_2(\delta_j + \delta_{j-1}).$$

Using the above estimates and $y_j = \min(\delta_j, \delta_{j-1})$, we derive

$$|\sigma_j| \leq 2\|a\|_\infty \gamma_2 \delta_j (\delta_j + \delta_{j-1}).$$

After a long but simple computation which we skip, we conclude

$$|\sum (\delta'_j)^2 - \sum (\delta''_j)^2| \leq C\lambda^2 \sum (\delta_j)^3,$$

with a constant C which depends only on γ_2, $\|a\|_\infty$ and $\|u^0\|_\infty$. Therefore, we proved that

$$\begin{aligned}
\delta'_{j+1} \quad &\sim \quad \frac{1}{2}(\delta_j + \delta_{j+1}) - \frac{1}{8}(\Delta y_{j+1} - \Delta y_j) - \lambda \Big[\bar{a}_{j+1/2}(\delta_{j+1} - \frac{\lambda}{2}\bar{a}_{j+1/2}\Delta y_{j+1}) \\
&- \quad \bar{a}_{j-1/2}(\delta_j - \frac{\lambda}{2}\bar{a}_{j-1/2}\Delta y_j) \Big].
\end{aligned}$$

We now shift the index ($j := j+1$) and regroup the terms the following way

$$\begin{aligned}
\delta'_j \quad &\sim \quad (\frac{1}{2} + \lambda \bar{a}_{j-3/2})\delta_{j-1} + (\frac{1}{2} - \lambda \bar{a}_{j-1/2})\delta_j - \frac{1}{8}(\Delta y_j - \Delta y_{j-1}) \\
&+ \quad \frac{\lambda^2}{2}\left((\bar{a}_{j-1/2})^2 \Delta y_j - (\bar{a}_{j-3/2})^2 \Delta y_{j-1} \right).
\end{aligned}$$

Let $\alpha_j := 1/2 + \lambda \bar{a}_{j-1/2}$ and $\varphi_j := \alpha_j(1-\alpha_j)$. With this notation, we have the following perturbation of the jump sequence

$$\delta'_j \sim \delta''_j = \alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j - \frac{1}{2}\left(\varphi_j \Delta y_j - \varphi_{j-1}\Delta y_{j-1}\right). \tag{21}$$

Let $\mathcal{D} := \sum \delta_j^2 - \sum (\delta'_j)^2$. Using the standard big O notation, we have

$$\mathcal{D} = \delta_j^2 - \sum (\delta''_j)^2 + O(\lambda^2 \sum (\delta_j)^3).$$

because of (21). As before, in all places the generic constants are positive and depend only on γ_2, $\|a\|_\infty$ and $\|u^0\|_\infty$. Instead of $O(\lambda^2 \sum(\delta_j)^3)$, we will use again the equivalence notation $\approx$ when manipulating sums. We have the following representation

$$\mathcal{D} \approx \sum \delta_j^2 - \sum (\delta''_j)^2 = I_1 + I_2 + I_3, \tag{22}$$

where

$$\begin{aligned} I_1 &= \sum \delta_j^2 - \sum (\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j)^2, \\ I_2 &= \sum (\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j)(\varphi_j \Delta y_j - \varphi_{j-1}\Delta y_{j-1}), \\ I_3 &= -\frac{1}{4}\sum (\varphi_j \Delta y_j - \varphi_{j-1}\Delta y_{j-1})^2 \end{aligned}$$

We transform the first term I_1 in the following way

$$\begin{aligned} I_1 &= \sum \delta_j^2 - \sum \left(\alpha_{j-1}^2 \delta_{j-1}^2 + 2\alpha_{j-1}(1-\alpha_j)\delta_{j-1}\delta_j + (1-\alpha_j)^2\delta_j^2\right) \\ &= \sum \left(\delta_j^2(1-\alpha_j^2 - (1-\alpha_j)^2) - 2\alpha_{j-1}(1-\alpha_j)\delta_{j-1}\delta_j\right) \\ &= \sum \left(\alpha_j(1-\alpha_j)\delta_j^2 + \alpha_{j-1}(1-\alpha_{j-1})\delta_{j-1}^2 - 2\alpha_{j-1}(1-\alpha_j)\delta_{j-1}\delta_j\right) \\ &= \sum \alpha_{j-1}(1-\alpha_j)(\delta_j - \delta_{j-1})^2 + \sum (\alpha_j - \alpha_{j-1})\left(\alpha_{j-1}\delta_{j-1}^2 + (1-\alpha_j)\delta_j^2\right) \end{aligned}$$

Therefore, we have

$$I_1 = \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + \sum \Delta\alpha_j \left(\alpha_{j-1}\delta_{j-1}^2 + (1-\alpha_j)\delta_j^2\right) \tag{23}$$

Recall that

$$\varphi_j = \alpha_j(1-\alpha_j) = \left(\frac{1}{2} + \lambda\bar{a}_{j-1/2}\right)\left(\frac{1}{2} - \lambda\bar{a}_{j-1/2}\right). \tag{24}$$

Similar to (20), we derive

$$|\Delta\varphi_j| = \lambda^2 \left|(\bar{a}_{j-3/2})^2 - (\bar{a}_{j-1/2})^2\right| \le 3\lambda^2 \|a\|_\infty \gamma_2 (\delta_{j-1} + \delta_j). \tag{25}$$

It is easy to see that the above estimate means that we can replace φ_j with φ_{j-1} in I_2 or I_3 and this will not affect the equivalence (22). In our notation, we make the perturbation $\varphi_j \sim \varphi_{j-1}$ and obtain

$$I_2 \approx I'_2 := \sum (\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j)\varphi_{j-1}\Delta^2 y_j \tag{26}$$

$$I_3 \approx I'_3 := -\frac{1}{4}\sum (\varphi_{j-1}\Delta^2 y_j)^2 \tag{27}$$

We use (26) and (27) in (22) and split $\mathcal{D}$ in the following way

$$\mathcal{D} \approx Q_1 + Q_2 + Q_3, \tag{28}$$

where

$$Q_1 := \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + I_2' - \frac{1}{2}\sum(\varphi_{j-1}\Delta^2\delta_j)^2, \tag{29}$$

$$Q_2 := \frac{1}{4}\left[2\sum(\varphi_{j-1}\Delta^2\delta_j)^2 - \sum(\varphi_{j-1}\Delta^2 y_j)^2\right], \tag{30}$$

and

$$Q_3 := \sum \Delta\alpha_j \left(\alpha_{j-1}\delta_{j-1}^2 + (1-\alpha_j)\delta_j^2\right). \tag{31}$$

If we take $\alpha_j = \alpha$ and denote $\beta := \frac{1}{2}\alpha(1-\alpha)$, we are going to get the split we used in [20] with Q_1 here equal to $2\beta Q_1$ in [20], Q_2 here equal to $\beta^2 Q_2$ in our notation from [20], and $Q_3 = 0$ in [20]. The new term Q_3 is due to the nonlinearity of the flux. Using that $\Delta\alpha_j = \lambda(\bar{a}_{j-1/2} - \bar{a}_{j-3/2})$, we derive

$$\Delta\alpha_j = \lambda f[v_j^{n+1/2}, v_{j-1}^{n+1/2}, v_{j-2}^{n+1/2}](v_j^{n+1/2} - v_{j-2}^{n+1/2}) = \frac{\lambda f''(\xi)}{2}(v_j^{n+1/2} - v_{j-2}^{n+1/2})$$

Under the CFL condition $\lambda\|a\|_\infty \leq 0.5$, we have

$$v_j^{n+1/2} = v_j^n - \frac{\lambda}{2}a(v_j^n)y_j \geq v_j^n - \frac{\delta_j}{4}$$

and

$$v_{j-2}^{n+1/2} = v_{j-2}^n - \frac{\lambda}{2}a(v_{j-2}^n)y_{j-2} \leq v_{j-2}^n + \frac{\delta_{j-1}}{4}.$$

Using the above inequalities, we obtain

$$\Delta\alpha_j \geq \frac{\lambda\gamma_1}{2}(v_j^{n+1/2} - v_{j-2}^{n+1/2}) \geq \frac{3}{8}\lambda\gamma_1(\delta_j + \delta_{j-1}), \tag{32}$$

where γ_1 is the minimum convexity of the flux. Hence, we derive a lower bound for the nonlinear term

$$Q_3 \geq C\gamma_1\lambda\sum(\delta_j)^3. \tag{33}$$

Note that, all perturbations using the relation $\approx$ are of order $O(\lambda^2\sum(\delta_j)^3)$ and can be dominated by $\lambda\sum\delta_j^3$ for a sufficiently small λ. Using three lemmas (see lemmas 2-4 in [20]), in the case of linear flux we proved that both terms Q_1 and Q_2 are nonnegative and their sum $\mathcal{D} = Q_1 + Q_2$ satisfies $\mathcal{D} \geq \frac{\beta^3}{4}\sum(\Delta^2\delta_j)^2$. Going through the same steps as in [20], we will prove the following lemma which concludes the proof of Theorem 1.

Lemma 2. *For any λ sufficiently small, we have*

$$\sum\delta_j^2 - \sum(\delta_j')^2 \approx Q_1 + Q_2 + Q_3 \geq C\left(\lambda\sum(\delta_j)^3 + \sum(\Delta^2\delta_j)^2\right) \tag{34}$$

with a constant C which depends only on γ_1, γ_2, $\|a\|_\infty$, and $\|u^0\|_{L^\infty}$.

Proof. We will transform Q_1 and Q_2 in the form needed to use the lower bound in Lemma 1 from [20]. We start with I_2'. Using $\varphi_j \sim \varphi_{j-1}$, we rearrange

$$\begin{aligned} I_2' &= \sum(\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j)\varphi_{j-1}(\Delta y_j - \Delta y_{j-1}) \\ &\approx \sum \varphi_j \Delta y_j \left(\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j - \alpha_j\delta_j - (1-\alpha_{j+1})\delta_{j+1}\right) \\ &= -\sum \varphi_j \Delta y_j \left(\alpha_{j-1}\Delta\delta_j + (1-\alpha_{j+1})\Delta\delta_{j+1} + \delta_j(\Delta\alpha_j - \Delta\alpha_{j+1})\right). \end{aligned}$$

We split I_2' in two parts

$$\begin{aligned} I_2' \approx & -\sum \varphi_j \Delta y_j \left(\alpha_{j-1}\Delta\delta_j + (1-\alpha_{j+1})\Delta\delta_{j+1}\right) \\ & + \sum \varphi_j \Delta y_j \delta_j \left(\Delta\alpha_{j+1} - \Delta\alpha_j\right). \end{aligned}$$

Using the above in (29), we get

$$\begin{aligned} Q_1 \approx & \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + \sum \varphi_j \Delta y_j \delta_j \left(\Delta\alpha_{j+1} - \Delta\alpha_j\right) \\ & - I_2'' - \frac{1}{2}\sum(\varphi_{j-1}\Delta^2\delta_j)^2, \end{aligned} \tag{35}$$

where

$$I_2'' := \sum \varphi_j \Delta y_j \left(\alpha_{j-1}\Delta\delta_j + (1-\alpha_{j+1})\Delta\delta_{j+1}\right) = A + B, \tag{36}$$

and we define

$$A = \sum_j \varphi_j(1-\alpha_{j+1})\Delta\delta_{j+1}\Delta y_j, \quad \text{and} \quad B = \sum_j \varphi_j \alpha_{j-1}\Delta\delta_j\Delta y_j.$$

We are going to split A and B in parts. We have used the same split in the linear case, see A and B in the proof of Lemma 2 in [20]. The new elements here are that we have $\sum_j \varphi_j$ instead of $\sum_j$ and the additional multipliers $(1-\alpha_{j+1})$ and α_{j-1} in each sum (in the linear case $\alpha_j = \alpha$ for all j). Using the equivalence $\varphi_j \sim \varphi_{j-1}$, exactly in the same way as in [20], we obtain the following representations

$$\begin{aligned} A \approx & \sum_{\Delta\delta_j<0} \varphi_j(1-\alpha_j)(\Delta\delta_j)^2 + \sum_{\Delta\delta_j\ge 0} \frac{\varphi_j}{2}\left((1-\alpha_{j+1}) + (1-\alpha_j)\right)(\Delta\delta_j)^2 \\ & - \frac{1}{2}\sum_{\Delta\delta_j\ge 0, \Delta\delta_{j+1}<0} \varphi_j(1-\alpha_{j+1})(\Delta\delta_j)^2 - \frac{1}{2}\sum_{\Delta\delta_j\ge 0, \Delta\delta_{j-1}<0} \varphi_j(1-\alpha_j)(\Delta\delta_j)^2 \\ & - \frac{1}{2}\sum_{\Delta\delta_{j-1}\ge 0, \Delta\delta_j\ge 0} \varphi_j(1-\alpha_j)(\Delta^2\delta_j)^2 + \sum_{\Delta\delta_j\ge 0, \Delta\delta_{j+1}<0} \varphi_j(1-\alpha_{j+1})\Delta\delta_j\Delta\delta_{j+1}, \end{aligned} \tag{37}$$

and

$$\begin{aligned} B \approx & \sum_{\Delta\delta_j\ge 0} \varphi_j\alpha_{j-1}(\Delta\delta_j)^2 + \sum_{\Delta\delta_j<0} \frac{\varphi_j}{2}\left(\alpha_{j-1} + \alpha_{j-2})\right)(\Delta\delta_j)^2 \\ & - \frac{1}{2}\sum_{\Delta\delta_j<0, \Delta\delta_{j+1}\ge 0} \varphi_j\alpha_{j-1}(\Delta\delta_j)^2 - \frac{1}{2}\sum_{\Delta\delta_j<0, \Delta\delta_{j-1}\ge 0} \varphi_j\alpha_{j-2}(\Delta\delta_j)^2 \\ & - \frac{1}{2}\sum_{\Delta\delta_{j-1}<0, \Delta\delta_j<0} \varphi_j\alpha_{j-2}(\Delta^2\delta_j)^2 + \sum_{\Delta\delta_j\ge 0, \Delta\delta_{j+1}<0} \varphi_j\alpha_{j-1}\Delta\delta_j\Delta\delta_{j+1}. \end{aligned} \tag{38}$$

Note that only the first two sums in each representation, see (37) and (38), are of order $\sum_j (\Delta\delta_j)^2$. All other sums either involve parts of $\sum_j (\Delta^2\delta_j)^2$ or the summation $\sum_{j\in\Lambda} (\Delta\delta_j)^2$ is over an index set Λ which is determined by two consecutive first differences with different signs: $\Delta\delta_j$ and $\Delta\delta_{j-1}$; or $\Delta\delta_j$ and $\Delta\delta_{j+1}$. Therefore, in all such cases we can dominate $(\Delta\delta_j)^2$ with $(\Delta^2\delta_j)^2$ or $(\Delta^2\delta_{j+1})^2$. Moreover, using (19) and (24), we derive the estimates

$$|\alpha_j - \frac{1}{2}| \leq \lambda \|a\|_\infty \quad \text{and} \quad |\varphi_j - \frac{1}{4}| \leq \lambda^2 \|a\|_\infty^2$$

Therefore, replacing φ_j with $\frac{1}{4}$ and α_j with $\frac{1}{2}$ in all such sums will change A and B at most by $O(\lambda \sum (\Delta^2\delta_j)^2)$. In order to simplify the presentation, we extend the equivalence $\approx$ in the following way. For any two terms a and b, we define

$$a \approx b \quad \text{if} \quad |a-b| \leq C\left(\lambda^2 \sum (\delta_j)^3 + \lambda \sum (\Delta^2\delta_j)^2\right)$$

and

$$a \gtrsim b \quad \text{if} \quad a - b \geq C\left(\lambda^2 \sum (\delta_j)^3 + \lambda \sum (\Delta^2\delta_j)^2\right).$$

That is, we add the new perturbation of order $O(\lambda \sum (\Delta^2\delta_j)^2)$ to the equivalence and introduce $a \gtrsim b$. With that notation, we have

$$\begin{aligned} (39) \quad A \quad \approx \quad & \sum_{\Delta\delta_j<0} \varphi_j (1-\alpha_j)(\Delta\delta_j)^2 + \sum_{\Delta\delta_j \geq 0} \frac{\varphi_j}{2} \left((1-\alpha_{j+1}) + (1-\alpha_j)\right)(\Delta\delta_j)^2 \\ - \quad & \frac{1}{2} \sum_{\Delta\delta_j\geq 0, \Delta\delta_{j+1}<0} \frac{1}{8}(\Delta\delta_j)^2 - \frac{1}{2} \sum_{\Delta\delta_j\geq 0, \Delta\delta_{j-1}<0} \frac{1}{8}(\Delta\delta_j)^2 \\ - \quad & \frac{1}{2} \sum_{\Delta\delta_{j-1}\geq 0, \Delta\delta_j\geq 0} \frac{1}{8}(\Delta^2\delta_j)^2 + \sum_{\Delta\delta_j\geq 0, \Delta\delta_{j+1}<0} \frac{1}{8}\Delta\delta_j \Delta\delta_{j+1}, \end{aligned}$$

and

$$\begin{aligned} (40) \qquad B \quad \approx \quad & \sum_{\Delta\delta_j\geq 0} \varphi_j \alpha_{j-1}(\Delta\delta_j)^2 + \sum_{\Delta\delta_j<0} \frac{\varphi_j}{2}\left(\alpha_{j-1} + \alpha_{j-2})\right)(\Delta\delta_j)^2 \\ - \quad & \frac{1}{2} \sum_{\Delta\delta_j<0, \Delta\delta_{j+1}\geq 0} \frac{1}{8}(\Delta\delta_j)^2 - \frac{1}{2} \sum_{\Delta\delta_j<0, \Delta\delta_{j-1}\geq 0} \frac{1}{8}(\Delta\delta_j)^2 \\ - \quad & \frac{1}{2} \sum_{\Delta\delta_{j-1}<0, \Delta\delta_j<0} \frac{1}{8}(\Delta^2\delta_j)^2 + \sum_{\Delta\delta_j\geq 0, \Delta\delta_{j+1}<0} \frac{1}{8}\Delta\delta_j \Delta\delta_{j+1}. \end{aligned}$$

We use (39) and (40) in (36) and (35), and obtain

$$(41) \qquad Q_1 \approx R_1 + Q_1^*,$$

where

$$(42) \quad R_1 \quad := \quad \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + \sum \varphi_j \Delta y_j \delta_j \left(\Delta\alpha_{j+1} - \Delta\alpha_j\right)$$

$$
\begin{aligned}
&- \sum_{\Delta\delta_j<0} \varphi_j(1-\alpha_j)(\Delta\delta_j)^2 - \sum_{\Delta\delta_j\geq 0} \frac{\varphi_j}{2}\left((1-\alpha_{j+1})+(1-\alpha_j)\right)(\Delta\delta_j)^2 \\
&- \sum_{\Delta\delta_j\geq 0} \varphi_j\alpha_{j-1}(\Delta\delta_j)^2 - \sum_{\Delta\delta_j<0} \frac{\varphi_j}{2}\left(\alpha_{j-1}+\alpha_{j-2})\right)(\Delta\delta_j)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
(43) Q_1^* \;:=\;& \frac{1}{8}\left(\frac{1}{2}\sum_{\Delta\delta_j\geq 0,\Delta\delta_{j+1}<0}(\Delta\delta_j)^2 + \frac{1}{2}\sum_{\Delta\delta_j\geq 0,\Delta\delta_{j-1}<0}(\Delta\delta_j)^2\right. \\
&+ \frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0,\Delta\delta_j\geq 0}(\Delta^2\delta_j)^2 - \sum_{\Delta\delta_j\geq 0,\Delta\delta_{j+1}<0}\Delta\delta_j\Delta\delta_{j+1} \\
&+ \frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j+1}\geq 0}(\Delta\delta_j)^2 + \frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j-1}\geq 0}(\Delta\delta_j)^2 \\
&+ \left.\frac{1}{2}\sum_{\Delta\delta_{j-1}<0,\Delta\delta_j<0}(\Delta^2\delta_j)^2 - \sum_{\Delta\delta_j\geq 0,\Delta\delta_{j+1}<0}\Delta\delta_j\Delta\delta_{j+1}.\right) - \frac{1}{32}\sum(\Delta^2\delta_j)^2.
\end{aligned}
$$

Similar to the perturbation of A and B we can replace φ_j with $\frac{1}{4}$ in Q_2, see (30), and the change is at most $O(\lambda^2\sum(\Delta^2\delta_j)^2)$. Then, we have

$$
Q_2 \approx Q_2^* := \frac{1}{64}\left(2\sum(\Delta^2\delta_j)^2 - \sum(\Delta^2 y_j)^2\right), \tag{44}
$$

We now observe that $Q_1^* + Q_2^*$ is identical to the term $2\beta Q_1 + \beta^2 Q_2$ from Lemma 4 in [20] with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{8}$. We use the lower bound of that lemma and obtain

$$
Q_1^* + Q_2^* \geq \frac{1}{2048}\sum(\Delta^2\delta_j)^2. \tag{45}
$$

We need to find a lower bound for R_1. Using $\varphi_j \sim \varphi_{j-1}$, we have

$$
\begin{aligned}
R_1 \;\approx\;& \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + \sum \varphi_j\Delta\alpha_j\left(\delta_{j-1}\Delta y_{j-1} - \delta_j\Delta y_j\right) \\
&- \sum_{\Delta\delta_j<0}\varphi_j\left(1-\alpha_j+\frac{\alpha_{j-1}+\alpha_{j-2}}{2}\right)(\Delta\delta_j)^2 \\
&- \sum_{\Delta\delta_j\geq 0}\varphi_j\left(1+\alpha_{j-1}-\frac{\alpha_j+\alpha_{j+1}}{2}\right)(\Delta\delta_j)^2.
\end{aligned} \tag{46}
$$

Recall that $\Delta\alpha_j = \alpha_j - \alpha_{j-1} \geq 0$, see (32). Then

$$
\begin{aligned}
R_1 \;\gtrsim\;& \sum \varphi_j\Delta\alpha_j\left(\delta_{j-1}\Delta y_{j-1} - \delta_j\Delta y_j\right) \\
&+ \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 - \sum \varphi_j\left(1-\Delta\alpha_j\right)(\Delta\delta_j)^2 \\
=\;& R_2 + R_3,
\end{aligned} \tag{47}
$$

where

$$R_2 := \sum \varphi_j \Delta\alpha_j \left(\delta_{j-1}\Delta y_{j-1} - \delta_j \Delta y_j\right)$$

and

$$R_3 := \sum \left(\alpha_{j-1}(1-\alpha_j) - \varphi_j(1-\Delta\alpha_j)\right)(\Delta\delta_j)^2.$$

Combining (28), (31), (41), (44), and (45), we derive

$$\sum \delta_j^2 - \sum (\delta_j')^2 \approx Q_1 + Q_2 + Q_3 \gtrsim Q_3 + R_2 + R_3 + \frac{1}{2048}\sum (\Delta^2\delta_j)^2 \tag{48}$$

Let $Q_3^* := Q_3 + R_2 + R_3$. Then

$$Q_3^* = Q_3 + R_2 - \sum (1-\alpha_j)^2 \Delta\alpha_j (\Delta\delta_j)^2.$$

Similar to (32), we estimate $\Delta\alpha_j \le C\lambda\gamma_2(\delta_j + \delta_{j-1})$. This means that we can replace φ_j with $\frac{1}{4}$ and α_j with $\frac{1}{2}$ in all terms of Q_3^* with a change at most $O(\lambda \sum (\delta_j)^3)$. Therefore,

$$Q_3^* \approx \frac{1}{4}\sum \Delta\alpha_j \left(2\delta_{j-1}^2 + 2\delta_j^2 - \delta_{j-1}\Delta y_{j-1} + \delta_j \Delta y_j - (\Delta\delta_j)^2\right) = \frac{1}{4}\sum \Delta\alpha_j z_j, \tag{49}$$

where

$$z_j := 2\delta_{j-1}^2 + 2\delta_j^2 - \delta_{j-1}\Delta y_{j-1} + \delta_j \Delta y_j - (\Delta\delta_j)^2.$$

Using that $\Delta y_j = y_j - y_{j-1}$, $\Delta\delta_j = \delta_j - \delta_{j-1}$, and $y_j = \min(\delta_j, \delta_{j+1}) \ge 0$, we derive

$$z_j \ge (\delta_{j-1} + \delta_j)^2 - \delta_{j-1}y_{j-2} - \delta_j y_j + y_{j-1}(\delta_{j-1} + \delta_j) \ge 2\delta_{j-1}\delta_j + 2y_{j-1}^2 \ge 4y_{j-1}^2.$$

Using the above bound in (49), we conclude

$$Q_3^* \gtrsim \sum \Delta\alpha_j y_{j-1}^2. \tag{50}$$

Let E be the following index set

$$E := \{j \mid \delta_j \text{ is a local maximum}\}. \tag{51}$$

Using (32), we estimate

$$\sum \Delta\alpha_j y_{j-1}^2 \ge C\gamma_1\lambda \sum_{j \notin E} (\delta_j)^3. \tag{52}$$

Combining (48), (50), and (52), we derive the lower bound

$$\begin{aligned} Q_1 + Q_2 + Q_3 &\ge C\gamma_1\lambda \sum_{j\notin E} (\delta_j)^3 + \frac{1}{2048}\sum (\Delta^2\delta_j)^2 \\ &- C\left(\lambda \sum (\Delta^2\delta_j)^2 + \lambda^2 \sum (\delta_j)^3\right), \end{aligned} \tag{53}$$

where C is a positive generic constant which could be different in every appearance but depends only on γ_2, $\|a\|_\infty$, and $\|u^0\|_{L^\infty}$. We split the set E in two disjoint subsets

$$E_1 := \{j \in E \mid \delta_j \geq 3\max(\delta_{j-1}, \delta_{j+1})\} \quad \text{and} \quad E_2 := \{j \in E \mid \delta_j < 3\max(\delta_{j-1}, \delta_{j+1})\}.$$

It is easy to verify that

$$\sum_{j\in E_1} (\delta_j)^3 \leq C \sum (\Delta^2 \delta_j)^2 \quad \text{and} \quad \sum_{j\in E_2} (\delta_j)^3 \leq C \sum_{j \notin E} (\delta_j)^3.$$

Using the above bounds in (53), we derive

$$Q_1 + Q_2 + Q_3 \geq C\gamma_1 \lambda \sum (\delta_j)^3 + C \sum (\Delta^2 \delta_j)^2 - C \left(\lambda \sum (\Delta^2 \delta_j)^2 + \lambda^2 \sum (\delta_j)^3 \right).$$

Therefore, when λ is sufficiently small (but fixed), we have

$$\sum \delta_j^2 - \sum (\delta_j')^2 \approx Q_1 + Q_2 + Q_3 \geq C\gamma_1 \lambda \sum (\delta_j)^3 + C \sum (\Delta^2 \delta_j)^2 \tag{54}$$

with a constant C which depends only on γ_1, γ_2, $\|a\|_\infty$, and $\|u^0\|_{L^\infty}$. This finishes the proof of Lemma 2. □

Using that the equivalence $\approx$ is of order $O(\lambda^2 \sum \delta_j^3)$ in (34) and (54), we prove that

$$\sum \delta_j^2 - \sum (\delta_j')^2 \geq C\gamma_1 \lambda \sum (\delta_j)^3 + C \sum (\Delta^2 \delta_j)^2 \geq 0 \tag{55}$$

for sufficiently small λ, i.e., for small but fixed CFL bound κ in (16). This completes the proof of Theorem 1. □

4 Convergence and error estimates

In this section we are going to use our onesided stability result, Theorem 1, to prove the convergence of the NT scheme to the entropy solution of

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = u^0(x), & x \in \mathbb{R}. \end{cases} \tag{56}$$

In [18], convergence was proven via a single cell entropy inequality. Unfortunately, in order to satisfy the inequality, the authors impose an additional restriction in all regions where the numerical solution is increasing. This reduces the formal order of the NT scheme in such regions to first order. They also note that the additional restriction is not necessary in the applications and one should use the true NT scheme for numerical computations. In order to describe the next result, we need to introduce some notation. A function g is of bounded variation, i.e., $g \in \mathrm{BV}(\mathbb{R})$, if

$$|g|_{\mathrm{BV}(\mathbb{R})} := \sup \sum_{i=1}^{n} |g(x_{i+1}) - g(x_i)| \, < \, \infty,$$

where the supremum is taken over all finite sequences $x_1 < \ldots < x_n$ in $\mathbb{R}$. Functions of bounded variation have at most countable many discontinuities, and their left and right limits $g(x^-)$ and $g(x^+)$ exist at each point $x \in \mathbb{R}$. Since the values of the initial condition u^0 on a set of measure zero have no influence on the numerical solution v and the entropy solution solution u, it is desirable to replace the seminorm $|\cdot|_{\mathrm{BV}(\mathbb{R})}$ by a similar quantity independent of the function values on sets of measure zero. The standard approach in conservation laws is to consider the space $\mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R}))$ of all functions $g \in L^1(\mathbb{R})$ such that the seminorm

$$|g|_{\mathrm{Lip}(1,\mathrm{L}^1(\mathbb{R}))} := \limsup_{y>0} \frac{1}{y} \int_{\mathbb{R}} |g(x+y) - g(x)| \, dx \tag{57}$$

is finite. It is clear that $|g|_{\mathrm{Lip}(1,\mathrm{L}^1(\mathbb{R}))}$ will not change if g is modified on a set of measure zero. At the same time the above two seminorms are equal for functions $g \in \mathrm{BV}(\mathbb{R})$ such that the value of g at a point of discontinuity lies between $g(x^-)$ and $g(x^+)$ (see Theorem 9.3 in [5]). Similarly, we define the space $\mathrm{Lip}(\mathrm{s}, \mathrm{L}^{\mathrm{p}}(\mathbb{R}))$, $1 \le p \le \infty$ and $0 < s \le 1$, which is the set of all functions $g \in L^p(\mathbb{R})$ for which

$$\|g(\cdot - y) - g(\cdot)\|_{L^p(\mathbb{R})} \le M y^s, \quad y > 0. \tag{58}$$

The smallest $M \ge 0$ for which (58) holds is $|g|_{\mathrm{Lip}(\mathrm{s},\mathrm{L}^{\mathrm{p}}(\mathbb{R}))}$. It is easy to see that in the case $p = 1$ and $s = 1$ the seminorm given in (58) is the same as the one in (57). In the case $p > 1$, the space $\mathrm{Lip}(1, L^p(\mathbb{R}))$ is essentially the same as $W^1(L^p(\mathbb{R}))$, see [5] for details. Because our stability result is onesided, for functions $g \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R}))$ we consider the classes $\mathrm{Lip}(\mathrm{s}, \mathrm{L}^{\mathrm{p}})+$ defined by

$$\| \left(g(\cdot - y) - g(\cdot)\right)_+ \|_{L^p(\mathbb{R})} \le M y^s, \quad y > 0. \tag{59}$$

The smallest $M \ge 0$ for which (59) holds is denoted by $|g|_{\mathrm{Lip}(\mathrm{s},\mathrm{L}^{\mathrm{p}})+}$. When we set $p = \infty$ and $s = 1$, we obtain the class $\mathrm{Lip}(1, L^\infty)+$ which is the usual onesided Lipschitz class used in conservation laws denoted by Lip+, see for example [19]. In our previous work [13], we proved that for any $u^0 \in \mathrm{Lip}(1, \mathrm{L}^2)$ the discrete l_2 norm of the jumps satisfies

$$\|\{\delta_j^0\}\|_2^2 = \sum_j (\delta_j^0)^2 \le h \|u^0\|^2_{\mathrm{Lip}(1,\mathrm{L}^2)},$$

see [13] for details. Similarly, it is easy to show the onesided analog

$$\sum_j (\delta_j^0)_+^2 \le h \|u^0\|^2_{\mathrm{Lip}(1,\mathrm{L}^2)+}.$$

We now use Theorem 1 and derive the following onesided bound

$$\sum_j (\delta_j^n)_+^2 \le \sum_j (\delta_j^0)_+^2 \le h \|u^0\|^2_{\mathrm{Lip}(1,\mathrm{L}^2)+} \tag{60}$$

for any $n = 0, 1, \ldots$. Using the estimate

$$\max_j (\delta_j^n) \le \left(\sum_j (\delta_j^n)_+^2 \right)^{\frac{1}{2}}$$

in (60), we derive the Onesided Lipschitz bound

$$\max_{n,j}(\delta_j^n) \leq h^{\frac{1}{2}} \|u^0\|_{\mathrm{Lip}(1,\mathrm{L}^2)+}. \tag{61}$$

For piecewise smooth solutions, it is well known that the fractional bound (61) is enough to guarantee convergence of the numerical method to the entropy solution. In the general case of initial data $u^0 \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R}))$, we refer the reader to [12] where the authors develop a convergence theory for numerical methods for conservation laws and Hamilton-Jacobi equations. They show that a class of TVD numerical methods with a weak onesided bound on second differences for Hamilton-Jacobi equations (see $(2.3)_\alpha$ in [12]) converges to the unique viscosity solution. The corresponding estimate we need here to guarantee convergence to the entropy solution of (56) is

$$\max_{n,j}(\delta_j^n) \leq Ch^{1-\alpha} \tag{62}$$

with $0 < 1 - \alpha \leq 1$. Therefore, we have the following theorem.

Theorem 3. *Let $u^0 \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R})) \cap \mathrm{Lip}(1, \mathrm{L}^2)+$. Then, there exists $\kappa > 0$ such that under the CFL condition $\lambda\|a\|_\infty \leq \kappa$ the NT scheme described in (14) converges to the unique entropy solution of (56).*

It should be possible to develop a theory for error estimates based on (60) for $u^0 \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R})) \cap \mathrm{Lip}(\mathrm{s}, \mathrm{L}^\mathrm{p})+$, $s > 1/2$. But the results do not immediately follow from the existing theory and are out of the scope of this paper. Here, we will discuss the case of nondecreasing initial data only. In our proof of Theorem 1, we considered only one approximation of the flux, see (18). In fact, one can derive the perturbation formula (21) for other approximations of the flux and even for exact evolution in time – that is, when we compute the integrals in (7) exactly. We leave the proof to the reader and only note that Theorem 1 is no longer valid for first order in time approximations of the flux. This implies that Theorem 1 is a *true* second order result. Using the stability result (62) for the NT scheme with exact evolution in time in our general error estimate from [14], we obtain

Theorem 4. *Let $u^0 \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R})) \cap \mathrm{Lip}(1, \mathrm{L}^2)+$ be a nondecreasing function. Then, there exists $\kappa > 0$ such that under the CFL condition $\lambda\|a\|_\infty \leq \kappa$ the NT scheme with exact evolution in time converges to the unique entropy solution of (56) and satisfies the error estimate*

$$\|u(\cdot, T) - v(\cdot, T)\|_{L^1(\mathbb{R})} \leq Ch^{1/4} |u^0|_{\mathrm{Lip}(1,\mathrm{L}^1(\mathbb{R}))}.$$

We include the above error estimate only to show that it is possible to derive error estimates from our new onesided stability. A general estimate for initial data $u^0 \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R})) \cap \mathrm{Lip}(1, \mathrm{L}^2)+$ requires a modification of our arguments in [14] or the dual Lip'-Lip+ arguments in [19] and will be addressed elsewhere.

5 Appendix: Proof of the localization argument

We need to show that the one-sided l_2 norms inequality (17) holds for *any* initial sequence $\{\delta_j\}$. Let $\{w_j\}$ be a generic sequence of cell averages and $\{\delta_j\}$ be its jump sequence. We need to show that

$$||\{(\delta'_j)_+\}||_{l_2} \leq ||\{(\delta_j)_+\}||_{l_2} \tag{63}$$

holds for *any* initial sequence $\{\delta_j\}$ with finite l_2 norm. Recall that we proved (63) for any monotone sequence. We follow our arguments in the case of linear flux in [13]. We consider the sequence $\{w_j\}$ and restrict the index j to a maximal subset Λ_m on which the piecewise constant function w is monotone, recall that $\delta_j = w_j - w_{j-1}$. Given a sequence $\{w_j\}$, we can decompose it into monotone subsequences. This decomposition also gives a decomposition of the sequence $\{\delta_j\}$ into subsequences such that in each subsequence all jumps have the same sign (non-negative or non-positive). Note that in the case of a sequence with non-positive jumps we have a trivial inequality in (63). Without any limitations, we assume that the jumps $\{\delta_j\}$ are non-negative for all $l \leq j \leq r$, $\delta_{l-1} < 0$ and $\delta_{r+1} < 0$. That is, w_{l-1} is a local minimum and w_r is a local maximum of the piecewise constant function w. Let w^m be the following piecewise constant correction of w

$$w_j^m := \begin{cases} w_j, & \text{if} \quad l \leq j \leq r, \\ w_{l-1}, & \text{if} \quad j < l, \\ w_r, & \text{if} \quad j > r. \end{cases} \tag{64}$$

Note that $\Lambda_m = \{j : l \leq j \leq r+1\}$ and the jumps sequence $\delta^m := \{\delta_j^m\}$ of w^m is given by

$$\delta_j^m := \begin{cases} w_j - w_{j-1}, & \text{if} \quad l \leq j \leq r, \\ 0, & \text{otherwise} \end{cases} \tag{65}$$

In the case of a non-increasing subsequence, we extend it with constant values analogous to (64). Hence, we have a sequence of monotone functions $\{w^m\}$ and the corresponding jump sequences $\{\delta^m\} := \{\delta_j^m\}_{j\in\mathbb{Z}}$ such that

$$\sum_m \sum_{j\in\mathbb{Z}} \|\{(\delta^m)_+\}\|_{l_2}^2 = \sum_m \sum_{j\in\Lambda_m} \|\{(\delta_j^m)_+\}\|_{l_2}^2 = \|\{(\delta_j)_+\}\|_{l_2}^2$$

because the sequence of the jumps of $\{\delta_j\}$ is decomposed into disjoint jump subsequences $\{\delta_j^m\}$. We only consider the nonnegative jumps because we are in the case of convex flux and the l_2 norm decreased only for nonnegative initial data. There are two types of jumps δ'_j. A jump δ'_j is of *type 1* if it is equal to the jump $\delta'_j(\delta^m)$ – that is the jump generated with the starting sequence $\{\delta_j^m\}$, where the index m such that $j \in \Lambda_m$. A jump is of *type 2* if it is not of *type 1*. Note that a *type 2* jump δ'_{j^*} occurs only inside an interval which contains a strict local extremum. Near a local extremum we have two nonzero jumps, say $\delta^l_{j^*}$ and $\delta^r_{j^*}$, generated by the two monotone w^m-s with index sets finishing/starting with j^*. It is easy to verify that

$$|\delta'_{j^*}| = \big||(\delta^l_{j^*})'| - |(\delta^r_{j^*})'|\big|$$

where $(\delta^l_{j^*})'$ and $(\delta^r_{j^*})'$ are the new jumps generated by the two monotone sequences ending/starting at the local extremum. Hence, for all nonnegative δ'_{j^*}, we have

$$(\delta'_{j^*})^2 = (\delta'_{j^*})^2_+ < \left((\delta^l_{j^*})'\right)^2_+ + \left((\delta^r_{j^*})'_+\right)^2,$$

where only one of the terms on the right can be positive because the jumps $(\delta^l_{j^*})'$ and $(\delta^r_{j^*})'$ have opposite signs. Therefore, we conclude that

$$\sum_j (\delta'_j)^2_+ \leq \sum_m \sum_{j\in\Lambda_m} (\delta'_j(\delta^m))^2_+ \leq \sum_m \sum_{j\in\Lambda_m} (\delta^m_j)^2_+ = \sum_j (\delta_j)^2_+,$$

where we use the notation $\delta'_j(\delta^m)$ for the new jumps generated by $\{\delta^m\}$. The above argument is similar to our argument in the case of linear flux. The main difference here is that (63) is onesided because the flux is convex and we need a stronger CFL condition to guarantee the TVD property of the NT scheme based on the Minmod limiter. A sufficient condition is

$$\lambda \max_{|u|\leq\|u^0\|_\infty} |f'(u)| \leq 0.32,$$

see [18] for details.

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