INDUSTRIAL MATHEMATICS INSTITUTE

2008:05

Sub-exponentially localized kernels and frames induced by orthogonal expansions

K. Ivanov, P. Petrushev and Y. Xu

IMI

Preprint Series

Department of Mathematics
University of South Carolina

SUB-EXPONENTIALLY LOCALIZED KERNELS AND FRAMES INDUCED BY ORTHOGONAL EXPANSIONS

KAMEN IVANOV, PENCHO PETRUSHEV AND YUAN XU

ABSTRACT. The aim of this paper is to construct sup-exponentially localized kernels and frames in the context of classical orthogonal expansions, namely, expansions in Jacobi polynomials, spherical harmonics, orthogonal polynomials on the ball and simplex, and Hermite and Laguerre functions.

1. INTRODUCTION

Orthogonal expansions have been recently used for the construction of kernels and frames (needlets) with localization that is faster than the reciprocal of any polynomial rate in non-standard settings such as on the sphere, interval and ball with weights, and in the context of Hermite and Laguerre expansions. The main purpose of this article is to show that the rapid decay of that sort of kernels and needlets can be improved to sub-exponential. In order to best present our results, it is perhaps suitable first to exhibit and illustrate the main principles and ideas which guided us in this undertaking.

1.1. **Localization principle.** Here we briefly revisit the "localization principle" described in [18]. Let (E, μ) be a measure space with E a metric space and suppose that there is an orthogonal decomposition $L^2(E, \mu) = \bigoplus_{n=0}^{\infty} \mathcal{V}_n$, where $\mathcal{V}_n$ are finite dimensional subspaces. Let P_n be the kernel of the orthogonal projector $\mathrm{Proj}_n : L^2(E, \mu) \to \mathcal{V}_n$, i.e.

$$(\mathrm{Proj}_n f)(x) = \int_E P_n(x, y) f(y) d\mu, \quad f \in L^2(E, \mu).$$

We are interested in kernels of the form

$$(1.1) \qquad L_n(x, y) := \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) P_j(x, y),$$

where the cutoff function $\hat{a}$ is compactly supported and in C^{∞}. For all our purposes it suffices to only consider cutoff functions obeying the following definition:

Definition 1.1. *A function $\hat{a} \in C^{\infty}[0, \infty)$ ($\hat{a} \geq 0$ if needed) is said to be admissible of type (a), (b) or (c) if it obeys the conditions:*
  (a) $\mathrm{supp}\, \hat{a} \subset [0, 2]$ *and* $\hat{a}(t) = 1$, $t \in [0, 1]$; *or*
  (b) $\mathrm{supp}\, \hat{a} \subset [1/2, 2]$; *or*
  (c) $\mathrm{supp}\, \hat{a} \subset [1/2, 2]$ *and* $|\hat{a}(t)|^2 + |\hat{a}(t/2)|^2 = 1$ *for* $t \in [1, 2]$.

1

Note that (c) is a subcase of (b). We list it separately as the additional requirement in (c) plays an essential role in our construction of tight frames.

The **localization principle** put forward in [18] says that for all "natural orthogonal systems" the kernels $\{L_n(x,y)\}$ decay at rates faster than any inverse polynomial rate away from the main diagonal $y = x$ in $E \times E$ with respect to the distance in E. This principle is very well-known in the case of the trigonometric system (and the Fourier transform) and not so long ago was established for spherical harmonics [15, 16], Jacobi polynomials [1, 17], orthogonal polynomials on the ball [18], and Hermite and Laguerre functions [2, 4, 19, 9].

Surprisingly, however, the localization principle as formulated above fails to be true for tensor product Jacobi polynomials and, in particular, for tensor product Legendre or Chebyshev polynomials, as will be shown in §10.

To grasp the notion of *rapidly decaying kernels* of form (1.1) let us illustrate them in the simple case of Chebyshev polynomials. Denote by $\tilde{T}_n$, $n = 0, 1, \ldots$, the weighted-L^2-normalized Chebyshev polynomials of first kind. Then for an admissible cutoff function $\hat{a}$ the kernels from (1.1) take the form

$$(1.2) \qquad L_n(x,y) := \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) \tilde{T}_j(x)\tilde{T}_j(y)$$

and satisfy (see §2)

$$(1.3) \qquad |L_n(x,y)| \le c_\sigma n(1 + n\rho(x,y))^{-\sigma}, \quad x, y \in [-1, 1],$$

for arbitrarily large $\sigma > 0$ but the constant c_σ depends on sigma; here ρ is the distance $\rho(x,y) := |\arccos x - \arccos y|$. The above estimate suggests that the localization of $L_n(x,y)$ can eventually be improved to a localization of exponential type. This kind of problems will be the main focus of this paper.

1.2. **General scheme for construction of frames from kernels.** The main application of the kernels $L_n(x,y)$ defined in (1.1) is to the construction of frames of rapidly decaying elements (needlets). We now briefly describe the main elements of this construction.

• *Semi-continuous Calderón type decomposition.* Suppose $\hat{a}$ is an admissible cutoff function of type (c) in the sense of Definition 1.1. Then $\sum_{\nu=0}^{\infty} |\hat{a}(2^{-\nu}t)|^2 = 1$, $t \in [1, \infty)$. In the general setup of §1.1, define

$$L_0(x,y) := P_0(x,y) \quad \text{and} \quad L_j(x,y) := \sum_{\nu=0}^{\infty} \hat{a}\left(\frac{\nu}{2^{j-1}}\right) P_\nu(x,y), \quad j = 1, 2, \ldots,$$

and denote briefly $(L_j * f)(x) := \int_E L_j(x,y)f(y)d\mu(y)$. The following decomposition follows readily from the conditions on $\hat{a}$

$$(1.4) \qquad f = \sum_{j=0}^{\infty} L_j * L_j * f \quad \text{for } f \in L^2(E, \mu).$$

• *Discretization via cubature formulas.* Suppose that there is a cubature formula

$$(1.5) \qquad \int_E f d\mu \sim \sum_{\xi \in \mathcal{X}_j} c_\xi f(\xi)$$

with $\mathcal{X}_j \subset E$ and $c_\xi > 0$, which is exact for all functions f of the form $f = gh$ with $g, h \in \bigoplus_{\nu=0}^{2^{2j}} \mathcal{V}_\nu$. Cubature formula (1.5) allows to rewrite (1.4) in the form

$$(1.6) \qquad f(x) = \sum_{j=0}^{\infty} \sum_{\xi \in \mathcal{X}_j} c_\xi^{1/2} L_j(\xi, x) \int_E f(y) c_\xi^{1/2} L_j(\xi, y) dy.$$

• *Definition of frame elements (needlets).* Now the frame elements are defined by

$$(1.7) \qquad \psi_\xi(x) := c_\xi^{1/2} \cdot L_j(\xi, x) \quad \text{for} \quad \xi \in \mathcal{X}_j, \; j = 0, 1, \ldots.$$

We write $\mathcal{X} := \cup_{j=0}^{\infty} \mathcal{X}_j$, where equal points from different levels $\mathcal{X}_j$ are considered as distinct point of $\mathcal{X}$, so that we can use $\mathcal{X}$ as an index set in the definition of the needlet system

$$\Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}.$$

From (1.6) and the definition of $\{\psi_\xi\}$ it readily follows that

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{in} \quad L^2(E, \mu) \quad \text{and} \quad \|f\|_{L^2(E,\mu)} = \left(\sum_{\xi \in \mathcal{X}} |\langle f, \psi_\xi \rangle|^2 \right)^{1/2},$$

i.e. Ψ is a tight frame for $L^2(E, \mu)$.

From definition (1.7) it is clear that the frame elements ψ_ξ inherit the rapid decay of the kernels L_j if this is the case. The superb localization of the building blocks $\{\psi_\xi\}$ is the reason for calling them *needlets.* The rapid decay of needlets makes them a powerful tool for decomposition of spaces of functions and distributions in various settings. The above scheme has already been utilized for construction of needlets and needlet decomposition of L^p, Sobolev, and the more general Triebel-Lizorkin and Besov spaces in the frameworks of spherical harmonics [15, 16], Jacobi polynomials [17, 12], orthogonal polynomials on the ball [18, 13], and Hermite and Laguerre functions [4, 19, 9].

1.3. Sub-exponentially localized wavelets, kernels, and needlets. In [3] Dziubański and Hernández constructed band-limited wavelets of sub-exponential decay. More precisely, they showed that for any $\varepsilon > 0$ there exists a C^∞ mother wavelet ψ such that its Fourier transform $\widehat{\psi}$ is compactly supported on $\mathbb{R}$ and

$$(1.8) \qquad |\psi(x)| \leq c_\varepsilon \exp\{-|x|^{1-\varepsilon}\}, \quad x \in \mathbb{R},$$

with c_ε a constant depending on ε. They also showed that in this estimate $\varepsilon > 0$ cannot be removed. However, as will be shown in §2, the localization of ψ can be improved to $|\psi(x)| \leq c \exp\{-\frac{c'|x|}{\ln(1+|x|)^{1+\varepsilon}}\}$ and beyond (see §2).

Our aim in this paper is to construct kernels and needlets with similar (sub-exponential) localization in the context of Jacobi polynomials, spherical harmonics, orthogonal polynomials on the d-dimensional ball and simplex with weights, and d-dimensional Hermite and Laguerre functions.

For instance, we shall show that for any $\varepsilon > 0$ there exists an admissible cutoff function $\hat{a}$ of type (a), (b) or (c) in the sense of Definition 1.1 such that the kernels $L_n(x, y)$ from (1.2) satisfy (see Theorem 2.1 below)

$$(1.9) \qquad |L_n(x, y)| \leq cn \exp \left\{ -\frac{c_\varepsilon n \rho(x, y)}{[\ln(e + n \rho(x, y))]^{1+\varepsilon}} \right\}, \quad x, y \in [-1, 1].$$

Evidently, this estimate yields (1.3). The above estimate can be further improved by replacing the term $[\ln(e + n\rho(x,y))]^{1+\varepsilon}$ by any product of the form

$$(1.10) \quad \ln(e + n\rho(x,y)) \cdots \underbrace{\ln\cdots\ln}_{\ell} \left(\underbrace{\exp\cdots\exp}_{\ell} 1 + n\rho(x,y) \right)$$

$$\times \Big[\underbrace{\ln\cdots\ln}_{\ell+1} \left(\underbrace{\exp\cdots\exp}_{\ell+1} 1 + n\rho(x,y) \right) \Big]^{1+\varepsilon}, \quad \ell > 1.$$

Estimate (1.9) leads to the following localization of the jth level Chebyshev needlets $|\psi_\xi(x)| \le c2^{j/2} \exp\left\{ -\frac{c_\varepsilon 2^j \rho(x,\xi)}{[\ln(e+2^j\rho(x,\xi))]^{1+\varepsilon}} \right\}$, which can be further improved as above. We shall also show that the above estimates are sharp in the sense that $\varepsilon > 0$ cannot be removed.

We would like to emphasize that according to the localization principle kernels of the form e.g. (1.2) are rapidly decaying for an arbitrary admissible cutoff function $\hat{a}$, while the sub-exponential localization of these kernels is only possible for exceptional cutoff functions $\hat{a}$. One of the main steps in constructing kernels and frames of sub-exponential localization is the construction of admissible cutoff functions $\hat{a}$, most importantly ones of type (c), with derivatives obeying $\|\hat{a}^{(k)}\|_\infty \le \tilde{c}(\tilde{c}/\varepsilon)^k k^k (\ln k)^{k(1+\varepsilon)}$, $k \ge 3$, see Theorem 3.1 below. Although we employ various techniques for proving our results, the proof of the sub-exponential localization of Jacobi kernels plays a prominent role in this paper.

An outline of this paper is as follows: In §2 we clarify the impact of the behavior of the cutoff functions around zero on the localization of the respective kernels in the simplest case of Chebyshev polynomials. We also illustrate the notion of sub-exponential localization of kernels in the case of Chebyshev polynomials. In §10 we show that the localization principle described in §1.1 is no longer valid in the case of tensor product Chebyshev or Legendre polynomials ($d = 2$) and products of Chebyshev and Legendre polynomials. In §3 we construct admissible cutoff functions of "small" derivatives. Sections 4, 5, 6, 7, 8, and 9 are devoted to the construction of sub-exponentially localized kernels and needlets in the context of Jacobi polynomials, spherical harmonics, orthogonal polynomials on the ball and simplex, and Hermite and Laguerre functions.

Throughout this paper we shall use the following notation: For $x \in \mathbb{R}^d$ we shall use the norms $\|x\| = \|x\|_\infty := \max_i |x_i|$, $\|x\|_2 := (\sum_i |x_i|^2)^{1/2}$, and $|x| = \|x\|_1 := \sum_i |x_i|$. Positive constants will be denoted by c, c_1, c', $\ldots$ and they may vary at every occurrence, $A \sim B$ will stand for $c_1 A \le B \le c_2 A$. Also, $\lfloor x \rfloor$ will denote the largest integer not exceeding x.

2. Localization principle and sub-exponential localization: Simple examples

We **first** would like to clarify the impact of the behavior of the cutoff function $\hat{a}$ at $t = 0$ on the localization properties of kernels as in (1.1). To this end, we shall use the simple example of normalized Chebyshev polynomials of first kind on $[-1, 1]$ defined by

$$\tilde{T}_n(x) := \sqrt{2/\pi} \cos n \arccos x \quad \text{for } n \ge 1 \text{ and } \quad \tilde{T}_0 := \sqrt{1/\pi}.$$

As is well-known $\{\tilde{T}_n\}_{n\ge0}$ is an orthonormal basis for $L^2([-1,1], (1 - x^2)^{-1/2})$. We are interested in the localization of the kernels $L_n(x,y)$, defined in (1.2) with

$\hat{a} \in C^\infty[0, \infty)$ and, say, supp $\hat{a} \subset [0, 2]$. Setting $x =: \cos\theta$ and $y =: \cos\phi$, we have

$$L_n(\cos\theta, \cos\phi) = \frac{2}{\pi}\left(\hat{a}(0)/2 + \sum_{j=1}^{\infty} \hat{a}\left(\frac{j}{n}\right)\cos\theta\cos\phi\right) = \frac{1}{\pi}(F_n(\theta - \phi) + F_n(\theta + \phi)),$$

where $F_n(\theta) := \hat{a}(0)/2 + \sum_{j=1}^{\infty} \hat{a}(\frac{j}{n})\cos\theta$. Let $\hat{a}$ be the even extension of $\hat{a}$, i.e. $\hat{a}(t) := \hat{a}(-t)$ for $t < 0$. Then

$$(2.1) \qquad F_n(\theta) = \frac{1}{2}\sum_{j\in\mathbb{Z}} \hat{a}\left(\frac{j}{n}\right)e^{ij\theta}$$

and the question of localization of $L_n(x, y)$ reduces to the localization of the trigonometric polynomial $F_n(\theta)$ around $\theta = 0$.

It is easy to see (see the proof of Theorem 2.1 below) that if $\hat{a} \in C^\infty$, then for any $\sigma > 0$ there exists a constant $c_\sigma > 0$ such that

$$(2.2) \qquad |F_n(\theta)| \le c_\sigma \frac{n}{(1 + n|\theta|)^\sigma}, \quad \theta \in [-\pi, \pi],$$

which readily leads to

$$(2.3) \qquad |L_n(x, y)| \le c_\sigma \frac{n}{(1 + n\rho(x, y))^\sigma} \quad \forall\sigma > 0,$$

where $\rho(x, y)$ is the distance on $[-1, 1]$ defined by $\rho(x, y) := |\arccos x - \arccos y|$. The point is that the localization from (2.3) is not possible if the even extension of $\hat{a}$ to $\mathbb{R}$ is not in C^∞. The precise argument is that (2.3) holds for $y = 1$ if and only if (2.2) is true. In turn (2.2) is valid if and only if $\hat{a} \in C^\infty$. This last claim can be justified as follows: Let f be defined with Fourier transform $\hat{f}(\xi) := \hat{a}(\xi/n)e^{i\xi t}$. Then $f(y) = na(n(y + t))$ and the Poisson summation formula

$$(2.4) \qquad \sum_{j\in\mathbb{Z}} g(2\pi j) = (2\pi)^{-1}\sum_{j\in\mathbb{Z}} \hat{g}(j) \quad \left(\hat{g}(\xi) := \int_{\mathbb{R}} g(t)e^{-i\xi t}dt\right)$$

gives

$$(2.5) \qquad F_n(t) := \frac{1}{2}\sum_{j\in\mathbb{Z}} \hat{a}\left(\frac{j}{n}\right)e^{ijt} = \pi n\sum_{j\in\mathbb{Z}} a(n(t + 2\pi j)).$$

Hence, the inverse Fourier transform a of $\hat{a}$, has to be in the Schwartz class, which is only possible if $\hat{a} \in C^\infty$. For this it suffices to have $\hat{a}^{(k)}(0) = 0$ for $k \ge 1$. For other orthogonal systems (see below), however, it is not completely clear what behavior of $\hat{a}$ at $t = 0$ would lead to rapid decay of the respective kernels and needlets.

In concluding, it is evident that the behavior of the cutoff function $\hat{a}$ at $t = 0$ is important and not every compactly supported C^∞ function $\hat{a}$ will give rise to rapidly decaying kernels $L_n(x, y)$. For our purposes, however, it suffices to restrict the selection of cutoff functions $\hat{a}$ to compactly supported C^∞ functions which are constants in a neighborhood of $t = 0$, which automatically resolves the issue about the behavior of $\hat{a}$ at $t = 0$ (see Definition 1.1).

Secondly, we want to illustrate with the next theorem the notion of a sub-exponential localization of kernels of type (1.1) on the simple example of Chebyshev polynomials of first kind.

Theorem 2.1. *For any $0 < \varepsilon \leq 1$ there exists an admissible cutoff function $\hat{a}$ of type* (a) *or* (b) *or* (c) *such that the kernels $L_n(x, y)$ from* (1.2) *satisfy*

$$(2.6) \qquad |L_n(x, y)| \leq cn \exp\left\{-\frac{c'\varepsilon n\rho(x, y)}{[\ln(e + n\rho(x, y))]^{1+\varepsilon}}\right\}, \quad x, y \in [-1, 1],$$

where $c' > 0$ is an absolute constant and c depends only on ε.

Proof. A key ingredient in this proof is the existence of an admissible cutoff function $\hat{a}$ of an arbitrary type such that

$$(2.7) \qquad \|\hat{a}^{(k)}\|_\infty \leq \tilde{c}(\tilde{c}/\varepsilon)^k k^k (\ln k)^{k(1+\varepsilon)} \quad \text{for } k \geq 3,$$

where $\tilde{c} > 0$ is an absolute constant (see Theorem 3.1 below). Denote again by $\hat{a}$ the even extension of $\hat{a}$ on $\mathbb{R}$.

Set $x =: \cos\theta$ and $y =: \cos\phi$. Exactly as above

$$(2.8) \qquad L_n(\cos\theta, \cos\phi) = \frac{1}{\pi}(F_n(\theta - \phi) + F_n(\theta + \phi)),$$

where F_n is defined in (2.1) and also (2.5) holds. Note that $F_n(\theta)$ is a 2π-periodic trigonometric polynomial of degree $< 2n$. Evidently,

$$t^\ell a(t) = \frac{i^\ell}{2\pi} \int_{\mathbb{R}} \hat{a}^{(\ell)}(\xi)e^{i\xi t}d\xi,$$

and hence $|t|^\ell |a(t)| \leq \|\hat{a}^{(\ell)}\|_\infty$ as $\operatorname{supp}\hat{a} \subset [-2, 2]$. This for $\ell = 0$ and $\ell = k$ gives

$$|a(t)| \leq 2^k \frac{\|\hat{a}^{(k)}\|_\infty + \|\hat{a}\|_\infty}{(1 + |t|)^k} \leq \frac{c(2\tilde{c}/\varepsilon)^k k^k (\ln k)^{k(1+\varepsilon)}}{(1 + |t|)^k}, \quad k \geq 3,$$

on account of (2.7). Combining this with (2.5) gives

$$(2.9) \qquad |F_n(\theta)| \leq cn(2\tilde{c}/\varepsilon)^k k^k (\ln k)^{k(1+\varepsilon)} \sum_{j\in\mathbb{Z}} \frac{1}{(1 + n|\theta + 2\pi j|)^k}$$

$$\leq cn\frac{(2\tilde{c}/\varepsilon)^k k^k (\ln k)^{k(1+\varepsilon)}}{(1 + n|\theta|)^k} = cn\left(\frac{(2\tilde{c}/\varepsilon)k(\ln k)^{(1+\varepsilon)}}{1 + n|\theta|}\right)^k.$$

We next use the above to show that

$$(2.10) \qquad |F_n(\theta)| \leq cn \exp\left\{-\frac{c'\varepsilon n|\theta|}{[\ln(e + n|\theta|)]^{1+\varepsilon}}\right\}, \quad |\theta| \leq \pi.$$

Indeed, if $1 + n|\theta| \leq 6e(2\tilde{c}/\varepsilon)[\ln(e + n|\theta|)]^{1+\varepsilon}$, then (2.10) follows from the obvious estimate $|F_n(\theta)| \leq 2n$.

Suppose $1 + n|\theta| > 6e(2\tilde{c}/\varepsilon)[\ln(e + n|\theta|)]^{1+\varepsilon}$. Choosing $k := \left\lfloor \frac{1+n|\theta|}{2e(2\tilde{c}/\varepsilon)[\ln(e+n|\theta|)]^{1+\varepsilon}} \right\rfloor$ one easily shows that $k \geq 3$ and $\frac{(2\tilde{c}/\varepsilon)k(\ln k)^{1+\varepsilon}}{1+n|\theta|} \leq e^{-1}$, and (2.10) follows by (2.9).

Note that θ, ϕ from (2.8) obey $0 \leq \theta, \phi \leq \pi$. We use (2.10) to estimate $|F_n(\theta - \phi)|$.

If $\theta + \phi \leq \pi$, then we use (2.10) and that $\theta + \phi \geq |\theta - \phi|$ to estimate $|F_n(\theta + \phi)|$.

If $\theta + \phi > \pi$, then $0 < 2\pi - \theta - \phi \leq \pi$ and since F_n is 2π-periodic and even, we have $F_n(\theta + \phi) = F_n(2\pi - \theta - \phi)$. We now use $2\pi - \theta - \phi \geq |\theta - \phi|$ and (2.10) to estimate $|F_n(\theta + \phi)| = |F_n(2\pi - \theta - \phi)|$. Putting together these estimates, we get

$$|L_n(\cos\theta, \cos\phi)| \leq \frac{1}{\pi}(|F_n(\theta - \phi)| + |F_n(\theta + \phi)|)$$

$$\leq cn \exp\left\{-\frac{c'\varepsilon n|\theta - \phi|}{[\ln(e + n|\theta - \phi|)]^{1+\varepsilon}}\right\},$$

which implies (2.6).   □

As a byproduct of the above proof we get the following localization result for trigonometric polynomials. For a compactly supported cutoff function $\hat{a}$ consider the trigonometric polynomial

$$(2.11) \qquad L_n(\theta) := \sum_{j \in \mathbb{Z}} \hat{a}\left(\frac{j}{n}\right) e^{ij\theta}.$$

Theorem 2.2. *For any $0 < \varepsilon \le 1$ there exists a C^∞ cutoff function $\hat{a} \not\equiv 0$ supported on $[-2, 2]$ such that the polynomial $L_n(\theta)$ from (2.11) satisfies*

$$(2.12) \qquad |L_n(\theta)| \le cn \exp\left\{ -\frac{c'\varepsilon n|\theta|}{[\ln(e + n|\theta|)]^{1+\varepsilon}} \right\}, \quad \theta \in [-\pi, \pi],$$

where $c' > 0$ is an absolute constant and c depends only on ε.

The construction of $\hat{a}$ is as above and the proof of (2.12) is exactly the same as the proof of (2.10) above. We omit it.

Several remarks are in order. Estimate (2.12) can be slightly improved, namely, the term $[\ln(e+n|\theta|)]^{1+\varepsilon}$ in (2.12) can be replaced by $\ln(e+n|\theta|)[\ln\ln(e^e+n|\theta|)]^{1+\varepsilon}$ or, in general, by any product of the form

$$(2.13) \ \ln(e+n|\theta|)\cdots\underbrace{\ln\cdots\ln}_{\ell}(\underbrace{\exp\cdots\exp}_{\ell} 1+n|\theta|)\Big(\underbrace{\ln\cdots\ln}_{\ell+1}\big(\underbrace{\exp\cdots\exp}_{\ell+1} 1+n|\theta|\big)\Big)^{1+\varepsilon}$$

with $\ell \ge 1$.

Estimate (2.6) can also be improved in a similar fashion (see Theorem 4.1 below). Note also that in these cases inequality (2.12) is trivially true for $\varepsilon = 0$.

On the other hand, Theorem 2.2 is sharp in the sense that (2.12) cannot be replaced by an estimate of this type

$$(2.14) \qquad |L_n(\theta)| \le c|L_n(0)|e^{-c'\varphi(n|\theta|)}, \quad \theta \in [-\pi, \pi],$$

with φ of the form $\varphi(t) = t\ln(e+t)^{-1}$ or $\varphi(t) = t\ln(e+t)^{-1}\ln\ln(e^e+t)^{-1}$ or etc. This follows by Example 2.2 in [8], which implies that the inequality

$$(2.15) \qquad \int_1^\infty \frac{\varphi(t)}{t^2}\, dt < \infty$$

is a necessary condition for the validity of (2.14) for an appropriate polynomial L_n of degree n. A similar observation applies to Theorem 2.1 as well.

Our **third** point is that the sub-exponential localization of the wavelet from (1.8), established in [3], can be improved as follows.

Theorem 2.3. *For any $0 < \varepsilon \le 1$ there exists a bandlimited orthogonal wavelet ψ such that*

$$(2.16) \qquad |\psi(x)| \le c \exp\left\{ -\frac{c'\varepsilon|x|}{[\ln(e + |x|)]^{1+\varepsilon}} \right\}, \quad x \in \mathbb{R},$$

where $c' > 0$ is an absolute constant and c depends only on ε. Furthermore, the term $[\ln(e+|x|)]^{1+\varepsilon}$ can be replaced by a product similar to the one in (2.13) above.

Proof. In essence, this proof is contained in the proof of Theorem 2.1. So, we only outline it here. Choose $\hat{a} \ge 0$ to be the admissible function of type (c) from the proof Theorem 3.1 below and such that its derivatives obey (2.7). Let us now denote again by $\hat{a}$ the even extension of $\hat{a}$ to all of $\mathbb{R}$ and set $\theta(\xi) := \hat{a}(\frac{3}{4\pi}\xi)$. From

the properties of $\hat{a}$ (see Theorem 3.1) it readily follows that the function ψ with Fourier transform

$$\hat{\psi}(\xi) := \theta(\xi)e^{-i\xi/2} = \hat{a}\left(\frac{3}{4\pi}\xi\right)e^{-i\xi/2}$$

is a band limited orthogonal wavelet (see [14]). It remains to show that ψ obeys (2.16). Indeed, we have

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{a}\left(\frac{3}{4\pi}\xi\right)e^{-i\xi/2}e^{ix\xi}d\xi,$$

which implies

$$(x-1/2)^{\ell}\psi(x) = \frac{1}{2\pi}\left(\frac{3i}{4\pi}\right)^{\ell}\int_{\mathbb{R}}\hat{a}^{(\ell)}\left(\frac{3}{4\pi}\xi\right)e^{i(x-1/2)\xi}d\xi \quad \forall\ell.$$

Similarly as in the proof of Theorem 2.1, this and (2.7) yield

$$|\psi(x)| \leq c\left(\frac{(2\tilde{c}/\varepsilon)k(\ln k)^{(1+\varepsilon)}}{1+|x|}\right)^{k}, \quad k \geq 3,$$

and one completes the proof by choosing k appropriately. $\qquad\square$

Finally, we remark that estimate (2.16) is sharp in the sense that it cannot be replaced by an estimate of the form $|\psi(x)| \leq c\exp\left\{-\frac{c'|x|}{\ln(e+|x|)}\right\}$. To see this, we shall use the argument we used to prove the sharpness of estimate (2.12). Precisely as in (2.5) with $a(t)$ replaced by $\psi(x)$ (using the Poisson summation formula (2.4)), we have

$$\phi_n(x) := \sum_{j\in\mathbb{Z}}\hat{\psi}\left(\frac{j}{n}\right)e^{ijx} = 2\pi n\sum_{j\in\mathbb{Z}}\psi\left(n(x+2\pi j)\right).$$

Note that since $\hat{\psi}$ is compactly supported, then ϕ_n from above is a trigonometric polynomial of degree $O(n)$. Assuming now that $|\psi(x)| \leq c\exp\left\{-\frac{c'|x|}{\ln(e+|x|)}\right\}$ leads to $|\phi_n(x)| \leq cn\exp\left\{-\frac{c'n|x|}{\ln(e+n|x|)}\right\}$. But this is impossible since the condition given in (2.14)-(2.15) would be violated.

3. Construction of C^∞ cutoff functions

The present section lays down some of the ground work that will be needed for the construction of sup-exponentially localized kernels and needlets. Admissible cutoff functions of "small" derivatives will be needed.

Theorem 3.1. *Let $0 < \varepsilon \leq 1$. Then there exists an admissible cutoff function $\hat{a}$ of any type: (a), (b) or (c) (see Definition 1.1) such that $0 \leq \hat{a} \leq 1$,*
(3.1)
$$\|\hat{a}^{(k)}\|_\infty \leq \tilde{c}(\tilde{c}/\varepsilon)^k k^k (\ln k)^{k(1+\varepsilon)}, \ k \geq 3, \ \text{and} \ \|\hat{a}^{(k)}\|_\infty \leq \tilde{c}(\tilde{c}/\varepsilon)^k k^k, \ k = 1, 2.$$

where $\tilde{c} > 1$ is an absolute constant (e.g. $\tilde{c} = 88$). Moreover, $\hat{a}^{(k)}(1) = 0$ for $k \geq 1$.

Furthermore, there exists an admissible function $\hat{a}$ of type (a), (b) or (c) such that the above estimates still hold with the term $(\ln k)^{k(1+\varepsilon)}$ replaced by $(\ln k)^k(\ln\ln k)^{k(1+\varepsilon)}$ or, in general, by a product of the form $(\ln k)^k \cdots (\underbrace{\ln\cdots\ln}_{\ell}k)^k(\underbrace{\ln\cdots\ln}_{\ell+1}k)^{k(1+\varepsilon)}$ with

$\ell \geq 1$, for sufficiently large k and $\tilde{c}$ depending on ℓ.

Proof. We first construct a C^∞ bump h of "small" derivatives. To this end let $\chi_\delta := \frac{1}{2\delta}\mathbb{1}_{[-\delta,\delta]}$ and choose $\delta_0 = \delta_1 := 1$ and $\delta_j := \frac{1}{j(\ln j)^{1+\varepsilon}}$ if $j \geq 2$. Note that

$$\sum_{j=0}^{\infty} \delta_j \leq 2 + \frac{1}{2} + \frac{1}{3} + \int_3^\infty \frac{dt}{t(\ln t)^{1+\varepsilon}} < 3 + \frac{1}{\varepsilon(\ln 3)^\varepsilon} \leq \frac{4}{\varepsilon}.$$

We define

$$h_m := \chi_{\delta_0} * \cdots * \chi_{\delta_m} \quad \text{and} \quad h(t) := \lim_{m \to \infty} h_m(t).$$

It is easy to see that (cf. [7, Theorem 1.3.5]) $h \in C^\infty$, $h \geq 0$, $\operatorname{supp} h \subset (-\frac{4}{\varepsilon}, \frac{4}{\varepsilon})$ and

$$\|h^{(k)}\|_\infty \leq \frac{1}{\prod_{j=0}^k \delta_j} \leq k^k (\ln k)^{k(1+\varepsilon)} \quad \text{for} \quad k \geq 3.$$

Also, since $\int_\mathbb{R} \chi_\delta = 1$, we have $\int_\mathbb{R} h = 1$.

Our second step is to rescale h, namely, we define $h_\varepsilon(t) := \frac{8}{\varepsilon} h(\frac{8t}{\varepsilon})$.

We next apply standard wavelet techniques. We first integrate h_ε, i.e. we define $g(t) := \frac{\pi}{2} \int_{-\infty}^t h_\varepsilon(s)ds$. Evidently, $g \in C^\infty$, $0 \leq g \leq \pi/2$, $\operatorname{supp} g' \subset (-\frac{1}{2}, \frac{1}{2})$, $g(t) + g(-t) = \frac{\pi}{2}$ for $t \in \mathbb{R}$, $\|g^{(k)}\|_\infty \leq \frac{\pi}{2}(\frac{8}{\varepsilon})^{k+1}$ for $0 \leq k \leq 3$, and

$$(3.2) \qquad \|g^{(k+1)}\|_\infty \leq \frac{\pi}{2}\|h_\varepsilon^{(k)}\|_\infty \leq \frac{\pi}{2}\left(\frac{8}{\varepsilon}\right)^{k+1} k^k (\ln k)^{k(1+\varepsilon)} \quad \text{for} \quad k \geq 3.$$

Observe that $\hat{a}(t) := \frac{2}{\pi}g(\frac{3}{2} - t)$ is an admissible function of type (a).

In order to construct an admissible function $\hat{a}$ of type (c) and, hence, of type (b) we define $\phi(t) := \sin g(t)$, $t \in \mathbb{R}$. From above, $\phi(t)^2 + \phi(-t)^2 = 1$ for $t \in \mathbb{R}$. We now set

$$\hat{a}(t) := \begin{cases} \phi(2t - \frac{3}{2}) & \text{if} \quad t \in [\frac{1}{2}, 1], \\ \phi(\frac{3}{2} - t) & \text{if} \quad t \in (1, 2], \\ 0 & \text{if} \quad \mathbb{R} \setminus [\frac{1}{2}, 2], \end{cases}$$

and claim that $\hat{a}$ has the properties of an admissible function of type (c). All of them are easy to verify but (3.1), which needs some care. Here we use some ideas from [3]. Fix $t_0 \in (-\frac{1}{2}, \frac{1}{2})$ and let $g_k(t) := \sum_{j=0}^k \frac{(t-t_0)^j}{j!} g^{(j)}(t_0)$ - the kth degree Taylor polynomial of g centered at t_0. Apparently, $\phi^{(k)}(t_0) = [\sin g_k]^{(k)}(t_0)$ and since $\sin g_k(z)$ is an entire function, by the Cauchy formula,

$$(3.3) \qquad \phi^{(k)}(t_0) = \frac{k!}{2\pi i} \int_\gamma \frac{\sin g_k(z)}{(z - t_0)^{k+1}} dz,$$

where $\gamma := \{z \in \mathbb{C} : |z - t_0| = r\}$ with $r = \frac{1}{2e(8/\varepsilon)(\ln k)^{1+\varepsilon}}$. By Stirling's formula $n! > (n/e)^n$ and using (3.2) we have for $z \in \gamma$ and $k \geq 3$

$$|g_k(z)| \leq \sum_{j=0}^k \frac{1}{j!} \frac{\pi}{2}\left(\frac{8}{\varepsilon}\right)^j \frac{j^j(\ln k)^{j(1+\varepsilon)}}{[2e(8/\varepsilon)(\ln k)^{1+\varepsilon}]^j} \leq \frac{\pi}{2} \sum_{j=0}^k \frac{1}{2^j} < \pi$$

and, therefore, $|\sin g_k(z)| \leq e^\pi$ for $z \in \gamma$. We use this in (3.3) to obtain

$$|\phi^{(k)}(t_0)| \leq \frac{k!}{2\pi} e^\pi 2\pi[2e(8/\varepsilon)(\ln k)^{1+\varepsilon}]^k \leq e^\pi (44/\varepsilon)^k k^k (\ln k)^{k(1+\varepsilon)},$$

which implies (3.1). Here we used that $k! \leq k^k$.

For the construction of $\hat{a}$ which satisfies (3.1) with $(\ln k)^{k(1+\varepsilon)}$ replaced by a product of the form $(\ln k)^k \cdots (\underbrace{\ln \cdots \ln}_{\ell} k)^k (\underbrace{\ln \cdots \ln}_{\ell+1} k)^{k(1+\varepsilon)}$ one proceeds as above with obvious modifications. $\square$

Remark 3.2. Theorem 3.1 cannot be improved in the sense that ε in the exponent in (3.1) cannot be removed (replaced by $\varepsilon = 0$). Indeed, by the Denjoy-Carleman theorem (see e.g. Theorem 1.3.8 in [7]) one has that if $\hat{a} \in C^\infty$ on a given interval and

$$\|\hat{a}^{(k)}\|_\infty \le c^{k+1} M_k, \quad k = 0, 1, \dots,$$

then the divergence of the series $\sum_{j=0}^\infty 1/L_j$ with $L_j = \inf_{k \ge j} M_k^{1/k}$ yields that $\hat{a}$ is quasi-analytic (see [7, Definition 1.3.7.]). Having in mind that the cutoff functions are not quasi-analytic we conclude that the above inequalities cannot be true with $M_k = [k \ln(e+k)]^k$ or $M_k = \left[k \ln(e+k) \cdots \underbrace{\ln \cdots \ln}_{\ell} \left(\underbrace{\exp \cdots \exp}_{\ell} 1 + k\right)\right]^k$, in general.

4. Sub-exponentially localized kernels and frames induced by Jacobi polynomials

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\ge 0}$, form an orthogonal basis for the weighted space $L^2([-1,1], w_{\alpha,\beta})$ with weight $w_{\alpha,\beta}(t) := (1-t)^\alpha (1+t)^\beta$, $\alpha, \beta > -1$. They are traditionally normalized by $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. It is well known that [21, (4.3.3)]

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) dt = \delta_{n,m} h_n^{(\alpha,\beta)},$$

where

$$(4.1) \qquad h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

We are interested in kernels of the form

$$(4.2) \qquad L_n^{\alpha,\beta}(x,y) = \sum_{j=0}^\infty \hat{a}\left(\frac{j}{n}\right) \left(h_j^{(\alpha,\beta)}\right)^{-1} P_j^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y),$$

where $\hat{a}$ is an admissible cutoff function in the sense of Definition 1.1. In [17] it is proved that this sort of kernels decay rapidly away from the main diagonal in $[-1,1]^d \times [-1,1]^d$. Our aim here is to show that for suitable admissible cutoff functions $\hat{a}$ the localization of $L_n^{\alpha,\beta}(x,y)$ can be improved to sub-exponential.

To state our result, we need the quantity

$$(4.3) \qquad w_{\alpha,\beta}(n;x) := (1-x+n^{-2})^{\alpha+1/2}(1+x+n^{-2})^{\beta+1/2}.$$

We shall also use again the distance $\rho(x,y) := |\arccos x - \arccos y|$ on $[-1,1]$.

Theorem 4.1. *Let $\alpha, \beta \ge -1/2$ and $0 < \varepsilon \le 1$. Then for any admissible cutoff function $\hat{a}$ obeying inequality (3.1) in Theorem 3.1 the kernels from (4.2) satisfy the following inequality for $x, y \in [-1,1]$*

$$(4.4) \qquad |L_n^{\alpha,\beta}(x,y)| \le \frac{cn}{\sqrt{w_{\alpha,\beta}(n;x)}\sqrt{w_{\alpha,\beta}(n;y)}} \exp\left\{-\frac{c^\diamond n\rho(x,y)}{[\ln(e+n\rho(x,y))]^{1+\varepsilon}}\right\},$$

where $c^\diamond = c'\varepsilon$ with $c' > 0$ an absolute constant, and c depends only on α, β, and ε.

Furthermore, for an appropriate cutoff function $\hat{a}$ the term $[\ln(e+n\rho(x,y))]^{1+\varepsilon}$ above can be replaced by any product of the form (1.10).

For the proof of Theorem 4.1 and the respective localization results on the sphere (Theorem 5.1) and ball (Theorem 6.1), we first need to establish the sub-exponential localization of $L_n^{\alpha,\beta}(x,1)$. Set

$$(4.5) \qquad \mathcal{Q}_n^{\alpha,\beta}(x) := L_n^{\alpha,\beta}(x,1) = \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right)\left(h_j^{(\alpha,\beta)}\right)^{-1} P_j^{(\alpha,\beta)}(1) P_j^{(\alpha,\beta)}(x).$$

It is easy to see that

$$(4.6) \qquad \mathcal{Q}_n^{\alpha,\beta}(x) = c^{\star} \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha,\beta)}(x),$$

where $c^{\star} := 2^{-\alpha-\beta-1}\Gamma(\alpha+1)^{-1}$.

Theorem 4.2. *Let $\alpha \geq \beta \geq -1/2$ and $0 < \varepsilon \leq 1$. Assume that the cutoff function $\hat{a}$ in (4.5) is given by Theorem 3.1. Then we have*

$$(4.7) \qquad \left|\frac{d^r}{dx^r}\mathcal{Q}_n^{\alpha,\beta}(\cos\theta)\right| \leq cn^{2\alpha+2r+2}\exp\left\{-\frac{c^{\diamond}n\theta}{[\ln(e+n\theta)]^{1+\varepsilon}}\right\}, \quad 0 \leq \theta \leq \pi,$$

where $c^{\diamond} = c'\varepsilon$ with $c' > 0$ an absolute constant and $c = c''8^r$ with $c'' > 0$ depending only on α, β, and ε.

Moreover, for an appropriate cutoff function $\hat{a}$ of type (a), (b), or (c) from Theorem 3.1 the above estimate holds with the term $[\ln(e+n\theta)]^{1+\varepsilon}$ replaced by any product of the form (2.13).

Proof. We first prove (4.7) for $r = 0$. We obtain a trivial estimate by simply using that $\Gamma(j+a)/\Gamma(j+1) \sim j^{a-1}$ and $\|P_n^{(\alpha,\beta)}\|_{L^{\infty}[-1,1]} \leq cn^{\alpha}$ with $c = c(\alpha,\beta)$ [21, (7.32.6)]. We get

$$(4.8) \qquad |\mathcal{Q}_n^{\alpha,\beta}(\cos\theta)| \leq c\sum_{j=0}^{2n} j^{2\alpha+1} \leq cn^{2\alpha+2},$$

which implies (4.7) ($r = 0$) for $0 \leq \theta \leq 1/n$.

A key role in proving a nontrivial estimate on $|\mathcal{Q}_n^{\alpha,\beta}(\cos\theta)|$ will play the identity [21, (4.5.3)]:

$$(4.9) \qquad \sum_{\nu=0}^{n} \frac{(2\nu+\alpha+k+\beta+1)\Gamma(\nu+\alpha+k+\beta+1)}{\Gamma(\nu+\beta+1)} P_{\nu}^{(\alpha+k,\beta)}(x)$$
$$= \frac{\Gamma(n+\alpha+k+1+\beta+1)}{\Gamma(n+\beta+1)} P_n^{(\alpha+k+1,\beta)}(x).$$

Applying summation by parts to the sum in (4.6) (using (4.9) with $k = 0$), we get

$$(4.10) \qquad \mathcal{Q}_n^{\alpha,\beta}(x) = c^{\star} \sum_{j=0}^{\infty} \left[\hat{a}\left(\frac{j}{n}\right) - \hat{a}\left(\frac{j+1}{n}\right)\right] \frac{\Gamma(j+\alpha+1+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha+1,\beta)}(x).$$

We now define the sequence of functions $(A_k(t))_{k=0}^{\infty}$ by

$$A_0(t) := (2t+\alpha+\beta+1)\hat{a}\left(\frac{t}{n}\right)$$

and inductively

$$(4.11) \qquad A_{k+1}(t) := \frac{A_k(t)}{2t+\alpha+k+\beta+1} - \frac{A_k(t+1)}{2t+\alpha+k+\beta+3}, \quad k \geq 0.$$

Notice that

$$(4.12) \qquad A_1(t) := \hat{a}\left(\frac{t}{n}\right) - \hat{a}\left(\frac{t+1}{n}\right)$$

and hence $\operatorname{supp} A_k \subset [n/2 - k, 2n] \subset [n/4, 2n]$ if $1 \le k \le n/4$.

Applying summation by parts k times starting from (4.6) (using every time (4.9)), we arrive at the identity:

$$(4.13) \qquad \mathcal{Q}_n^{\alpha,\beta}(x) = c^\star \sum_{j=0}^{\infty} A_k(j) \frac{\Gamma(j + \alpha + k + \beta + 1)}{\Gamma(j + \beta + 1)} P_j^{(\alpha+k,\beta)}(x).$$

We next show that for all $1 \le k \le n/4$ and $m \ge 0$

$$(4.14) \quad \|A_k^{(m)}\|_\infty \le \tilde{c}12^{k-1}(e\tilde{c}/\varepsilon)^{k+m}(k+m)^{1/2}(m+1)_{k-1}3^m m! n^{-2k-m+1}$$
$$\times [\ln(k+m+2)]^{(k+m)(1+\varepsilon)},$$

where $\tilde{c} > 1$ and ε are as in Theorem 3.1 and $(a)_r := a(a+1)\cdots(a+r-1)$. In particular,

$$\|A_k\|_\infty \le \tilde{c}(12e\tilde{c}/\varepsilon)^k k^{1/2}(k-1)![\ln(k+2)]^{k(1+\varepsilon)}n^{-2k+1}$$
$$(4.15) \qquad\qquad \le \tilde{c}(12e\tilde{c}/\varepsilon)^k k^k [\ln(k+2)]^{k(1+\varepsilon)}n^{-2k+1} \quad \text{for} \quad 1 \le k \le n/4.$$

To make the proof of (4.14) more transparent we denote

$$B_k^m := \frac{n^{2k+m-1}}{3^m m!} \|A_k^{(m)}\|_\infty.$$

Then (4.14) is the same as

$$(4.16) \quad B_k^m \le \tilde{c}12^{k-1}(e\tilde{c}/\varepsilon)^{k+m}(k+m)^{1/2}(m+1)_{k-1}[\ln(k+m+2)]^{(k+m)(1+\varepsilon)}.$$

We shall proceed by induction on k. By (4.12) and Theorem 3.1 it follows that

$$\|A_1^{(m)}\|_\infty \le n^{-m-1}\|\hat{a}^{(m+1)}\|_\infty$$
$$\le \tilde{c}(\tilde{c}/\varepsilon)^{m+1}(m+1)^{m+1}[\ln(m+3)]^{(m+1)(1+\varepsilon)}n^{-m-1}.$$

But by Stirling's formula it follows that $(m+1)^{m+1} < (m+1)^{-1/2}e^{m+1}(m+1)!$ and hence from above

$$B_1^m \le \tilde{c}(e\tilde{c}/\varepsilon)^{m+1}(m+1)^{1/2}[\ln(m+3)]^{(m+1)(1+\varepsilon)},$$

which shows that (4.16) holds for $k = 1$ and $m \ge 0$.

Suppose (4.16) holds for some $1 \le k \le n/4$ and all $m \ge 0$. We shall show that it holds with k replaced by $k+1$ and for all $m \ge 0$. Denote $G_k(t) := \frac{A_k(t)}{2t+\alpha+k+\beta+1}$. Then by (4.11) $A_{k+1}^{(m)}(t) = -\int_0^1 G_k^{(m+1)}(t+s)ds$ and evidently

$$G_k^{(m+1)}(t) = \sum_{\nu=0}^{m+1} \binom{m+1}{\nu} A_k^{(\nu)}(t) \frac{(-2)^{m+1-\nu}(m+1-\nu)!}{(2t+\alpha+k+\beta+1)^{m+2-\nu}}.$$

Therefore,

$$\|A_{k+1}^{(m)}\|_\infty \le \|G_k^{(m+1)}\|_\infty \le \sum_{\nu=0}^{m+1} \binom{m+1}{\nu} \|A_k^{(\nu)}\|_\infty \frac{2^{m+1-\nu}(m+1-\nu)!}{(n-k)^{m+2-\nu}}$$
$$\le 2\sum_{\nu=0}^{m+1} \frac{(m+1)!3^{m+1-\nu}}{\nu! n^{m+2-\nu}} \|A_k^{(\nu)}\|_\infty,$$

where we used that since $k \leq n/4$ we have $n - k > 3n/4$. It is readily seen that the above is the same as

$$B_{k+1}^m \leq 6(m+1) \sum_{\nu=0}^{m+1} B_k^\nu.$$

We now use the inductive assumption (see (4.16)) to obtain the following upper bound for B_{k+1}^m:

$$6(m+1) \sum_{\nu=0}^{m+1} \tilde{c} 12^{k-1} (e\tilde{c}/\varepsilon)^{k+\nu} (k+\nu)^{1/2} (\nu+1)_{k-1} [\ln(k+\nu+2)]^{(k+\nu)(1+\varepsilon)}$$

$$\leq 6\tilde{c} 12^{k-1} (k+m+1)^{1/2} (m+1)_k [\ln(k+m+3)]^{(k+m+1)(1+\varepsilon)} \sum_{\nu=0}^{m+1} (e\tilde{c}/\varepsilon)^{k+\nu}$$

For the last sum above we have

$$\sum_{\nu=0}^{m+1} (e\tilde{c}/\varepsilon)^{k+\nu} < \frac{(e\tilde{c}/\varepsilon)^{k+m+2}}{e\tilde{c}/\varepsilon - 1} < 2(e\tilde{c}/\varepsilon)^{k+m+1}$$

using that $e\tilde{c}/\varepsilon > 2$. Putting the above estimates together we get

$$B_{k+1}^m \leq \tilde{c} 12^k (e\tilde{c}/\varepsilon)^{k+m+1} (k+m+1)^{1/2} (m+1)_k [\ln(k+m+3)]^{(k+m+1)(1+\varepsilon)},$$

which shows that (4.16) holds when k is replaced by $k+1$. Therefore, (4.16) and hence (4.14) hold for all $1 \leq k \leq n/4$ and $m \geq 0$.

We shall need the following estimate for Jacobi polynomials [6, Theorem 1]: For $\alpha, \beta \geq -1/2$ and $n \geq 1$,

$$(4.17) \quad \sup_{x \in [-1,1]} (1-x)^{\alpha+1/2} (1+x)^{\beta+1/2} |P_n^{(\alpha,\beta)}(x)|^2 \leq \frac{2e}{\pi} \big(2 + \sqrt{\alpha^2 + \beta^2}\big) h_n^{(\alpha,\beta)},$$

where $h_n^{(\alpha,\beta)}$ is from (4.1).

We next prove (4.7) ($r = 0$) for $1/n \leq \theta \leq \pi/2$. By (4.1) it readily follows that $h_n^{(\alpha+k,\beta)} \leq c2^k/n$. Using this, (4.17), and the obvious inequality $1 - \cos\theta = 2\sin^2\theta/2 \geq \theta^2$ for $0 \leq \theta \leq \pi/2$, we infer

$$(4.18) \quad |P_n^{(\alpha+k,\beta)}(\cos\theta)| \leq \frac{ck^{1/2} 2^{k/2}}{n^{1/2} \theta^{k+\alpha+1/2}} \leq \frac{c2^k}{n^{1/2} \theta^{k+\alpha+1/2}}, \quad 0 < \theta \leq \pi/2.$$

We now use this and (4.15) in (4.13) to obtain for $1/n \leq \theta \leq \pi/2$ and $1 \leq k \leq n/4$

$$|\mathcal{Q}_n^{\alpha,\beta}(\cos\theta)| \leq c(12e\tilde{c}/\varepsilon)^k k^k [\ln(k+2)]^{k(1+\varepsilon)} n^{-2k+1} \sum_{n/2-k<j<2n} \frac{2^k (j+k)^{\alpha+k}}{j^{1/2} \theta^{k+\alpha+1/2}}$$

$$(4.19) \qquad \leq c(54e\tilde{c}/\varepsilon)^k k^k [\ln(k+2)]^{k(1+\varepsilon)} \frac{n^{2\alpha+2}}{(n\theta)^{k+\alpha+1/2}}$$

$$\leq cn^{2\alpha+2} \Big(\frac{c_\varepsilon k [\ln(k+2)]^{1+\varepsilon}}{n\theta}\Big)^k, \quad c_\varepsilon := 54e\tilde{c}/\varepsilon.$$

Here we used that $\Gamma(j+\alpha+k+\beta+1)/\Gamma(j+\beta+1) \leq c(j+k)^{\alpha+k}$.

If $n\theta < 2ec_\varepsilon [\ln(e+n\theta)]^{1+\varepsilon}$, then (4.7) (in the case $r = 0$) follows by (4.8) with $c^\diamond = (2ec_\varepsilon)^{-1}$. Assume that $n\theta \geq 2ec_\varepsilon [\ln(e+n\theta)]^{1+\varepsilon}$ and choose $k :=$

$\left\lfloor \frac{n\theta}{ec_\varepsilon[\ln(e+n\theta)]^{1+\varepsilon}} \right\rfloor$. Evidently, $k \le n/4$. We claim that

$$(4.20) \qquad \frac{c_\varepsilon k[\ln(k+2)]^{1+\varepsilon}}{n\theta} \le e^{-1},$$

Indeed, by the definition of k we have $k \le \frac{n\theta}{ec_\varepsilon[\ln(e+n\theta)]^{1+\varepsilon}}$ and then (4.20) follows by the obvious inequality $k + 2 \le e + n\theta$. Combining (4.20) with (4.19) leads to (4.7) for $r = 0$ with $c^\diamond = (ec_\varepsilon)^{-1}$. Therefore, (4.7) ($r = 0$) holds for $1/n \le \theta \le \pi/2$ with $c^\diamond = (2ec_\varepsilon)^{-1} = c'\varepsilon$.

Let now $\pi/2 < \theta \le \pi - 1/n$. Similarly as in the proof of estimate (4.18) we use that $1 + \cos\theta = 2\sin^2\frac{\pi-\theta}{2} \ge (\pi-\theta)^2$ for $\pi/2 \le \theta \le \pi$ to obtain

$$|P_n^{(\alpha+k,\beta)}(\cos\theta)| \le \frac{c2^k}{n^{1/2}(\pi-\theta)^{\beta+1/2}} \le c2^k n^\beta, \quad \pi/2 \le \theta \le \pi - 1/n.$$

Combining this with (4.13) and (4.15) we get

$$|\mathcal{Q}_n^{\alpha,\beta}(\cos\theta)| \le c(12e\tilde{c}/\varepsilon)^k k^k[\ln(k+2)]^{k(1+\varepsilon)} n^{-2k+1} \sum_{n/2-k<j<2n} 2^k n^\beta (j+k)^{\alpha+k}$$

$$\le c(54e\tilde{c}/\varepsilon)^k k^k[\ln(k+2)]^{k(1+\varepsilon)} n^{-k+\alpha+\beta+2}$$

$$\le cn^{\alpha+\beta+2} \left(\frac{c_\varepsilon k[\ln(k+2)]^{1+\varepsilon}}{n} \right)^k, \quad c_\varepsilon := 54e\tilde{c}/\varepsilon.$$

Exactly as above this leads to the estimate

$$(4.21) \qquad \left| \mathcal{Q}_n^{\alpha,\beta}(\cos\theta) \right| \le cn^{\alpha+\beta+2} \exp\left\{ -\frac{c^\diamond n}{[\ln(e+n)]^{1+\varepsilon}} \right\}, \quad \pi/2 \le \theta \le \pi - 1/n,$$

and, since $\alpha \ge \beta$, estimate (4.7) ($r = 0$) holds in this case as well.

Finally, let $\pi - 1/n \le \theta \le \pi$. In this case (4.7) follows easily from (4.21). Indeed, as is well known for any polynomial $P \in \Pi_n$ one has

$$|P(x)| \le \|P\|_{L^\infty[-1,1]} T_n(x) \le \|P\|_{L^\infty[-1,1]} \left(x + \sqrt{x^2-1} \right)^n, \quad x \ge 1,$$

where T_n is the nth degree Chebyshev polynomial of first kind, and hence

$$|P(x)| \le \|P\|_{L^\infty[-1,1]} \left(1 + 2\sqrt{x-1} \right)^n \quad \text{for} \quad 1 \le x \le 5/4.$$

By changing variables we obtain

$$(4.22) \quad |P(t)| \le \|P\|_{L^\infty[a,b]} \left(1 + 2^{3/2}\sqrt{(t-b)/(b-a)} \right)^n, \quad b \le t \le b + (b-a)/8.$$

If $t := -\cos\theta$, then $t \in [0, \cos\frac{1}{n}]$ for $\theta \in [\pi - 1/n, \pi]$. On the other hand, $\cos\frac{1}{n} = 1 - 2\sin^2\frac{1}{n} \ge 1 - 2/n^2$. Hence

$$(4.23) \qquad \|\mathcal{Q}_n^{\alpha,\beta}(-\cdot)\|_{L^\infty[0,1-2/n^2]} \le \|\mathcal{Q}_n^{\alpha,\beta}(\cos\cdot)\|_{L^\infty[\pi/2,\pi-1/n]}.$$

Since $\mathcal{Q}_n^{\alpha,\beta} \in \Pi_{2n}$, then we can apply (4.22) on the interval $[a,b] = [0, 1-2/n^2]$ to obtain using (4.23)

$$\|\mathcal{Q}_n^{\alpha,\beta}(\cos\cdot)\|_{L^\infty[\pi-1/n,\pi]} \le \|\mathcal{Q}_n^{\alpha,\beta}(-\cdot)\|_{L^\infty[1-2/n^2,1]}$$

$$\le \|\mathcal{Q}_n^{\alpha,\beta}(-\cdot)\|_{L^\infty[0,1-2/n^2]} \left(1 + 2^{3/2}\sqrt{(2/n^2)/(1-2/n^2)} \right)^{2n}$$

$$\le (1 + 2^{5/2}/n)^{2n} \|\mathcal{Q}_n^{\alpha,\beta}(\cos\cdot)\|_{L^\infty[\pi/2,\pi-1/n]} \le c\|\mathcal{Q}_n^{\alpha,\beta}(\cos\cdot)\|_{L^\infty[\pi/2,\pi-1/n]},$$

provided $2/n^2 \le (1/8)(1-2/n^2)$ which is the same as $n \ge 5$ (the case $n < 5$ is trivial). Combining the above with (4.21) implies that the estimate in (4.21) holds

for $\pi - 1/n \leq \theta \leq \pi$ as well, but perhaps with a different constant c. This completes the proof of estimate (4.7) when $r = 0$.

For $r \geq 1$, estimate (4.7) is an easy consequence of Markov's inequality: If $Q \in \Pi_m$, then $\|Q'\|_{L^\infty[a,b]} \leq 2m^2(b-a)^{-1}\|Q\|_{L^\infty[a,b]}$. Indeed, substituting $x = \cos\theta$ shows that (4.7) with $r = 0$ is the same as

$$(4.24) \qquad |\mathcal{Q}_n^{\alpha,\beta}(x)| \leq \frac{cn^{2\alpha+2}}{E(\arccos x)}, \quad \text{where } E(\theta) := \exp\left\{\frac{c^\diamond n\theta}{[\ln(e+n\theta)]^{1+\varepsilon}}\right\}.$$

It is easy to see that the function $E(\theta)$ is increasing on $[0,\pi]$ and hence $1/E(\arccos x)$ is also increasing on $[-1,1]$. Note that $\mathcal{Q}_n^{\alpha,\beta} \in \Pi_{2n}$ and by Markov's inequality we have, for $x \in [0,1]$,

$$\left|\frac{d}{dx}\mathcal{Q}_n^{\alpha,\beta}(x)\right| \leq 8n^2(1+x)^{-1}\|\mathcal{Q}_n^{\alpha,\beta}\|_{L^\infty[-1,x]} \leq 8n^2\frac{cn^{2\alpha+2}}{E(\arccos x)},$$

where we used (4.24) and the monotonicity of $1/E(\arccos x)$. Iterating we get

$$\left|\frac{d^r}{dx^r}\mathcal{Q}_n^{\alpha,\beta}(x)\right| \leq c8^r n^{2r}\frac{n^{2\alpha+2}}{E(\arccos x)}, \quad r \geq 1,$$

which yields (4.7) for $0 \leq \theta \leq \pi/2$.

To estimate $|(d/dx)^r \mathcal{Q}_n^{\alpha,\beta}(x)|$ for $x \in [-1,0)$ one applies Markov's inequality on $[-1,0]$ which leads readily to the same estimate with a different constant $c^\diamond$.

We finally observe that one proves estimate (4.7) with $[\ln(e+n\theta)]^{1+\varepsilon}$ replaced by a product of the form (2.13) exactly as above using the respective estimate for $|\hat{a}^{(k)}(t)|$ from Theorem 3.1. The proof of Theorem 4.2 is complete. $\square$

Proof of Theorem 4.1. We shall use the notation and scheme of the proof of Theorem 2 in [17]. Consider first the case when $\alpha > \beta > -1/2$. By the product formula of Jacobi polynomials [10] $(\alpha > \beta > -1/2)$, one has

$$(4.25) \qquad \frac{P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} = c_{\alpha,\beta}\int_0^\pi\int_0^1 P_n^{(\alpha,\beta)}(t(x,y,r,\psi))dm(r,\phi),$$

where

$$t(x,y,r,\psi) = \tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 + r\sqrt{1-x^2}\sqrt{1-y^2}\cos\psi - 1,$$

the integral is against

$$dm(r,\phi) = (1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}drd\psi,$$

and the constant $c_{\alpha,\beta}$ is selected so that $c_{\alpha,\beta}\int_0^\pi\int_0^1 1\,dm(r,\phi) = 1$. Therefore, as in [17, (2.15)] we have

$$(4.26) \qquad L_n^{\alpha,\beta}(x,y) = c_{\alpha,\beta}\int_0^\pi\int_0^1 \mathcal{Q}_n^{\alpha,\beta}(t(x,y,r,\psi))dm(r,\psi).$$

Evidently, if $t = \cos\theta$ with $0 \leq \theta \leq \pi$, then $\theta \sim \sin\theta/2 \sim \sqrt{1-t}$. Consequently, estimate (4.7) implies

$$(4.27) \qquad \left|\mathcal{Q}_n^{\alpha,\beta}(t)\right| \leq cn^{2\alpha+2}\exp\left\{-\frac{c'\varepsilon n\sqrt{1-t}}{[\ln(e+n\sqrt{1-t})]^{1+\varepsilon}}\right\}, \quad -1 \leq t \leq 1.$$

Precisely as for the proof of (2.18) in [17], this yields

$$(4.28) \quad \left|L_n^{\alpha,\beta}(\cos\theta,\cos\phi)\right| \leq cn^{2\alpha+2}\exp\left\{-\frac{c'\varepsilon n|\theta-\phi|}{[\ln(e+n|\theta-\phi|)]^{1+\varepsilon}}\right\}, \quad 0 \leq \theta,\phi \leq \pi.$$

But $\exp\left\{\frac{-\frac{1}{2}c'\varepsilon t}{[\ln(e+t)]^{1+\varepsilon}}\right\} \le c(1+t)^{-3\alpha-3\beta-2}$, $t > 0$, for a sufficiently large constant $c > 0$, and hence (4.28) implies
(4.29)
$$\left|L_n^{\alpha,\beta}(\cos\theta,\cos\phi)\right| \le \frac{cn^{2\alpha+2}}{(1+n|\theta-\phi|)^{3\alpha+3\beta+2}} \exp\left\{-\frac{\frac{1}{2}c'\varepsilon n|\theta-\phi|}{[\ln(e+n|\theta-\phi|)]^{1+\varepsilon}}\right\}.$$

Just as in the proof of estimate (2.14) in [17] we use this estimate to prove (4.4) with $c^\diamond = c'/2$. We skip the further details.

In the case when $\alpha = \beta = -1/2$ (the case of Chebyshev polynomials of first kind), estimate (4.4) was already proved in Theorem 2.1.

In the case when $\alpha = \beta > -1/2$, as in [17] and above, one uses the product formula for Gegenbauer polynomials and (4.29) to prove estimate (4.4). We omit the details.

Finally, let $\alpha > \beta = -1/2$. Evidently, for a continuous function f on $[-1,1]$
$$\lim_{\mu\to-1^+} \frac{\int_0^\pi f(\cos\theta)(\sin\theta)^\mu d\theta}{\int_0^\pi (\sin\theta)^\mu d\theta} = \frac{f(1) + f(-1)}{2}.$$

On the other hand, $\lim_{\beta\to-1/2} P_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,-1/2)}(x)$ uniformly on every compact interval, and in particular on $[-1,1]$, since as is well know the coefficients of $P_n^{(\alpha,\beta)}$ converge to the respective coefficients of $P_n^{(\alpha,-1/2)}$ as $\beta \to -1/2$. Therefore, passing to the limit in (4.25) as $\beta \to -1/2$ the product formula of Jacobi polynomials takes the form

$$\frac{P_n^{(\alpha,-1/2)}(x)P_n^{(\alpha,-1/2)}(y)}{P_n^{(\alpha,-1/2)}(1)}$$
(4.30)
$$= \frac{1}{2}c_\alpha \int_0^1 \left[P_n^{(\alpha,-1/2)}(t(x,y,r,0)) + P_n^{(\alpha,-1/2)}(t(x,y,r,\pi))\right] dm(r),$$

where $dm(r) = (1-r^2)^{\alpha-1/2}dr$ and $c_\alpha \int_0^1 dm(r) = 1$. Using this product formula, one can carry over the entire proof of Theorem 2 in [20] in the case when $\alpha > \beta = -1/2$. In a similar fashion, one uses (4.29)-(4.30) to prove estimate (4.4) whenever $\alpha > \beta = -1/2$. We skip the details.

Due to symmetry, in all other cases for α, β estimate (4.4) follows from the cases considered above.   □

Sub-exponentially localized needlets induced by Jacobi polynomials. Given $j \ge 0$ let $\mathcal{X}_j$ be the set of all zeros of the Jacobi polynomial $P_m^{(\alpha,\beta)}$ of degree $m := 2^j$ and let $\{c_\xi\}_{\xi\in\mathcal{X}_j}$ be the coefficients of the Gaussian quadrature formula on $[-1,1]$, which is exact for all polynomials of degree $2m-1$ (cf. (1.5)). Suppose $\hat{a}$ is an admissible cutoff function of type (c) obeying (3.1). According to the general scheme from §1.2 the jth level needlets are defined by

$$\psi_\xi(x) := c_\xi^{1/2} L_n^{\alpha,\beta}(\xi,x), \quad \xi \in \mathcal{X}_j,$$

where $L_n^{\alpha,\beta}$ with $n := 2^{j-1}$ is the kernel defined in (4.2). The needlet system $\Psi := \{\psi_\xi\}$ is a tight frame for $L^2([-1,1], w_{\alpha,\beta})$ and it is easy to see that the jth level needlet inherit from the kernel $L_n^{\alpha,\beta}$ the localization:

$$(4.31) \qquad |\psi_\xi(x)| \le \frac{c2^{j/2}}{\sqrt{w_{\alpha,\beta}(2^j;\xi)}} \exp\left\{-\frac{c'\varepsilon 2^j \rho(x,\xi)}{[\ln(e + 2^j\rho(x,\xi))]^{1+\varepsilon}}\right\}, \quad \xi \in \mathcal{X}_j.$$

Here we used the notation from Theorem 4.1. This is an improvement compared with the localization of the Jacobi needlets constructed in [17] and [12].

5. Sub-exponentially localized kernels and frames on the sphere

Denote by $\mathcal{H}_n$ the space of all spherical harmonics of degree n on the d-dimensional unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$. As is well known (see e.g. [20]) the kernel of the orthogonal projector onto $\mathcal{H}_n$ is given by

$$(5.1) \qquad \mathsf{P}_n(\xi \cdot \eta) = \frac{n+\lambda}{\lambda \omega_d} C_n^\lambda(\xi \cdot \eta),$$

where $\lambda := \frac{d-1}{2}$, $\omega_d := \int_{\mathbb{S}^d} 1 d\sigma$ is the hypersurface area of $\mathbb{S}^d$, and $\xi \cdot \eta$ stands for the inner product of $\xi, \eta \in \mathbb{S}^d$. Here C_n^λ is the Gegenbauer polynomial of degree n normalized with $C_n^\lambda(1) = \binom{\nu+2\lambda-1}{\nu}$ [5, p. 174].

Our aim is to construct sub-exponentially localized kernels of the form

$$(5.2) \qquad L_n(\xi \cdot \eta) = \sum_{j=0}^\infty \hat{a}\left(\frac{j}{n}\right) \mathsf{P}_j(\xi \cdot \eta),$$

where $\hat{a}$ is a C^∞ cutoff function.

Denote by $\rho(\xi, \eta) := \arccos(\xi \cdot \eta)$ the geodetic distance between $\xi, \eta \in \mathbb{S}^d$ on $\mathbb{S}^d$.

Theorem 5.1. *Let $0 < \varepsilon \leq 1$. Then for any admissible cutoff function $\hat{a}$ obeying inequality (3.1) in Theorem 3.1 the kernels from (5.2) satisfy*

$$(5.3) \qquad |L_n(\xi \cdot \eta)| \leq cn^d \exp\left\{-\frac{c^\diamond n\rho(\xi,\eta)}{[\ln(e+n\rho(\xi,\eta))]^{1+\varepsilon}}\right\}, \quad \xi, \eta \in \mathbb{S}^d,$$

where $c^\diamond = c'\varepsilon$ with $c' > 0$ an absolute constant and $c > 0$ depends only on d and ε. Moreover, for an appropriate cutoff function $\hat{a}$ the above estimate can be improved similarly as in Theorem 4.1.

Proof. By (4.6) with $\alpha = \beta = \lambda - 1/2$ ($\lambda := (d-1)/2$) we get

$$\mathcal{Q}_n^{\lambda-1/2,\lambda-1/2}(t) = c^\star \sum_{j=0}^\infty \hat{a}\left(\frac{j}{n}\right) \frac{(j+\lambda)\Gamma(j+2\lambda)}{\Gamma(j+\lambda+1/2)} P_j^{(\lambda-1/2,\lambda-1/2)}(t),$$

where $c^\star := 2^{-2\lambda+1}\Gamma(\lambda+1/2)^{-1}$. On the other hand, using the relation between Gegenbauer and Jacobi polynomials [21, (4.7.1)] we have

$$(5.4) \qquad C_n^\lambda(t) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1/2)} P_n^{(\lambda-1/2,\lambda-1/2)}(t).$$

Therefore, $L_n(t) = c(d)\mathcal{Q}_n^{\lambda-1/2,\lambda-1/2}(t)$ and hence estimate (5.3) follows immediately from Theorem 4.2. $\square$

Sub-exponentially localized needlets on the sphere. As in [16] for the construction of the jth level needlets on $\mathbb{S}^d$ we use a cubature formula with nodes in $\mathcal{X}_j \subset \mathbb{S}^d$ consisting of $O(2^{jd})$ almost uniformly distributed points on the sphere and positive coefficients $\{c_\xi\}_{\xi \in \mathcal{X}_j}$ of size $c_\xi \sim 2^{-jd}$, which is exact for all spherical harmonics of degree $\leq 2^{j+1}$ (for more details, see [16]). Then choosing an admissible cutoff function $\hat{a}$ of type (c) satisfying (3.1), we define

$$\psi_\xi(x) := c_\xi^{1/2} L_n(\xi \cdot x), \quad \xi \in \mathcal{X}_j,$$

where L_n with $n := 2^{j-1}$ is the kernel defined in (5.2). Setting $\mathcal{X} := \cup_{j\geq 0}\mathcal{X}_j$, we define the needlet system by $\Psi := \{\psi_\xi\}_{\xi\in\mathcal{X}}$. This is a tight frame for $L^2(\mathbb{S}^d)$ and Theorem 5.1 implies the sub-exponential localization of the spherical needlets:

$$(5.5) \qquad |\psi_\xi(x)| \leq c2^{jd/2} \exp\left\{-\frac{c'\varepsilon 2^j \rho(x,\xi)}{[\ln(e + 2^j\rho(x,\xi))]^{1+\varepsilon}}\right\}, \quad x \in \mathbb{S}^d, \quad \xi \in \mathcal{X}_j.$$

Here, $\rho(x,\xi)$ is the geodesic distance between $x,\xi \in \mathbb{S}^d$. This is a natural improvement of the localization of the needlets constructed in [16].

6. Sub-exponentially localized kernels and frames on the ball

Here, we consider orthogonal polynomials on the unit ball $B^d := \{x \in \mathbb{R}^d : \|x\|_2 < 1\}$ in $\mathbb{R}^d$ ($d > 1$) with weight

$$w_\mu(x) := (1 - \|x\|_2^2)^{\mu-1/2}, \quad \mu \geq 0.$$

We shall use the notation and results from [18]. Denote by $\mathcal{V}_n^d$ the space of all polynomials of total degree n which are orthogonal to lower degree polynomials in $L^2(B^d, w_\mu)$. As is shown in [24], if $\mu > 0$ the orthogonal projector $\mathrm{Proj}_n : L^2(B^d, w_\mu) \mapsto \mathcal{V}_n^d$ can be written in the form

$$(6.1) \qquad (\mathrm{Proj}_n f)(x) = \int_{B^d} f(y)P_n(w_\mu; x, y)w_\mu(y)dy,$$

where

$$P_n(w_\mu; x, y) = c_{\mu,d}\frac{\lambda + n}{\lambda} \int_{-1}^{1} C_n^\lambda\left(\langle x, y\rangle + u\sqrt{1 - \|x\|_2^2}\sqrt{1 - \|y\|_2^2}\right)(1 - u^2)^{\mu-1}du.$$

Here $\langle x, y\rangle$ is the Euclidean inner product in $\mathbb{R}^d$, C_n^λ is the nth degree Gegenbauer polynomial, and

$$\lambda = \mu + \frac{d-1}{2}.$$

In the case $\mu = 0$ the kernel $P_n(w_\mu; x, y)$ is simpler, see [18].

As before, we are interested in the construction of sup-exponentially localized kernels of the form

$$(6.2) \qquad L_n^\mu(x, y) = \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right)P_j(w_\mu; x, y), \quad x, y \in B^d,$$

where $\hat{a}$ is a C^∞ cutoff function. As in [18], we shall need the following distance on B^d:

$$(6.3) \qquad \rho(x, y) := \arccos\left\{\langle x, y\rangle + \sqrt{1 - \|x\|_2^2}\sqrt{1 - \|y\|_2^2}\right\}$$

and the quantity

$$W_\mu(n; x) := \left(\sqrt{1 - \|x\|_2^2} + n^{-1}\right)^{2\mu}.$$

Theorem 6.1. *Let $0 < \varepsilon \leq 1$. Then for any admissible cutoff function $\hat{a}$ obeying inequality (3.1) in Theorem 3.1 the kernels from (6.2) satisfy*

$$(6.4) \qquad |L_n^\mu(x, y)| \leq \frac{cn^d}{\sqrt{W_\mu(n; x)}\sqrt{W_\mu(n; y)}} \exp\left\{-\frac{c^\diamond n\rho(x, y)}{[\ln(e + n\rho(x, y))]^{1+\varepsilon}}\right\},$$

where $c^\diamond = c'\varepsilon$ with $c' > 0$ an absolute constant, and c depends only on μ, d, and ε.

Moreover, for an appropriate cutoff function $\hat{a}$ the term $[\ln(e + n\rho(x,y))]^{1+\varepsilon}$ above can be replaced by any product of the form (1.10).

Proof. Assume $\mu > 0$ (the case $\mu = 0$ is easer). From (6.1) it follows that

$$(6.5) \quad L_n^\mu(x,y) = c(\mu, d) \int_{-1}^{1} \mathcal{Q}_n^\lambda(\langle x,y\rangle + u\sqrt{1 - \|x\|_2^2}\sqrt{1 - \|y\|_2^2})(1 - u^2)^{\mu-1} du,$$

where $\mathcal{Q}_n^\lambda(t) := \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) \frac{j+\lambda}{\lambda} C_j^\lambda(t)$. Combining identities (5.4) and (4.6) gives $\mathcal{Q}_n^{\lambda-1/2,\lambda-1/2}(t) = c\mathcal{Q}_n^\lambda(t)$. Therefore, by (4.27) we get

$$(6.6) \qquad \left|\mathcal{Q}_n^\lambda(t)\right| \leq cn^{2\lambda+1} \exp\left\{-\frac{c'\varepsilon n\sqrt{1-t}}{[\ln(e + n\sqrt{1-t})]^{1+\varepsilon}}\right\}, \quad -1 \leq t \leq 1,$$

and since $\exp\left\{\frac{-(c'/2)\varepsilon u}{\ln(e+u)^{1+\varepsilon}}\right\} \leq c(1+u)^{-3\mu-1}$ for sufficiently large constant $c > 0$

$$(6.7) \qquad \left|\mathcal{Q}_n^\lambda(t)\right| \leq \frac{cn^{2\lambda+1}}{(1 + n\sqrt{1-t})^{3\mu+1}} \exp\left\{-\frac{\frac{1}{2}c'\varepsilon n\sqrt{1-t}}{[\ln(e + n\sqrt{1-t})]^{1+\varepsilon}}\right\}.$$

Now, just as in the proof of Theorem 4.2 in [18], we use estimates (6.6)-(6.7) to obtain (6.4). $\square$

Sub-exponentially localized needlets on the ball. For any $j \geq 0$ we shall utilize the cubature formula on B^d from [18] with nodes in $\mathcal{X}_j \subset B^d$ consisting of $O(2^{jd})$ almost uniformly distributed points on B^d with respect to the distance $\rho(\cdot, \cdot)$ defined in (6.3) and with positive coefficients $\{c_\xi\}_{\xi \in \mathcal{X}_j}$, which is exact for algebraic polynomials of degree 2^{j+1}. The second step is to select an admissible cutoff function $\hat{a}$ of type (c) as in Theorem 6.1. Then we define

$$\psi_\xi(x) := c_\xi^{1/2} L_n^\mu(\xi, x), \quad \xi \in \mathcal{X}_j,$$

where L_n^μ with $n := 2^{j-1}$ is the kernel from (6.2). Set $\mathcal{X} := \cup_{j\geq0}\mathcal{X}_j$. Then we define the needlet system on the ball B^d by $\Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}$. The sub-exponential localization of the needlets on the ball follows by Theorem 6.1:

$$(6.8) \quad |\psi_\xi(x)| \leq \frac{c2^{j/2}}{\sqrt{W_\mu(2^j; \xi)}} \exp\left\{-\frac{c'\varepsilon 2^j \rho(x,\xi)}{[\ln(e + 2^j\rho(x,\xi))]^{1+\varepsilon}}\right\}, \quad x \in B^d, \quad \xi \in \mathcal{X}_j.$$

This is an improvement in comparison with the localization of the needlets in [19].

7. SUB-EXPONENTIALLY LOCALIZED KERNELS AND FRAMES ON THE SIMPLEX

Here, we consider orthogonal polynomials on the simplex

$$T^d = \{x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, 1 - |x|_1 \geq 0\}, \qquad |x|_1 := x_1 + \ldots + x_d,$$

in $\mathbb{R}^d$ $(d \geq 1)$ with weight function

$$w_\kappa(x) = x_1^{\kappa_1 - \frac{1}{2}} \cdots x_d^{\kappa_d - \frac{1}{2}}(1 - |x|_1)^{\kappa_{d+1} - \frac{1}{2}}, \qquad \kappa_i \geq 0.$$

Denote by $\mathcal{V}_n^d$ the space of all polynomials of total degree n that are orthogonal to lower degree polynomials in $L^2(T^d, w_\kappa)$. As is shown in [23] the orthogonal projector $\mathrm{Proj}_n : L^2(T^d, w_\kappa) \mapsto \mathcal{V}_n^d$ can be written in the form

$$(7.1) \qquad (\mathrm{Proj}_n f)(x) = \int_{T^d} f(y) P_n(w_\kappa; x, y) w_\kappa(y) dy,$$

where, if all $\kappa_i > 0$,

$$(7.2) \quad P_n(w_\kappa; x, y) = c(\kappa, d) \frac{2n + \lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^{d+1}} C_{2n}^{\lambda_\kappa}(z(x, y, t)) \prod_{i=1}^{d+1} (1 - t_i^2)^{\kappa_i - 1} dt.$$

Here

$$(7.3) \qquad z(x, y, t) := \sqrt{x_1 y_1} t_1 + \cdots + \sqrt{x_d y_d} t_d + \sqrt{x_{d+1} y_{d+1}} t_{d+1}$$

with $x_{d+1} := 1 - \|x\|_1$ and $y_{d+1} := 1 - \|y\|_1$, C_n^λ is the nth degree Gegenbauer polynomial, and

$$\lambda_\kappa := |\kappa| + \frac{d-1}{2}, \qquad |\kappa| := \kappa_1 + \cdots + \kappa_{d+1}.$$

If some $\kappa_i = 0$, then the identity in (7.2) holds with the integral in t_i replaced according to the limit relation (see [23])

$$\lim_{\kappa \to 0} \int_{-1}^1 f(t)(1 - t^2)^{\kappa - 1} dt \Big/ \int_{-1}^1 (1 - t^2)^{\kappa - 1} dt = \frac{f(1) + f(-1)}{2}.$$

We are interested in kernels of the form

$$(7.4) \qquad L_n^\kappa(x, y) = \sum_{j=0}^\infty \hat{a}\left(\frac{j}{n}\right) P_j(w_\kappa; x, y), \quad x, y \in T^d,$$

where $\hat{a}$ is a C^∞ cutoff function. Our aim is to show that for any admissible cutoff function $\hat{a}$ the kernels $L_n^\kappa(x, y)$ decay rapidly away from the main diagonal in $T^d \times T^d$ and for any cutoff function $\hat{a}$ from Theorem 3.1 the decay is sub-exponential.

In analogy with the kernels on the ball, we shall need the distance $\rho(\cdot, \cdot)$ on T^d defined by

$$\rho(x, y) := \arccos\left\{\sqrt{x_1 y_1} + \cdots + \sqrt{x_d y_d} + \sqrt{x_{d+1} y_{d+1}}\right\}$$

and the quantity

$$W_\kappa(n; x) := \prod_{i=1}^{d+1} (x_i + n^{-2})^{\kappa_i}, \qquad x_{d+1} := 1 - \|x\|_1.$$

Given $x, y \in T^d$, set $a_j := \sqrt{x_j} =: \cos\theta_j$ and $b_j := \sqrt{y_j} =: \cos\phi_j$, where $0 \le \theta_j, \phi_j \le \pi/2$. Applying the Cauchy-Schwartz inequality we get, for $1 \le j \le d$,

$$\sum_{i=1}^d a_i b_i + \sqrt{1 - a_1^2 - \cdots - a_d^2} \sqrt{1 - b_1^2 - \cdots - b_d^2} \le a_j b_j + \sqrt{1 - a_j^2} \sqrt{1 - b_j^2}.$$

Hence,

$$\rho(x, y) \ge \arccos(\cos\theta_j \cos\phi_j + \sin\theta_j \sin\phi_j) = \arccos(\cos(\theta_j - \phi_j)),$$

which yields $\rho(x, y) \ge |\theta_j - \phi_j|$ and as a consequence

$$(7.5) \qquad |\sqrt{x_j} - \sqrt{y_j}| \le \rho(x, y), \quad 1 \le j \le d + 1.$$

Our localization results take the form:

Theorem 7.1. *Let $\hat{a}$ be an admissible cutoff function according to Definition 1.1. Then for any $\sigma > 0$ there exists a constant c_σ depending on σ, $|\kappa|$, and d such that the kernels from (7.4) satisfy*

$$(7.6) \qquad |L_n^\kappa(x,y)| \leq \frac{c_\sigma n^d}{\sqrt{W_\kappa(n;x)}\sqrt{W_\kappa(n;y)}}\big(1 + n\rho(x,y)\big)^{-\sigma}, \quad x, y \in T^d.$$

The next theorem show that for suitable cutoff functions $\hat{a}$ the localization of the kernels $L_n^\kappa(x,y)$ can be improved to sub-exponential.

Theorem 7.2. *Suppose $0 < \varepsilon \leq 1$ and let $\hat{a}$ in (7.4) be an admissible cutoff function obeying inequality (3.1) in Theorem 3.1. Then*

$$(7.7) \qquad |L_n^\kappa(x,y)| \leq \frac{cn^d}{\sqrt{W_\kappa(n;x)}\sqrt{W_\kappa(n;y)}}\exp\Big\{-\frac{c^\diamond n\rho(x,y)}{[\ln(e + n\rho(x,y))]^{1+\varepsilon}}\Big\},$$

where $c^\diamond = c'\varepsilon$ with $c' > 0$ an absolute constant, and c depends only on κ, d, and ε.

As elsewhere in this article, for an appropriate cutoff function $\hat{a}$ estimate (7.7) can be improved by replacing the term $[\ln(e + n\rho(x,y))]^{1+\varepsilon}$ by any product of logarithms as in (1.10).

We shall only present the proof of Theorem 7.2 since the proof of Theorem 7.1 follows along the same lines but is simpler.

Proof of Theorem 7.2. For a given $0 < \varepsilon \leq 1$, let $\hat{a}$ be the cutoff function from Theorem 3.1. Consider the case when $\kappa_i > 0$ for $1 \leq i \leq d+1$ (the case when some κ_i's are zeros is treated in the same way with appropriate modifications). We begin with the relation

$$C_{2n}^\lambda(x) = \frac{\Gamma(n+\lambda)\Gamma(\tfrac{1}{2})}{\Gamma(\lambda)\Gamma(n+\tfrac{1}{2})}P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2 - 1),$$

which follows readily combining identities (4.1.5) and (4.7.1) in [21]. This allows us to express $L_n^\kappa(x,y)$ in terms of the univariate kernel $\mathcal{Q}_n^{\alpha,\beta}$ from (4.5)-(4.6), namely,

$$L_n^\kappa(x,y) = c(\kappa,d)\int_{[-1,1]^{d+1}} \mathcal{Q}_n^{\lambda_\kappa - \frac{1}{2}, -\frac{1}{2}}\big(2z(x,y,t)^2 - 1\big)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i - 1}dt.$$

Let $\theta(x,y,t) := \arccos(2z(x,y,t)^2 - 1)$ with $z(x,y,t)$ given by (7.3). We use the fact that

$$1 - z(x,y,t)^2 = \frac{1}{2}(1 - \cos\theta(x,y,t)) = \sin^2\frac{\theta(x,y,t)}{2} \sim \theta(x,y,t)^2$$

and apply estimate (4.27) for $\mathcal{Q}_n^{\alpha,\beta}$ with $\alpha = \kappa - 1/2$, $\beta = -1/2$ to obtain

$$|L_n^\kappa(x,y)| \leq cn^{2\lambda_\kappa + 1}\int_{[-1,1]^{d+1}} \exp\Big\{\frac{-c'\varepsilon n\sqrt{1 - z(x,y,t)^2}}{[\ln(e + n\sqrt{1 - z(x,y,t)^2})]^{1+\varepsilon}}\Big\}\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i - 1}dt.$$

Evidently,

$$1 - z(x,y,t)^2 \geq 1 - |z(x,y,t)| \geq 1 - \sqrt{x_1 y_1}|t_1| - \ldots - \sqrt{x_{d+1}y_{d+1}}|t_{d+1}|$$

and using the symmetry of the integrand with respect to $t \in [-1,1]^{d+1}$, we get

$$|L_n^\kappa(x,y)| \leq cn^{2\lambda_\kappa + 1}\int_{[0,1]^{d+1}} \exp\Big\{\frac{-c'\varepsilon n\sqrt{1 - z(x,y,t)}}{\big[\ln\big(e + n\sqrt{1 - z(x,y,t)}\big)\big]^{1+\varepsilon}}\Big\}\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i - 1}dt.$$

Furthermore, we have the lower bound estimate

$$1 - z(x, y, t) \geq 1 - \sqrt{x_1 y_1} - \ldots - \sqrt{x_{d+1} y_{d+1}} = 1 - \cos \rho(x, y)$$

$$= 2 \sin^2 \frac{\rho(x, y)}{2} \geq \frac{2}{\pi^2} \rho(x, y)^2,$$

which enables us to deduce the estimate

$$(7.8) \qquad |L_n^\kappa(x, y)| \leq c n^{2\lambda_\kappa + 1} \exp \left\{ -\frac{\frac{1}{2} c' \varepsilon n \rho(x, y)}{[\ln(e + n\rho(x, y))]^{1+\varepsilon}} \right\}$$

$$\times \frac{1}{(1 + n\rho(x, y))^{|\kappa|}} \int_{[0,1]^{d+1}} \frac{1}{[1 + n\sqrt{1 - z(x, y, t)}]^\gamma} \prod_{i=1}^{d+1} (1 - t_i^2)^{\kappa_i - 1} dt,$$

where $\gamma = 2|\kappa| + d + 1$. Here we used that $\exp \left\{ \frac{-(c'/2)\varepsilon u}{\ln(e+u)^{1+\varepsilon}} \right\} \leq c(1+u)^{-|\kappa|-\gamma}$, $u \geq 0$, for sufficiently large constant $c > 0$. From the definition of $z(x, y, t)$, we have

$$1 - z(x, y, t) = 1 - \cos \rho(x, y) + \sum_{i=1}^{d+1} \sqrt{x_i y_i} (1 - t_i) \geq \sum_{i=1}^{d+1} \sqrt{x_i y_i} (1 - t_i).$$

Denote by J the integrals in (7.8). From above, we have

$$J \leq \int_{[0,1]^{d+1}} \frac{\prod_{i=1}^{d+1} (1 - t_i^2)^{\kappa_i - 1} dt}{\left(1 + n \left[\sum_{i=1}^{d+1} \sqrt{x_i y_i} (1 - t_i) \right]^{1/2} \right)^\gamma} =: I_{d+1}(\gamma)$$

and our next goal is to estimate $I_{d+1}(\gamma)$. To this end we first establish the following inequality for $A > 0$, $B \geq 0$, $\gamma \geq 2\kappa + 1$, $\kappa > 0$:

$$(7.9) \qquad \int_0^1 \frac{(1 - t^2)^{\kappa - 1} dt}{(1 + n\sqrt{B + A(1-t)})^\gamma} \leq \frac{c n^{-2\kappa}}{A^\kappa (1 + n\sqrt{B})^{\gamma - 2\kappa - 1}}.$$

Indeed, substituting $s = n^2 A(1 - t)$, we see that

$$\int_0^1 \frac{(1 - t^2)^{\kappa - 1} dt}{(1 + n\sqrt{B + A(1-t)})^\gamma} \leq \frac{2^{\kappa - 1}}{(An^2)^\kappa} \int_0^{An^2} \frac{s^{\kappa - 1} ds}{(1 + \sqrt{n^2 B + s})^\gamma}$$

$$\leq \frac{2^{\kappa - 1}}{(An^2)^\kappa (1 + n\sqrt{B})^{\gamma - 2\kappa - 1}} \int_0^\infty \frac{s^{\kappa - 1} ds}{(1 + \sqrt{s})^{2\kappa + 1}} \leq \frac{c n^{-2\kappa}}{A^\kappa (1 + n\sqrt{B})^{\gamma - 2\kappa - 1}}.$$

We now set $B := 1 + n \sum_{i=1}^d \sqrt{x_i y_i} t_i$ and $A := \sqrt{x_{d+1} y_{d+1}}$, and apply inequality (7.9) to the integral in $I_{d+1}(\gamma)$ with respect to t_{d+1}. We get

$$I_{d+1}(\gamma) \leq \frac{c n^{-2\kappa_{d+1}}}{(\sqrt{x_{d+1} y_{d+1}})^{\kappa_{d+1}}} \int_{[0,1]^d} \frac{\prod_{i=1}^{d+1} (1 - t_i^2)^{\kappa_i - 1} dt}{\left(1 + n \left[\sum_{i=1}^d \sqrt{x_i y_i} (1 - t_i) \right]^{1/2} \right)^{\gamma - 2\kappa_{d+1} - 1}}$$

$$= \frac{c n^{-2\kappa_{d+1}}}{(\sqrt{x_{d+1} y_{d+1}})^{\kappa_{d+1}}} I_d(\gamma - 2\kappa_{d+1} - 1).$$

Iterating this we obtain

$$I_{d+1}(\gamma) \leq \frac{c n^{-2|\kappa|}}{\prod_{i=1}^{d+1} (\sqrt{x_i y_i})^{\kappa_i}}.$$

One the other hand, we trivially have $I_{d+1}(\gamma) \le 1$ and hence

$$I_{d+1}(\gamma) \le \frac{cn^{-2|\kappa|}}{\prod_{i=1}^{d+1}(\sqrt{x_iy_i} + n^{-2})^{\kappa_i}}.$$

Therefore,

$$(7.10) \qquad \frac{J}{(1 + n\rho(x,y))^{|\kappa|}} \le \frac{cn^{-2|\kappa|}}{(1 + n\rho(x,y))^{|\kappa|}\prod_{i=1}^{d+1}(\sqrt{x_iy_i} + n^{-2})^{\kappa_i}}.$$

The simple inequality

$$(a + n^{-1})(b + n^{-1}) \le 3(ab + n^{-2})(1 + n|a - b|), \quad a, b \ge 0, \quad n \ge 1,$$

and (7.5) imply

$$\sqrt{x_i + n^{-2}}\sqrt{y_i + n^{-2}} \le (\sqrt{x_i} + n^{-1})(\sqrt{y_i} + n^{-1})$$

$$\le 3(\sqrt{x_iy_i} + n^{-2})(1 + n|\sqrt{x_i} - \sqrt{y_i}|) \le 3(\sqrt{x_iy_i} + n^{-2})(1 + n\rho(x,y)).$$

This coupled with (7.10) gives

$$\frac{J}{(1 + n\rho(x,y))^{|\kappa|}} \le \frac{cn^{-2|\kappa|}}{\sqrt{W_\kappa(n;x)}\sqrt{W_\kappa(n;y)}}.$$

and inserting the above in (7.8) leads to (7.7). $\square$

Sub-exponentially localized needlets on the simplex. Needlet systems on the simplex T^d have not been developed yet. To this end one should follow the well established scheme from e.g. [16, 17, 18, 19]. The main ingredient of this development are Theorems 7.1 and 7.2 from above, which provide the needed localization results. Since some other elements of this theory are not completely developed yet we shall not speculate here and leave the subject open.

8. Sub-exponentially localized kernels and frames induced by Hermite functions

The Hermite polynomials are defined by

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right), \qquad n = 0, 1, \ldots.$$

These polynomials are orthogonal on $\mathbb{R}$ with weight e^{-t^2}. We will denote by h_n the L^2-normalized Hermite functions, i.e.

$$h_n(t) := \left(2^n n!\sqrt{\pi}\right)^{-1/2} H_n(t)e^{-t^2/2}.$$

Then the d-dimensional Hermite functions $\mathcal{H}_\alpha(x)$ are defined by

$$\mathcal{H}_\alpha(x) := h_{\alpha_1}(x_1)\cdots h_{\alpha_d}(x_d), \quad x = (x_1, \ldots, x_d),$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Denoting by Proj_n the orthogonal projector onto $\mathcal{V}_n := \mathrm{span}\{\mathcal{H}_\alpha : |\alpha| = n\}$, we have

$$(\mathrm{Proj}_n f)(x) = \int_{\mathbb{R}^d} f(y)\mathcal{H}_n(x,y)dy \quad \text{with} \quad \mathcal{H}_n(x,y) := \sum_{|\alpha|=n} \mathcal{H}_\alpha(x)\mathcal{H}_\alpha(y).$$

It was shown in [19] that for admissible cutoff functions $\hat{a}$ the kernels

$$(8.1) \qquad\qquad L_n(x,y) := \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right)\mathcal{H}_j(x,y)$$

decay rapidly away from the main diagonal in $\mathbb{R}^d \times \mathbb{R}^d$. Our goal here is to show that for a suitable admissible cutoff functions $\hat{a}$ the localization of $L_n(x, y)$ can be improved to sup-exponential.

Theorem 8.1. *Let $0 < \varepsilon \le 1$ and assume that $\hat{a}$ in (8.1) is an admissible cutoff function obeying inequality (3.1) in Theorem 3.1. Then*

$$(8.2) \qquad |L_n(x, y)| \le c n^{d/2} \exp \left\{ - \frac{c^\diamond n^{1/2} \|x - y\|}{[\ln(e + n^{1/2} \|x - y\|)]^{1+\varepsilon}} \right\}, \quad x, y \in \mathbb{R}^d,$$

where $c^\diamond = c' \varepsilon$ with $c' > 0$ an absolute constant, and c depends only on d and ε. Recall that $\|x\| := \max_j |x_j|$.

As before, for a suitable cutoff function $\hat{a}$ the term $[\ln(e + n^{1/2} \|x - y\|)]^{1+\varepsilon}$ above can be replaced by any product of multiple logarithmic terms as in (1.10).

Furthermore, for some constant $c'' > 0$

$$(8.3) \qquad |L_n(x, y)| \le c e^{-c'' \max\{\|x\|^2, \|y\|^2\}} \quad \text{if} \ \ \max\{\|x\|, \|y\|\} \ge (8n + 2)^{1/2}.$$

Proof. As is well known (see [22, Lemma 1.5.1]) for some constants $\gamma, c > 0$

$$|h_n(t)| \le e^{-\gamma t^2} \quad \text{for} \quad t \ge (4n + 2)^{1/2} \quad \text{and} \quad \|h_n\|_\infty \le c n^{-1/12} \le c,$$

which readily yields (8.3). In turn (8.3) immediately implies (8.2) in the case when $\max\{\|x\|, \|y\|\} \ge (8n + 2)^{1/2}$.

To obtain a nontrivial estimate on $|L_n(x, y)|$ we follow the approach from [19]. Let $A_j^{(x)} := -\frac{\partial}{\partial x_j} + x_j$. Then the following identity holds (see the proof of Theorem 1 in [19]):

$$(8.4) \quad 2^k (x_j - y_j)^k L_n(x, y) = \sum_{k/2 \le l \le k} c_{l,k} \sum_{\nu=0}^\infty \Delta^l \hat{a}\left(\frac{\nu}{n}\right) \left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k} \mathcal{H}_\nu(x, y),$$

where $\Delta^l \hat{a}(\frac{\nu}{n})$ is the lth forward difference applied to the sequence $\{\hat{a}(\frac{\nu}{n})\}_{\nu=0}^\infty$, and the coefficients $c_{l,k}$ are given by

$$(8.5) \quad c_{l,k} = (-1)^{k-l} 4^{k-l} (2k - 2l - 1)!! \binom{k}{2l - k} = (-1)^{k-l} 2^{k-l} \frac{k!}{(k-l)!(2l-k)!}.$$

Identity (8.4) follows from Lemma 3.2.3 in [22]; the constants $c_{l,k}$ are given explicitly in [19].

Let us assume that $3 \le k \le n/4$. Using the estimate [22, Lemma 3.2.2]

$$(8.6) \qquad \qquad \mathcal{H}_n(x, x) \le c n^{d/2-1}, \qquad x \in \mathbb{R}^d,$$

one can follow the proof of Theorem 2.2 in [19] to show that

$$\left| \left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k} \mathcal{H}_\nu(x, y) \right| \le c(2\nu + 4l - 2k)^{(2l-k)/2}$$

$$\times \sum_{i=0}^{2l-k} \binom{2l-k}{i} (\nu + i)^{(\frac{d}{2}-1)/2} (\nu + 2l - k + i)^{(\frac{d}{2}-1)/2}$$

$$\le 2^{(2l-k)/2} (\nu + 2l - k)^{(2l-k+d-2)/2} \sum_{i=0}^{2l-k} \binom{2l-k}{i}$$

$$\le 2^{3(2l-k)/2} (\nu + 2l - k)^{(2l-k+d-2)/2} \le 2^{3k/2} (\nu + k)^{(2l-k+d-2)/2}.$$

Evidently, $\left|\Delta^l \hat{a}\left(\frac{\nu}{n}\right)\right| \leq n^{-l}\|\hat{a}^{(l)}\|_\infty$ and $\Delta^l \hat{a}(\frac{\nu}{n}) = 0$ if $0 \leq \nu \leq n/2 - l$ or $\nu \geq 2n$. Using the above and the representation of $c_{l,k}$ from (8.5) in (8.4) we obtain

$$2^k|x_j - y_j|^k |L_n(x,y)| \leq c 2^{3k/2} \sum_{k/2 \leq l \leq k} \sum_{\nu=n/2-k}^{2n} (\nu+k)^{(2l-k+d-2)/2} |c_{l,k}| n^{-l} \|\hat{a}^{(l)}\|_\infty$$

$$\leq c 2^{2k} 3^{k+d} n^{(d-k)/2} \sum_{k/2 \leq l \leq k} \frac{k!}{(k-l)!(2l-k)!} \|\hat{a}^{(l)}\|_\infty,$$

where we used that $\nu + k \leq 2n + n/4 < 3n$. This along with estimate (3.1) in Theorem 3.1 leads to

$$2^k|x_j - y_j|^k |L_n(x,y)| \leq c(12)^k n^{(d-k)/2} \sum_{k/2 \leq l \leq k} \frac{k!}{(k-l)!(2l-k)!} (\tilde{c}/\varepsilon)^l l^l (\ln l)^{l(1+\varepsilon)}$$

$$(8.7) \qquad\qquad \leq c n^{(d-k)/2} (12\tilde{c}/\varepsilon)^k (\ln k)^{k(1+\varepsilon)} k! \sum_{k/2 \leq l \leq k} \frac{k^l}{(k-l)!(2l-k)!}.$$

We next establish the following estimate for the last sum above:

$$(8.8) \qquad\qquad \sum_{k/2 \leq l \leq k} \frac{k^l}{(k-l)!(2l-k)!} \leq 2(2e^2)^k.$$

To prove this we split the sum into two sums. For $k/2 \leq l \leq 3k/4$, by Stirling's formula, we have $(k-l)! \geq [(k-l)/e]^{k-l} \geq [k/(4e)]^{k-l}$ and hence

$$\sum_{k/2 \leq l \leq 3k/4} \frac{k^l}{(k-l)!(2l-k)!} \leq (4e)^{k/2} \sum_{k/2 \leq l \leq 3k/4} \frac{k^{2l-k}}{(2l-k)!}$$

$$\leq (4e)^{k/2} \sum_{m=0}^{\infty} \frac{k^m}{m!} = (4e^3)^{k/2},$$

whereas for $3k/4 < l \leq k$, we have $(2l-k)! \geq [k/(2e)]^{2l-k}$, and hence

$$\sum_{3k/4 < l \leq k} \frac{k^l}{(k-l)!(2l-k)!} \leq (2e)^k \sum_{3k/4 < l \leq k} \frac{k^{k-l}}{(k-l)!} \leq (2e^2)^k,$$

and (8.8) follows. Substituting (8.8) in (8.7) we get

$$2^k|x_j - y_j|^k |L_n(x,y)| \leq c n^{(d-k)/2} c_\varepsilon^k k^k (\ln k)^{k(1+\varepsilon)}, \quad 1 \leq j \leq k,$$

where $c_\varepsilon = 24e^2 \tilde{c}/\varepsilon \geq 24e^2$; we used that $k! < k^k$. Therefore,

$$(8.9) \qquad\qquad |L_n(x,y)| \leq c n^{d/2} \left(\frac{c_\varepsilon k (\ln k)^{1+\varepsilon}}{n^{1/2}\|x-y\|} \right)^k, \quad 3 \leq k \leq n/4.$$

We use (8.6) and the Cauchy-Schwartz inequality to obtain the following trivial estimate

$$(8.10) \qquad |L_n(x,y)| \leq c \sum_{j=0}^{2n} \Big(\sum_{|\alpha|=j} |\mathcal{H}_\alpha(x)|^2 \Big)^{1/2} \Big(\sum_{|\alpha|=j} |\mathcal{H}_\alpha(x)|^2 \Big)^{1/2}$$

$$= c \sum_{j=0}^{2n} \mathcal{H}_j(x,x)^{1/2} \mathcal{H}_j(y,y)^{1/2} \leq c n^{d/2}.$$

Let $\max\{\|x\|, \|y\|\} \le (8n+2)^{1/2}$. From (8.10) it follows immediately that estimate (8.2) holds if $n^{1/2}\|x-y\| \le 3c_\varepsilon[\ln(e+n^{1/2}\|x-y\|)]^{1+\varepsilon}$.

Assume now that $n^{1/2}\|x-y\| > 3c_\varepsilon[\ln(e+n^{1/2}\|x-y\|)]^{1+\varepsilon}$ and choose

$$k := \left\lfloor \frac{n^{1/2}\|x-y\|}{c_\varepsilon[\ln(e+n^{1/2}\|x-y\|)]^{1+\varepsilon}} \right\rfloor.$$

Then $3 \le k \le n/4$ as $\|x-y\| \le 2(8n+2)^{1/2}$ and evidently $\frac{c_\varepsilon k(\ln k)^{1+\varepsilon}}{n^{1/2}\|x-y\|} \le e^{-1}$. Now, estimate (8.2) follows by (8.9) with $c^\diamond = c_\varepsilon^{-1} = \varepsilon(24e^2\tilde{c})^{-1}$. $\square$

Sub-exponentially localized needlets in the context of Hermite functions. For this construction, we shall utilize the cubature formula from [19] with nodes in $\mathcal{X}_j \subset \mathbb{R}^d$ consisting of $O(4^{jd})$ points on $\mathbb{R}^d$ obtained as a product of nodal sets of univariate Gaussian quadrature formulas and with positive coefficients $\{c_\xi\}_{\xi \in \mathcal{X}_j}$, which is exact for functions in $\bigoplus_{m=0}^{N} \mathcal{V}_m$ with $N := c4^{j+1}$ (for more details, see [19]). As before, we choose an admissible cutoff function $\hat{a}$ of type (c) obeying (3.1) and define

$$\psi_\xi(x) := c_\xi^{1/2} L_n(\xi, x), \quad \xi \in \mathcal{X}_j,$$

where $L_n(x, y)$ with $n := 4^{j-1}$ is the kernel from (8.1). Setting $\mathcal{X} := \cup_{j \ge 0} \mathcal{X}_j$ we define the Hermite needlet system by $\Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}$. The sub-exponential localization of the Hermite needlets is inherited from Theorem 8.1: For $\xi \in \mathcal{X}_j$, $j \ge 0$,

$$(8.11) \qquad |\psi_\xi(x)| \le c2^{jd} \exp\left\{ -\frac{c'\varepsilon 2^j\|x-\xi\|}{[\ln(e+2^j\|x-\xi\|)]^{1+\varepsilon}} \right\}, \qquad x \in \mathbb{R}^d,$$

and

$$(8.12) \qquad |\psi_\xi(x)| \le ce^{-c''\|x\|^2} \quad \text{if } \|x\| \ge c_*2^j$$

for an appropriate constant c_*. This is an improvement compared with the localization of the needlets from [19].

9. SUB-EXPONENTIALLY LOCALIZED KERNELS AND FRAMES INDUCED BY LAGUERRE FUNCTIONS

The Laguerre polynomials, defined by

$$L_n^\alpha(t) = \frac{1}{n!} t^{-\alpha} e^t \frac{d^n}{dt^n}(t^{n+\alpha}e^{-t}), \quad \alpha > -1, \quad n = 0, 1, \ldots.$$

are orthogonal on $\mathbb{R}_+ = (0, \infty)$ with weight $t^\alpha e^{-t}$. There are three types of Laguerre functions considered in the literature (see [22]), defined by

$$(9.1) \qquad \mathcal{F}_n^\alpha(t) := \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-t^2/2} L_n^\alpha(t^2),$$

$$(9.2) \qquad \mathcal{L}_n^\alpha(t) := \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-t/2} t^{\alpha/2} L_n^\alpha(t),$$

and

$$(9.3) \qquad \mathcal{M}_n^\alpha(t) := (2t)^{1/2} \mathcal{L}_n^\alpha(t^2).$$

It is well known that $\{\mathcal{F}_n^\alpha\}_{n \ge 0}$ is an orthonormal basis for the weighed space $L^2(\mathbb{R}_+, t^{2\alpha+1})$, while $\{\mathcal{L}_n^\alpha\}_{n \ge 0}$ and $\{\mathcal{M}_n^\alpha\}_{n \ge 0}$ are orthogonal bases for $L^2(\mathbb{R}_+)$.

Here we only consider the Laguerre functions $\{\mathcal{F}_n^\alpha\}$ for $\alpha \geq 0$. Analogous results for $\{\mathcal{L}_n^\alpha\}_{n\geq 0}$ and $\{\mathcal{M}_n^\alpha\}_{n\geq 0}$ follow immediately as in [9].

The d-dimensional tensor product Laguerre functions associated to $\{\mathcal{F}_n^\alpha\}$ are defined by

$$\mathcal{F}_\nu^\alpha(x) := \mathcal{F}_{\nu_1}^{\alpha_1}(x_1) \cdots \mathcal{F}_{\nu_d}^{\alpha_d}(x_d), \quad x = (x_1, \ldots, x_d),$$

where $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$. The kernel of the orthogonal projector onto $\mathcal{V}_n := \text{span}\{\mathcal{F}_\nu^\alpha : |\nu| = n\}$ is given by

$$\mathcal{F}_n^\alpha(x, y) := \sum_{|\nu|=n} \mathcal{F}_\nu^\alpha(x) \mathcal{F}_\nu^\alpha(y).$$

As elsewhere in this paper, we are interested in constructing sup-exponential localized kernels of the form

$$(9.4) \qquad L_n^\alpha(x, y) := \sum_{j=0}^\infty \hat{a}\left(\frac{j}{n}\right) \mathcal{F}_j^\alpha(x, y), \quad x, y \in \mathbb{R}_+^d,$$

where $\hat{a}$ is an admissible cutoff function (see Definition 1.1). Note that in [9] it is proved that for admissible cutoff functions $\hat{a}$ the kernels $L_n^\alpha(x, y)$ have faster than the reciprocal of any polynomial decay away from the main diagonal in $\mathbb{R}_+^d \times \mathbb{R}_+^d$.

We shall use some of the notation and results from [9]. Recall our standing notation for norms in $\mathbb{R}^d$: $\|x\| := \max_i |x_i|$, $\|x\|_2 := (\sum_i |x_i|^2)^{1/2}$, and $|x| = \|x\|_1 := \sum_i |x_i|$. We shall also need the quantity

$$W_\alpha(n; x) := \prod_{j=1}^d (x_j + n^{-1/2})^{2\alpha_j+1}, \qquad x \in \mathbb{R}_+^d.$$

Theorem 9.1. *Let $0 < \varepsilon \leq 1$ and assume that $\hat{a}$ in (9.4) is an admissible cutoff function obeying inequality (3.1) in Theorem 3.1. Then for $x, y \in \mathbb{R}_+^d$,*

$$(9.5) \quad |L_n^\alpha(x, y)| \leq \frac{cn^{d/2}}{\sqrt{W_\alpha(n; x)} \sqrt{W_\alpha(n; y)}} \exp\left\{ -\frac{c'\varepsilon n^{1/2}\|x - y\|}{[\ln(e + n^{1/2}\|x - y\|)]^{1+\varepsilon}} \right\},$$

where $c' > 0$ is a constant depending only on d and α, and c depends on d, α, ε.

As in similar situations before, for an appropriate cutoff function $\hat{a}$ the term $[\ln(e + n^{1/2}\|x - y\|)]^{1+\varepsilon}$ above can be replaced by any product of multiple logarithms as in (1.10).

In addition, for some constant $c'' > 0$

$$(9.6) \quad |L_n^\alpha(x, y)| \leq ce^{-c'' \max\{\|x\|^2, \|y\|^2\}} \quad \text{if } \max\{\|x\|, \|y\|\} \geq (12n + 3\|\alpha\| + 3)^{1/2}.$$

Besides Theorem 3.1 a main new ingredient in the proof of this theorem will be the following result for Laguerre polynomials:

Proposition 9.2. *For $n \geq 1$ and $-1/2 \leq \alpha \leq n$, we have*

$$(9.7) \qquad |L_n^\alpha(t)|e^{-t/2} \leq c2^\alpha (n/t)^{\alpha/2}, \quad 0 < t < \infty,$$

where $c > 0$ is an absolute constant. The lower bound $-1/2$ for α can be replaced by any constant $\alpha_0 > -1$ but c will depend on α_0.

Proof. We shall utilize some estimates on $|L_n^\alpha(t)|$ established in [11]. As is shown in [11, Theorem 2] for $\alpha \geq 24$ and $n \geq 1$

$$(9.8) \qquad \max_{0 < t \leq \frac{2(\alpha+1)^2}{2n+\alpha+1}} t^{\alpha+1} e^{-t} [L_n^\alpha(t)]^2 < 650 \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \frac{(\alpha+1)^{4/3}}{n^{1/6}(n+\alpha+1)^{5/6}}.$$

For the next estimate, let us denote briefly

$$s_n := \sqrt{n+\alpha+1} + \sqrt{n} \quad \text{and} \quad q_n := \sqrt{n+\alpha+1} - \sqrt{n}.$$

Then, for $\alpha \geq 24$ and $n \geq 35$ [11, Theorem 4],

$$(9.9) \qquad t^{\alpha+1} e^{-t} [L_n^\alpha(t)]^2 < 680 \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \frac{t}{\sqrt{(t-q_n^2)(s_n^2-t)}} \quad \text{for } t \in (q_n^2, s_n^2).$$

We shall consider several cases for α, n, and t.

Case 1: $-1/2 \leq \alpha < 24$ and $n \geq 1$ or $1 \leq n < 35$ (then $\alpha \leq n < 35$). In this case, estimate (9.7) ($0 < t < \infty$) follows from the standard estimate in [21, §8.22] with a constant depending on α (see also [22, Lemma 1.5.3] or [19, (2.11)]).

Case 2: $n \geq 35$, $24 \leq \alpha \leq n$, and $0 < t \leq 1/n$. A standard upper bound for $|L_n^\alpha(t)|$ ($\alpha \geq 0$) is given by (see e.g. [11, (3)])

$$(9.10) \qquad |L_n^\alpha(t)| e^{-t/2} \leq \frac{(\alpha+1)_n}{n!} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \quad 0 < t < \infty.$$

Note that by Stirling's formula $\Gamma(t+1) \sim (t/e)^t \sqrt{t}$ on $[1, \infty)$ and hence

$$(9.11) \qquad \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \leq c \frac{((n+\alpha)/e)^{n+\alpha}\sqrt{n+\alpha}}{(n/e)^n \sqrt{n}} \leq c 2^\alpha n^\alpha,$$

where $c > 0$ is an absolute constant. Using this in (9.10) we get

$$|L_n^\alpha(t)| e^{-t/2} \leq c n^\alpha \leq c(n/t)^{\alpha/2}, \quad 0 < t \leq 1/n,$$

which gives (9.7) in the case under consideration.

Case 3: $n \geq 35$, $24 \leq \alpha \leq n$, and $\frac{1}{n} < t \leq \frac{2(\alpha+1)^2}{2n+\alpha+1}$. From (9.8) and (9.11) we get

$$e^{-t} [L_n^\alpha(t)]^2 \leq \frac{c(\alpha+1)^{4/3}}{n^{1/6}(n+\alpha+1)^{5/6}} \frac{2^\alpha n^\alpha}{t^{\alpha+1}} \leq \frac{c\alpha^{4/3} 2^\alpha}{nt} \left(\frac{n}{t}\right)^\alpha \leq c\alpha^{4/3} 2^\alpha \left(\frac{n}{t}\right)^\alpha,$$

which implies (9.7).

Case 4: $\frac{2(\alpha+1)^2}{2n+\alpha+1} < t \leq n$. We have $q_n^2 < \frac{(\alpha+1)^2}{4n} < \frac{1}{2} \frac{2(\alpha+1)^\alpha}{2n+\alpha+1}$ and $s_n^2 > 4n$. We use these and (9.11) in (9.9) to obtain

$$t^\alpha e^{-t} [L_n^\alpha(t)]^2 \leq \frac{c 2^\alpha n^\alpha}{\sqrt{(t-q_n^2)(s_n^2-t)}} \leq \frac{c 2^\alpha n^\alpha}{\sqrt{n/(n+\alpha+1)}} \leq c 2^\alpha n^\alpha,$$

which again gives (9.7). The proof of the proposition is complete. $\qquad \square$

Proof of Theorem 9.1. Note first that from Lemma 1.5.3 in [22] or [21, §8.22] it follows that

$$(9.12) \quad |L_n^\alpha(t)| e^{-t/2} \leq c n^\alpha \quad \text{for } t \geq 0 \quad \text{and} \quad |L_n^\alpha(t)| e^{-t/2} \leq c e^{-\gamma n} \quad \text{for } t \geq 3N/2,$$

where $\gamma > 0$ is a constant and $N := 4n + 2\alpha + 2$. These immediately lead to the estimates (see also [9]):

$$(9.13) \qquad \|\mathcal{F}_n^\alpha\|_\infty \leq c n^{\alpha/2} \quad \text{and} \quad |\mathcal{F}_n^\alpha(t)| \leq c e^{-\gamma' t^2} \quad \text{for } t \geq (3N/2)^{1/2},$$

which readily imply (9.6). In turn estimate (9.6) obviously implies that (9.5) holds if $\max\{\|x\|, \|y\|\} \geq (12n + 3\|\alpha\| + 3)^{1/2}$.

The proof of (9.5), when $\max\{\|x\|, \|y\|\} < (12n + 3\|\alpha\| + 3)^{1/2}$, will rely on the developments in [9], where in particular it is shown that $L_n^\alpha(x, y)$ has the representation (see [9, (8.3)-(8.4)])

$$(9.14) \qquad L_n^\alpha(x, y) = c \int_{[0,\pi]^d} \mathcal{K}_n^\alpha \Big(\|x\|_2^2 + \|y\|_2^2 + 2 \sum_{i=1}^d x_i y_i \cos \theta_i \Big)$$

$$\times \prod_{i=1}^d j_{\alpha_i - 1/2}(x_i y_i \cos \theta_i) \sin^{2\alpha_i} \theta_i \, d\theta,$$

where $j_\alpha(t) := t^{-\alpha} J_\alpha(t)$ with $J_\alpha(t)$ being the Bessel function, and the kernel $\mathcal{K}_n^\alpha$ is given by

$$(9.15) \qquad \mathcal{K}_n^\alpha(t) = \sum_{m=0}^\infty \Delta^{k+1} \hat{a} \Big(\frac{m}{n} \Big) L_m^{|\alpha|+k+d}(t) e^{-t/2}.$$

Here $k \geq 0$ is arbitrary and the finite difference Δ^{k+1} is with respect to m.

We claim that $\mathcal{K}_n^\alpha$ satisfies the estimate

$$(9.16) \quad |\mathcal{K}_n^\alpha(t)| \leq cn^{|\alpha|+d} \exp \left\{ -\frac{c^\diamond (nt)^{1/2}}{[\ln(e + (nt)^{1/2})]^{1+\varepsilon}} \right\}, \quad t \leq 4d(12n + 3\|\alpha\| + 3),$$

where $c^\diamond = c'\varepsilon$ with $c' > 0$ independent of ε.

Suppose $t \leq 4d(12n + 3\alpha + 3)$ and let $2 \leq k \leq n/4$. Evidently, we have $\left| \Delta^{k+1} \hat{a} \left(\frac{\nu}{n} \right) \right| \leq n^{-k-1} \|\hat{a}^{(k+1)}\|_\infty$ and $\Delta^{k+1} \hat{a}(\frac{\nu}{n}) = 0$ if $0 \leq \nu \leq n/2 - k - 1$ or $\nu \geq 2n$. From these, estimate (3.1) from Theorem 3.1, and (9.7) it follows that

$$|\mathcal{K}_n^\alpha(t)| \leq c \sum_{n/2-k-1 < m < 2n} \frac{1}{n^{k+1}} \Big(\frac{4m}{t} \Big)^{(|\alpha|+k+d)/2} \|\hat{a}^{(k+1)}\|_\infty$$

$$(9.17) \qquad \leq cn^{-k} \Big(\frac{8n}{t} \Big)^{(|\alpha|+k+d)/2} (\tilde{c}/\varepsilon)^{k+1} (k+1)^{k+1} [\ln(k+1)]^{(k+1)(1+\varepsilon)}$$

$$\leq cn^{|\alpha|+d} \Big(\frac{8}{nt} \Big)^{(|\alpha|+d-1)/2} \left[\frac{c_\varepsilon (k+1)[\ln(k+1)]^{1+\varepsilon}}{(nt)^{1/2}} \right]^{k+1},$$

where $c_\varepsilon = 64\tilde{c}/\varepsilon$, $\tilde{c} \geq 1$.

Exactly as above, but using the first estimate in (9.12) instead of (9.7) and (9.15) with $k = 0$ we get

$$(9.18) \qquad |\mathcal{K}_n^\alpha(t)| \leq c \sum_{n/2-1 < m < 2n} n^{-1} m^{|\alpha|+d} \|\hat{a}'\|_\infty \leq cn^{|\alpha|+d}.$$

Suppose $(nt)^{1/2} \geq 2c^\flat c_\varepsilon [\ln(e + \sqrt{nt})]^{1+\varepsilon}$, where $c^\flat := (12d(5 + \|\alpha\|))^{1/2}$, and choose

$$k := \left\lfloor \frac{\sqrt{nt}}{c^\flat c_\varepsilon [\ln(e + \sqrt{nt})]^{1+\varepsilon}} \right\rfloor - 1.$$

Then $3 \leq k + 1 \leq n/4$ as $t \leq 4d(12n + 3\|\alpha\| + 3) \leq (c^\flat)^2 n$, and it is easy to see that $\frac{c_\varepsilon(k+1)[\ln(k+1)]^{1+\varepsilon}}{\sqrt{nt}} \leq e^{-1}$. Hence, by (9.17) it follows that (9.16) holds with $c^\diamond = (c^\flat c_\varepsilon)^{-1}$, where we used that $nt \geq (c^\flat c_\varepsilon)^2$.

If $(nt)^{1/2} < c^\flat c_\varepsilon [\ln(e + \sqrt{nt})]^{1+\varepsilon}$, then (9.16) is immediate from (9.18). This completes the proof of (9.16).

To prove (9.5), we will use (9.16) with $t = \|x\|_2^2 + \|y\|_2^2 + 2\sum_{i=1}^d x_i y_i \cos\theta_i$. Evidently, $t \geq \|x - y\|^2$ and $t \leq d(\|x\| + \|y\|)^2 \leq 4d(12n + 3\|\alpha\| + 3)$ as in (9.16). Also, $\exp\left\{\frac{-\frac{1}{2}c^\diamond x}{[\ln(e+x)]^{1+\varepsilon}}\right\} \leq c(1+x^2)^{-2|\alpha|-d}$, $x > 0$, where $c > 0$ is a sufficiently large constant. Therefore, from (9.16) it follows that

$$|\mathcal{K}_n^\alpha(t)| \leq cn^{|\alpha|+d} \exp\left\{-\frac{\frac{1}{2}c^\diamond n^{1/2}\|x - y\|}{[\ln(e + n^{1/2}\|x - y\|)]^{1+\varepsilon}}\right\} \left(\frac{1}{1 + nt}\right)^\tau$$

with $\tau = 2|\alpha| + d$. Combining this with (9.14) and the fact that $|j_{\alpha-1/2}(t)| \leq c_\alpha$ for $t \in \mathbb{R}_+$, $\alpha \geq 0$, (see [9]) we arrive at the estimate

$$|L_n^\alpha(x, y)| \leq cn^{|\alpha|+d} \exp\left\{-\frac{\frac{1}{2}c^\diamond n^{-1/2}\|x - y\|}{[\ln(e + n^{1/2}\|x - y\|)]^{1+\varepsilon}}\right\}$$
$$\times \int_{[0,\pi]^d} \frac{\prod_{i=1}^d \sin^{2\alpha_i}\theta_i \, d\theta}{\left[1 + n\left(\|x\|_2^2 + \|y\|_2^2 + 2\sum_{i=1}^d x_i y_i \cos\theta_i\right)\right]^\tau}.$$

Denote the last integral above by J. We now estimate J quite similarly as in the proof of Theorem 3.2 in [9]. Substituting $\theta_i := \pi - t_i$ and using that $1 - \cos t = 2\sin^2\frac{t}{2} \sim t^2$ we get

$$J \leq c \int_{[0,\pi]^d} \frac{\prod_{i=1}^d t^{2\alpha_i} \, dt}{\left[1 + n\left(\|x - y\|^2 + \sum_{i=1}^d x_i y_i t_i^2\right)\right]^\tau}$$
$$\leq c \int_{[0,\pi]^\ell} \frac{\prod_{i=1}^\ell t^{2\alpha_i} \, dt}{\left[1 + n\left(\|x - y\|^2 + \sum_{i=1}^\ell x_i y_i t_i^2\right)\right]^\tau}, \quad 0 \leq \ell \leq d.$$

For a fixed $1 \leq \ell \leq d$ denote $\alpha' := (\alpha_1, \ldots, \alpha_\ell)$. Then substituting $u_i = t_i(x_i y_i)^{1/2}$ above, we get

$$J \leq \frac{c}{n^{|\alpha'|}\prod_{i=1}^\ell (x_i y_i)^{\alpha_i+1/2}} \prod_{i=1}^\ell \int_0^{\pi(x_i y_i)^{1/2}} \frac{du}{\left[1 + n(\|x - y\|^2 + \sum_{i=1}^\ell u_i^2)\right]^{\tau-|\alpha'|}},$$

where we used $|\alpha'|$ terms from the denominator to cancel the numerator. We now enlarge the domain of integration to $\mathbb{R}^\ell$ and use spherical coordinates to bound the product of integrals above by

$$\int_0^\infty \frac{r^{\ell-1}dr}{\left[1 + n(\|x - y\|^2 + r^2)\right]^{\tau-|\alpha'|}} \leq \frac{c}{n^{\ell/2}\left(1 + n\|x - y\|^2\right)^{\tau-|\alpha'|-\ell/2}},$$

where we used that $\tau := 2|\alpha| + d > |\alpha| + d/2$. Therefore, we have for $0 \leq \ell \leq d$ (the case $\ell = 0$, $|\alpha'| := 0$, is trivial)

$$n^{|\alpha|+d/2}J \leq \frac{cn^{|\alpha|+d/2}}{n^{|\alpha'|+\ell/2}\prod_{i=1}^\ell (x_i y_i)^{\alpha_i+1/2}(1 + n\|x - y\|^2)^{\tau-|\alpha|-d/2}}$$
$$\leq \frac{c}{\prod_{i=1}^\ell (x_i y_i)^{\alpha_i+1/2}\prod_{i=\ell+1}^d (n^{-1})^{\alpha_i+1/2}(1 + n^{1/2}\|x - y\|)^{|\alpha|+d/2}},$$

which readily implies

$$n^{|\alpha|+d/2}J \leq \frac{c}{\prod_{i=1}^{\ell}(x_iy_i + n^{-1})^{\alpha_i+1/2}(1 + n^{1/2}\|x-y\|)^{|\alpha|+d/2}}.$$

Using now the simple inequalities

$$(x_i + n^{-1/2})(y_i + n^{-1/2}) \leq 3(x_iy_i + n^{-1})(1 + n^{1/2}\|x-y\|), \quad 1 \leq i \leq d,$$

we obtain

$$n^{|\alpha|+d/2}J \leq \frac{c}{\sqrt{W_\alpha(n;x)}\sqrt{W_\alpha(n;y)}}$$

and estimate (9.5) follows. $\qquad\square$

Sub-exponentially localized needlets induced by Laguerre functions. The construction here is similar as in previous sections, in particular, see the case of Hermite functions. We start with the cubature formula from [9] with nodes in $\mathcal{X}_j \subset \mathbb{R}_+^d$ consisting of $O(4^{jd})$ points in $\mathbb{R}^d$ obtained as product of the nodal sets of univariate Gaussian quadrature formulas on $\mathbb{R}_+$ and with positive coefficients $\{c_\xi\}_{\xi \in \mathcal{X}_j}$, which is exact for functions in $\bigoplus_{m=0}^{N}\mathcal{V}_m$ with $N := c4^{j+1}$ (see [9]). We next choose an admissible cutoff function $\hat{a}$ of type (c) obeying (3.1) and then define

$$\psi_\xi(x) := c_\xi^{1/2}L_n(\xi, x), \quad \xi \in \mathcal{X}_j,$$

where $L_n(x,y)$ with $n := 4^{j-1}$ is the kernel from (8.1). Notice that here the dilation on the frequency side is by a factor of 4. Setting $\mathcal{X} := \cup_{j\geq 0}\mathcal{X}_j$, we define the Laguerre needlet system by $\Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}$. The sub-exponential localization of the Laguerre needlets follows by Theorem 9.1 and takes the form: For $\xi \in \mathcal{X}_j$, $j \geq 0$,

$$(9.19) \qquad |\psi_\xi(x)| \leq \frac{c2^{jd}}{\sqrt{W_\alpha(n;\xi)}} \exp\left\{ -\frac{c'\varepsilon 2^j\|x-\xi\|}{[\ln(e + 2^j\|x-\xi\|)]^{1+\varepsilon}}\right\}, \quad x \in \mathbb{R}_+^d,$$

and

$$(9.20) \qquad |\psi_\xi(x)| \leq ce^{-c''\|x\|^2} \quad \text{if } \|x\| \geq c_*2^j$$

for an appropriate constant c_*. This is a natural improvement of the localization of the needlets from [9].

10. THE LOCALIZATION PRINCIPLE FROM §2 FAILS FOR PRODUCT JACOBI POLYNOMIALS

In this section, we show that in contrast to the cases considered in previous sections surprisingly the localization principle described in §2 is no longer valid for 2-d tensor product Legendre or Chebyshev polynomials and products of Legendre and Chebyshev polynomials.

The nth degree Legendre polynomial P_n is defined by

$$P_n(x) := \frac{1}{n!2^n}\left(\frac{d}{dx}\right)^n(x^2 - 1)^n$$

and it is known that $P_n(1) = 1$, $P_n(-1) = (-1)^n$ and $\int_{-1}^{1}P_n^2(x)dx = (n+1/2)^{-1}$. Denote $\tilde{P}_n := (n+1/2)^{1/2}P_n$ and note that $\{\tilde{P}_n\}_{n\geq 0}$ is an orthonormal basis for $L^2[-1,1]$.

The 2-d tensor product Legendre polynomials are defined by

$$\tilde{P}_\nu(x) := \tilde{P}_{\nu_1}(x_1)\tilde{P}_{\nu_2}(x_2), \quad \nu = (\nu_1, \nu_2), \ x = (x_1, x_2).$$

For an admissible univariate cutoff function $\hat{a} \geq 0$ (see Definition 1.1) define

$$(10.1) \quad L_n(x,y) := \sum_{m=0}^{\infty} \hat{a}\left(\frac{m}{n}\right)\tilde{P}_m(x,y), \quad \text{where} \quad \tilde{P}_m(x,y) := \sum_{|\nu|=m} \tilde{P}_\nu(x)\tilde{P}_\nu(y).$$

Here $|\nu| := \nu_1 + \nu_2$ and $x,y \in [-1,1]^2$. In more detail

$$\tilde{P}_m(x,y) = \sum_{\nu_1+\nu_2=m} (\nu_1+1/2)(\nu_2+1/2)P_{\nu_1}(x_1)P_{\nu_2}(x_2)P_{\nu_1}(y_1)P_{\nu_2}(y_2).$$

In order to show that $L_n(x,y)$ has very poor localization at some points on $[-1,1]^2 \times [-1,1]^2$, which are away from the main diagonal, we fix $y = (1,1)$. Using that $P_n(-1) = (-1)^n$ one easily verifies the identity

$$(10.2) \qquad \tilde{P}_m(1,-1,1,1) = \sum_{j=0}^{m}(-1)^j(j+1/2)(m-j+1/2) = (1+(-1)^m)/8.$$

Indeed, for odd m this follows from the symmetry of the terms in the sum and in the case $m = 2k$ it follows by simple manipulations from the case $m = 2k-1$.

By (10.1)-(10.2) it follows that

$$L_n(1,-1,1,1) = \frac{1}{4}\sum_{j=0}^{\infty}\hat{a}\left(\frac{2j}{n}\right) = \frac{n}{8}\int_0^{\infty}\hat{a}(t)dt + \frac{\hat{a}(0)}{8} + \mathcal{O}(n^{-1}),$$

which shows that $L_n(x,y)$ has no localization whatsoever for $x = (1,-1)$, $y = (1,1)$. The behavior of the sequence $\{L_n(x,(1,1))\}_{n=0}^{\infty}$ is similar when x is any other point on the lines $\{(x_1,1)\}$ and $\{(1,x_2)\}$. However, the kernels $L_n(x,y)$ are very well localized in large portion of $[-1,1]^2 \times [-1,1]^2$.

It is slightly more complicated to show that the situation is quite the same for Chebyshev polynomials. In this case (10.1) holds with

$$\tilde{P}_m(x,y) = \frac{4}{\pi^2}\sum_{j=0}^{m}\left(1 - \frac{\delta_{j,0}}{2}\right)\left(1 - \frac{\delta_{j,m}}{2}\right)T_j(x_1)T_{m-j}(x_2)T_j(y_1)T_{m-j}(y_2),$$

where $T_n(v) = \cos n\arccos v$ and $\delta_{j,k}$ is the Kronecker delta (see the definition of Chebyshev polynomials given in §2). Using that $T_n(-1) = (-1)^n$ one easily verifies the identity

$$\tilde{P}_m(1,-1,1,1) = \frac{4}{\pi^2}\sum_{j=0}^{m}\left(1 - \frac{\delta_{j,0}}{2}\right)\left(1 - \frac{\delta_{j,m}}{2}\right)(-1)^{m-j} = \frac{\delta_{m,0}}{\pi^2}.$$

Hence

$$L_n(1,-1,1,1) = \frac{\hat{a}(0)}{\pi^2},$$

which implies that there is no localization for cutoff functions of type (a) in Definition 1.1.

The above identity is inconclusive for cutoff functions $\hat{a}$ of type (b) and (c). In fact, for every such $\hat{a}$ and every segment

$$\{x(t) = (1 - 2t\cos\theta, 1 - 2t\sin\theta) \ : \ t \in [0,\min\{\sec\theta,\csc\theta\}]\}$$

with fixed $\theta \in [0,\pi/2]$ the sequence $\{L_n(x(t),(1,1))\}_{n=0}^{\infty}$ has excellent localization when t increases. This localization, however, is not uniform over $\theta \in [0,\pi/2]$ and

even $\{L_n(x,(1,1))\}_{n=0}^\infty$ is not localized at all for $x \in [-1,1]^2$! In order to demonstrate this fact we consider the function $F_n(x_1) = L_n(x_1, -1, 1, 1)$. Straightforward calculations give

$$F_n(x_1) = \frac{\hat{a}(0)}{\pi^2} - \sum_{j=1}^\infty \frac{2}{\pi^2} \hat{a}\left(\frac{j}{n}\right) \sin(j \arccos x_1) \tan\left(\frac{1}{2} \arccos x_1\right).$$

In particular, F_n is an algebraic polynomial of degree at most ℓn, where $\mathrm{supp}\,\hat{a} \subset [0, \ell]$. The above identity implies for its derivative

$$F_n'(1) = \frac{4}{\pi^2} \sum_{j=1}^\infty \frac{j}{2} \hat{a}\left(\frac{j}{n}\right) = \frac{2n^2}{\pi^2} \int_0^\infty t\hat{a}(t)dt + \mathcal{O}(1).$$

Now, because of $\int_0^\infty t\hat{a}(t)dt > 0$, Markov's inequality implies that

$$\|F_n\|_{L^\infty[-1,1]} > const,$$

which validates our assertion.

Another example is the weight $(1 - x_1^2)^{-1/2}$ in $[-1,1]^2$, which is associated with products of Chebyshev and Legendre polynomials. In this case (10.1) holds with

$$\tilde{P}_m(x, y) = \frac{2}{\pi} \sum_{j=0}^m \left(1 - \frac{\delta_{j,0}}{2}\right)\left(m - j + \frac{1}{2}\right) T_j(x_1) T_{m-j}(x_2) P_j(y_1) P_{m-j}(y_2).$$

Now

$$\tilde{P}_m(1, -1, 1, 1) = \frac{2}{\pi} \sum_{j=0}^m \left(1 - \frac{\delta_{j,0}}{2}\right)\left(m - j + \frac{1}{2}\right)(-1)^{m-j} = \frac{(-1)^m}{2\pi}$$

and hence

$$L_n(1, -1, 1, 1) = \frac{\hat{a}(0)}{4\pi} + \mathcal{O}(n^{-1}).$$

As for tensor products of Chebyshev polynomials the localization of the cutoff functions of type (a) is ruled out by this argument, but for those of types (b) and (c) we consider the derivative of the function $F_n(x_1) = L_n(x_1, -1, 1, 1)$. Here we have

$$F_n'(1) = \frac{2}{\pi} \sum_{j=1}^\infty \frac{j}{2}\left(\hat{a}\left(\frac{2j-1}{n}\right) + \hat{a}\left(\frac{2j}{n}\right)\right) = \frac{n^2}{2\pi} \int_0^\infty t\hat{a}(t)dt + \frac{n}{4\pi} \int_0^\infty \hat{a}(t)dt + \mathcal{O}(1)$$

and hence

$$\|F_n\|_{L^\infty[-1,1]} > const.$$

The above facts lead us to the **conclusion** that one cannot expect good localization of kernels of the form (10.1) for tensor product Jacobi polynomials or cross product bases. This kind of bases apparently have completely different nature compared to e.g. orthogonal polynomials on the simplex (§7) and the ball (§6), or multivariate Hermite and Laguerre functions (§8-9) for which the localization principle is valid.

In the case of tensor product Jacobi polynomials truly multivariate cutoff functions need to be employed. This sort of cutoff functions and the associated kernels and needlets will be developed in a follow up paper.

References

[1] G. Brown, F. Dai, Approximation of smooth functions on compact two-point homogeneous spaces, J. Funct. Anal. 220 (2005), 401–423.

[2] J. Dziubański, Triebel-Lizorkin spaces associated with Laguerre and Hermite expansions, Proc. Amer. Math. Soc. 125 (1997), 3547–3554.

[3] J. Dziubański, E. Hernándedez, Band-limited wavelets with subexponential decay, Canad. Math. Bull. 41 (1998), 398–403.

[4] J. Epperson, Hermite and Laguerre wave packet expansions, Studia Math. 126 (1997), 199–217.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions, Vol. II,* McGraw-Hill, New York, 1953.

[6] T. Erdelyi, A, Magnus and P. Nevai, Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994), 602–614.

[7] L. Hörmander, The analysis of linear partial differential operators I, Springer-Verlag, 1983.

[8] K. Ivanov, V. Totik, Fast decreasing polynomials, Constr. Approx. 6 (1990), 1–20.

[9] G. Kerkyacharian, P. Petrushev, D. Picard and Yuan Xu, Decomposition of Triebel-Lizorkin and Besov spaces in the context of Laguerre expansions, J. Funct. Anal. (to appear). (arXiv:0804.4648v1)

[10] T. Koornwinder, Jacobi polynomials. II. An analytic proof of the product formula. SIAM J. Math. Anal. 5 (1974), 125–137.

[11] I. Krasikov, Inequalities for orthonormal Laguerre polynomials, J. Approx. Theory, 144 (2007), 1- 26.

[12] G. Kyriazis, P. Petrushev and Yuan Xu, Jacobi decomposition of weighted Triebel-Lizorkin and Besov spaces, Studia Math. 186 (2008), 161–202.

[13] G. Kyriazis, P. Petrushev and Yuan Xu, Decomposition of weighted Triebel-Lizorkin and Besov spaces on the ball, Proc. London Math. Soc. 97 (2008), 477–513.

[14] Y. Meyer, *Ondelettes et Opérateurs. I*, Hermann, Paris, 1990.

[15] F. J. Narcowich, P. Petrushev and J. D. Ward, Localized tight frames on spheres, SIAM J. Math. Anal. 38 (2006), 574–594.

[16] F. J. Narcowich, P. Petrushev and J. D. Ward, Decomposition of Besov and Triebel-Lizorkin spaces on the sphere, J. Funct. Anal. 238 (2006), 530–564.

[17] P. Petrushev, Yuan Xu, Localized polynomial frames on the interval with Jacobi weights, J. Fourier Anal. and Appl. 11 (2005), 557–575.

[18] P. Petrushev, Yuan Xu, Localized polynomial frames on the ball, Constr. Approx. 27 (2008), 121–148.

[19] P. Petrushev, Yuan Xu, Decomposition of spaces of distributions induced by Hermite expansion, J. Fourier Anal. and Appl. 14 (2008), 372-414.

[20] E. Stein, G. Weiss, *Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, New Jessey, 1971.

[21] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Vol.23, Providence, 4th edition, 1975.

[22] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Princeton Univ. Press, 1993.

[23] Yuan Xu, Summability of Fourier orthogonal series for Jacobi weight on the simplex in $\mathbb{R}^d$, Proc. Amer. Math. Soc. 126 (1998), 3027-3036.

[24] Yuan Xu, Summability of Fourier orthogonal series for Jacobi weight on a ball in $\mathbb{R}^d$, Trans. Amer. Math. Soc. 351 (1999), 2439-2458.

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria
   E-mail address: ivanov@math.sc.edu

Department of Mathematics, University of South Carolina, Columbia, SC 29208, and Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
   E-mail address: pencho@math.sc.edu

Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222.
   E-mail address: yuan@math.uoregon.edu