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A GENERALIZED CURVELET TRANSFORM. APPROXIMATION PROPERTIES

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Abstract. Some modifications are made to the original definitions of the three Curvelet Transforms (continuous, semi-discrete and discrete—see [2], [3] and [4]), which improves and simplifies the expressions of the related Parseval-Plancherel formula and Calderón resolution of the identity. The results presented in this article cast new light on the structure and further properties of curvelet-like schemes of approximation.

The discrete curvelet transform obtained here gives rise to a tight frame for the space of square-integrable functions on the plane. Analysis based on manipulation of the corresponding curvelet coefficients (with respect to this frame) helps measure the regularity of functions in different smoothness spaces. This information is used to offer characterizations of Lipschitz and Besov spaces, as well as approximation spaces for sequences of finite-dimensional linear spaces spanned by curvelets.

1. THE CURVELET TRANSFORM

The basic curvelets proposed by Candès and Donoho in [3] and [4] have Fourier transforms obtained from tensor products of real-valued window functions for amplitude and phase, weighted accordingly with respect to a scaling parameter in order to satisfy Calderón and Parseval-Plancherel integral identities (in a similar fashion to the treatment of wavelets by Daubechies in [6]). But their choice allows only representation of functions for which there exists $\varepsilon > 0$ such that $\widehat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}^2$ with $|\xi| < \varepsilon$. To be able to treat general functions in $L_2(\mathbb{R}^2)$, they introduce in the framework an auxiliary low-frequency radial wavelet together with its shifts (but not its dilations).

We use this tensor product construction of curvelets while modifying slightly the treatment of the shape of the support of their Fourier transforms. We introduce an aspect-ratio weight function, which permits an alternative framework where no auxiliary non-curvelet functions are necessary for decomposition and reconstruction in $L_2(\mathbb{R}^2)$, and simpler expressions are obtained for the corresponding curvelet transforms. Proposition 1.2, its corollary 1.2.1 and theorem 1.1 present improvements to the results of Candés and Donoho in this direction. Also notice how the integration measure in the integral identities presented in the results of this paper $(d\beta\, d\sigma(\theta)\, d\alpha)$ is much simpler than the corresponding in [3, Theorem 1] $(d\beta\, d\sigma(\theta)\, \alpha^{-3} d\alpha)$.

In §1.1 we present this new definition of curvelets, the construction of the corresponding curvelet transform, and we explore some basic properties of both regarding norm estimates and shape considerations. In §1.4 we perform a discretization of this curvelet transform by means of very simple quadrature formulas performed

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evaluating the triple integrals in sequences α_n in $\mathbb{R}$, $\theta_{nk} \in \mathbb{S}^1$, and $\beta_{nk} \in \mathbb{R}^2$. By carefully choosing these sequences, the discretizations are shown to offer tight frames for the space $L_2(\mathbb{R}^2)$. We explore some properties of these particular frames needed in section 2 for approximation purposes.

We point out that the conditions of admissibility for the windows W and V in the construction of the tight frames presented in this dissertation are much simpler than the ones obtained in [4]. In their case, the windows are required to satisfy

$$\sum_{j=-\infty}^{\infty} W(2^j r)^2 = 1, \quad r \in (3/4, 3/2); \qquad \sum_{\ell=-\infty}^{\infty} V(t-\ell)^2 = 1, \quad t \in (-1/2, 1/2).$$

In our case, the windows are required to satisfy, for some $\alpha_0 > 1$ fixed,

$$W(\rho)^2 + W(\alpha_0 \rho)^2 = \frac{1}{\log \alpha_0}, \quad 1/\alpha_0 < \rho \le 1; \qquad V(\omega)^2 + V(\omega-1)^2 = 1.$$

1.1. **The Continuous Curvelet Transform.**

1.1.1. *Definition of Curvelets. Basic Properties.* Define *amplitude windows* as dilations of a common real-valued function $W \in C_c^\infty(0,\infty)$ with $\int_0^\infty W(t)^2 \, \frac{dt}{t} = 1$ and $\operatorname{supp} W \subset [\frac{1}{\alpha_0}, \alpha_0]$ for some $\alpha_0 > 1$: For each $\lambda > 0$, set $W_\lambda(t) = \lambda^{-1/2} W(t/\lambda)$.

Define *phase windows* as functions on the circle $\mathbb{S}^1$ parameterized as dilations of a common real-valued smooth function $V \in C_c^\infty(\mathbb{R})$ with $\|V\|_{L_2(\mathbb{R})} = 1$ and $\operatorname{supp} V \subset [-1,1]$: Given $0 < \lambda < \frac{\pi}{2}$ and $\theta \in \mathbb{S}^1$, let $V_{\lambda,\theta} \colon \mathbb{S}^1 \to \mathbb{R}$ parameterized by $\lambda^{-1/2} V\big(\frac{\omega-\omega_0}{\lambda}\big)$, for $\omega \in [\omega_0 - \pi, \omega_0 + \pi)$ where ω_0 is the only value in $[-\pi, \pi)$ for which $\theta = e^{i\omega_0}$.

We then define the curvelets $\gamma_{\alpha\beta\theta}$ in the frequency domain as follows.

Definition. A curvelet is a complex-valued function $\gamma_{\alpha\beta\theta} \colon \mathbb{R}^2 \to \mathbb{C}$ defined for each *scale* $0 < \alpha < \infty$, *location* $\beta \in \mathbb{R}^2$ and *direction* $\theta \in \mathbb{S}^1$ by its Fourier transform as

$$\widehat{\gamma}_{\alpha\beta\theta}(\xi) = W_\alpha(|\xi|) V_{\tau(\alpha),\theta}(\xi/|\xi|) e^{2\pi i \beta\cdot\xi}, \tag{1}$$

where the *aspect-ratio weight function* $\tau \colon (0,\infty) \to (0, \frac{\pi}{4})$ has an absolute maximum at $m_\tau > 0$ and satisfies:

(i) $\tau|_{(0,m_\tau)}$ is non-decreasing, with $\lim_{\alpha\to 0} \tau(\alpha) = 0$.
(ii) $\tau|_{(m_\tau,\infty)}$ is non-increasing, with $\lim_{\alpha\to\infty} \tau(\alpha) = 0$.

Remark 1.1. The support of the Fourier transform of a curvelet $\gamma_{\alpha\beta\theta}$ is the annular wedge $\big\{\xi \in \mathbb{R}^2 : \alpha_0^{-1}\alpha \le |\xi| \le \alpha_0\alpha, |\arg\xi - \omega_0| < \tau(\alpha)\big\}$; the aspect ratio between the difference of its radii and the angular span is $\big[\big(\frac{\alpha_0^2-1}{\alpha_0}\big)\alpha \colon 2\tau(\alpha)\big]$. The size of these regions is $|\operatorname{supp}\widehat{\gamma}_{\alpha\beta\theta}| = \big(\alpha_0^2 - \frac{1}{\alpha_0^2}\big)\alpha^2\,\tau(\alpha)$.

Notice that, according to this construction, there is an obvious relation among curvelets within the same scaling factor α: $\gamma_{\alpha\beta\theta}(x) = \gamma_{\alpha\mathbf{0}\mathbf{1}}\big(R_\theta(x-\beta)\big)$, where R_θ is the rotation with center the origin sending $\mathbf{1}$ to θ. But unlike wavelets, no relation exists between different scaled basic curvelets $\gamma_{\alpha\mathbf{0}\mathbf{1}}$ and $\gamma_{\alpha'\mathbf{0}\mathbf{1}}$ a priori.

Some useful bounds and basic properties of the curvelets $\gamma_{\alpha\beta\theta}$ are stated below.

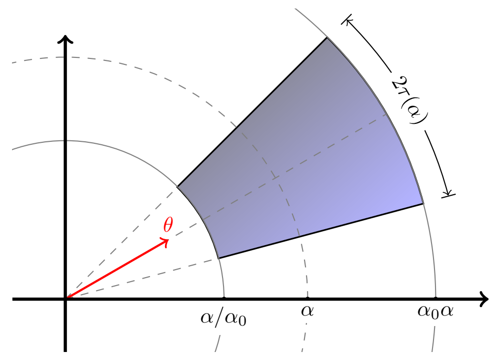

FIGURE 1. The support of the Fourier transform of a curvelet $\gamma_{\alpha\beta\theta}$ for $\alpha > 0$, $\beta \in \mathbb{R}^2$ and $\theta \in \mathbb{S}^1$.

1.1.2. *Norm estimates of curvelets.*

Lemma 1.1. *Given admissible phase and amplitude windows V, W (respectively), the following estimate holds for the frequency-domain k-th Laplacian of curvelets, for all $k \geq 0$:*

$$\|\Delta^k \widehat{\gamma}_{\alpha\mathbf{01}}\|_{L_\infty(\mathbb{R}^2)} \leq C_{k,V,W} \left(\alpha\tau(\alpha)\right)^{-2k-1/2}, \tag{2}$$

for some constant $C_{k,V,W} > 0$.

Proof. Estimate (2) follows trivially for $k = 0$ from the definition of curvelets, since for $\rho > 0$, $0 \leq \omega < 2\pi$,

$$\widehat{\gamma}_{\alpha\mathbf{01}}\left(\rho e^{i\omega}\right) = \alpha^{-1/2}\tau(\alpha)^{-1/2} W(\rho/\alpha) V\left(\omega/\tau(\alpha)\right).$$

For $k > 0$, recall that the expression in polar coordinates of the k-th Laplacian of differentiable functions $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\Delta^k \varphi = \sum_{j=1}^{2k} \sum_{m=0}^{\lfloor k+\frac{1-j}{2}\rfloor} \frac{c_{m,j}}{\rho^{j+2m-1}} \frac{\partial^{2k-j+1}\varphi}{\partial\rho^{2k-j+1-2m}\partial\omega^{2m}},$$

for some real values $c_{m,j}$.

For the Fourier transform of curvelets $\gamma_{\alpha\mathbf{01}}$, it is

$$\begin{aligned}
&\frac{\partial^{2k-j+1}\widehat{\gamma}_{\alpha\mathbf{01}}\left(\rho e^{i\omega}\right)}{\partial\rho^{2k-j+1-2m}\partial\omega^{2m}} \\
&\quad= \left(\alpha\tau(\alpha)\right)^{-1/2} \alpha^{2m-1+j-2k} \frac{\partial^{2k-j+1-2m} W\left(\frac{\rho}{\alpha}\right)}{\partial\rho^{2k-j+1-2m}} \tau(\alpha)^{-2m} \frac{\partial^{2m} V\left(\frac{\omega}{\tau(\alpha)}\right)}{\partial\omega^{2m}},
\end{aligned}$$

and thus, the k-th Laplacians $\Delta^k\widehat{\gamma}_{\alpha\mathbf{01}}$ are given by the expression:

$$\begin{aligned}
&\left(\alpha\tau(\alpha)\right)^{1/2} \Delta^k \widehat{\gamma}_{\alpha\mathbf{01}} \\
&= \sum_{j=1}^{2k} \sum_{m=0}^{\lfloor k+\frac{1-j}{2}\rfloor} \tfrac{c_{m,j}}{\rho^{j+2m-1}} \alpha^{2m-1+j-2k} W^{(2k-j+1-2m)}\left(\tfrac{\rho}{\alpha}\right) \tau(\alpha)^{-2m} V^{(2m)}\left(\tfrac{\omega}{\tau(\alpha)}\right) \\
&= \alpha^{-2k} \sum_{j=1}^{2k} \sum_{m=0}^{\lfloor k+\frac{1-j}{2}\rfloor} c_{m,j} \tau(\alpha)^{-2m} \left[\left(\tfrac{\alpha}{\rho}\right)^{j+2m-1} W^{(2k-j+1-2m)}\left(\tfrac{\rho}{\alpha}\right)\right] V^{(2m)}\left(\tfrac{\omega}{\tau(\alpha)}\right).
\end{aligned}$$

As $\tau(\alpha) < 1$ for all $\alpha > 0$, it is $\tau(\alpha)^{-2m} \leq \tau(\alpha)^{-2k}$ for all $m = 0, \dots, \lfloor k + \frac{1-j}{2} \rfloor$; $j = 1, \dots, 2k$; and thus the following estimate holds:

$$|\Delta^k \widehat{\gamma}_{\alpha\mathbf{01}}(\xi)| \leq \big(\alpha\tau(\alpha)\big)^{-2k-1/2} \underbrace{\big(\max|c_{m,j}|\big) \sum_{j=1}^{2k} \sum_{m=0}^{\lfloor k+\frac{1-j}{2} \rfloor} \sup_{t>0} \left| \tfrac{W^{(2k-j+1-2m)}(t)}{t^{j+2m-1}} \right| \|V^{(2m)}\|_{L_\infty(\mathbb{R})}}_{C_{k,V,W}},$$

which gives the statement. □

Lemma 1.2. *Curvelets belong to the Schwartz class $\mathcal{S}(\mathbb{R}^2)$. In particular, for all $\alpha > 0$, $\beta \in \mathbb{R}^2$, $\theta \in \mathbb{S}^2$ and $k \geq 0$,*

$$|x - \beta|^{2k} \, |\gamma_{\alpha\beta\theta}(x)| \leq \big(\alpha_0^2 - \tfrac{1}{\alpha_0^2}\big) \, C_{k,V,W} \, \alpha^{3/2-2k} \tau(\alpha)^{1/2-2k}. \tag{3}$$

Proof. Since their Fourier transforms are $C_c^\infty(\mathbb{R}^2)$ functions, curvelets belong trivially to the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. As the inverse Fourier transform of each partial derivative satisfies $\big(\frac{\partial^{2k}\widehat{\gamma}_{\alpha\mathbf{01}}}{\partial\xi_1^{2j}\partial\xi_2^{2(k-j)}}\big)^\vee(x) = (2\pi i)^{2k} x_1^{2j} x_2^{2(k-j)} \gamma_{\alpha\mathbf{01}}(x)$, the following identity for the inverse Fourier transform of the k-th Laplacian holds:

$$\big(\Delta^k \widehat{\gamma}_{\alpha\mathbf{01}}\big)^\vee(x) = \sum_{j=0}^{k} (2\pi i)^{2k} \tbinom{k}{j} x_1^{2j} x_2^{2(k-j)} \gamma_{\alpha\mathbf{01}}(x) = (-1)^k (2\pi)^{2k} |x|^{2k} \gamma_{\alpha\mathbf{01}}(x).$$

Thus,

$$|x - \beta|^{2k} \, \big|\gamma_{\alpha\mathbf{01}}\big(R_\theta(x-\beta)\big)\big| \leq (2\pi)^{-2k} \int_{\mathbb{R}^2} |\Delta^k \widehat{\gamma}_{\alpha\mathbf{01}}(\xi)| \, d\xi.$$

Upper bounds for the integral on the right hand side of the estimate above may be found using the size of the support of the Fourier transform of curvelets, $|\operatorname{supp} \widehat{\gamma}_{\alpha\beta\theta}|$ (see remark 1.1), and the L_∞ norm of the k-th Laplacian:

$$|x - \beta|^{2k} \, |\gamma_{\alpha\beta\theta}(x)| \leq (2\pi)^{-2k} \big(\alpha_0^2 - \tfrac{1}{\alpha_0^2}\big) \, \alpha^2 \tau(\alpha) \, \|\Delta^k \widehat{\gamma}_{\alpha\mathbf{01}}\|_{L_\infty(\mathbb{R}^2)}.$$

Using the estimate (2) on the inequality above yields (3). □

Lemma 1.3. *Curvelets belong to the space $L_p(\mathbb{R}^2)$ for all $0 < p \leq \infty$. An estimate for the L_p–(quasi)norms is given by*

$$\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} \leq C(V, W, \alpha_0, p) \, \alpha^{3/2-2/p} \tau(\alpha)^{1/2-2/p}. \tag{4}$$

Proof. For all $x \in \mathbb{R}^2$, $\alpha > 0$, $\beta \in \mathbb{R}^2$ and $\theta \in \mathbb{S}^1$,

$$\begin{aligned} |\gamma_{\alpha\beta\theta}(x)| &\leq \int_{\mathbb{R}^2} |\widehat{\gamma}_{\alpha\beta\theta}(\xi)| \, d\xi \leq |\operatorname{supp} \widehat{\gamma}_{\alpha\beta\theta}| \|\widehat{\gamma}_{\alpha\beta\theta}\|_{L_\infty(\mathbb{R}^2)} \\ &\leq \big(\alpha_0^2 - \tfrac{1}{\alpha_0^2}\big) \|W\|_{L_\infty(0,\infty)} \, \|V\|_{L_\infty(\mathbb{R})} \alpha^{3/2} \tau(\alpha)^{1/2}. \end{aligned}$$

This gives (4) for $p = \infty$. For $0 < p \leq 1$, given $\alpha > 0$, set $\zeta(\alpha) = \alpha^{-1}\tau(\alpha)^{-1}$, and consider the following decomposition:

$$\int_{\mathbb{R}^2} |\gamma_{\alpha\mathbf{01}}(x)|^p \, dx = \underbrace{\int_{B_2(\mathbf{0},\zeta(\alpha))} |\gamma_{\alpha\mathbf{01}}(x)|^p \, dx}_{\mathbf{I}} + \underbrace{\int_{\mathbb{R}^2 \setminus B_2(\mathbf{0},\zeta(\alpha))} |\gamma_{\alpha\mathbf{01}}(x)|^p \, dx}_{\mathbf{II}}.$$

Term **I** above is estimated by means of the essential supremum of $|\gamma_{\alpha\mathbf{0}\mathbf{1}}|$ in the integration domain, $B_2\big(\mathbf{0},\zeta(\alpha)\big)$:

$$\begin{aligned}\mathbf{I} &= \int_{B_2(\mathbf{0},\zeta(\alpha))} |\gamma_{\alpha\mathbf{0}\mathbf{1}}(x)|^p\,dx \le |B_2\big(\mathbf{0},\zeta(\alpha)\big)|\,\|\gamma_{\alpha\mathbf{0}\mathbf{1}}\|^p_{L_\infty(\mathbb{R}^2)}\\ &\le \pi\zeta(\alpha)^2\big(\alpha_0^2-\tfrac{1}{\alpha_0^2}\big)^p\|W\|^p_{L_\infty(0,\infty)}\|V\|^p_{L_\infty(\mathbb{R})}\alpha^{3p/2}\tau(\alpha)^{p/2} \quad \big(\text{by (3) with } k=0\big)\\ &= \pi\big(\alpha_0^2-\tfrac{1}{\alpha_0^2}\big)^p\|W\|^p_{L_\infty(0,\infty)}\|V\|^p_{L_\infty(\mathbb{R})}\alpha^{3p/2-2}\tau(\alpha)^{p/2-2}.\end{aligned}$$

To estimate **II**, set $k(p)=\lceil 1/p\rceil+1$, and let

$$\widetilde{C}_{p,V,W,\alpha_0} = (2\pi)^{-2k(p)}\big(\alpha_0^2-\alpha_0^{-2}\big)C_{k(p),V,W}$$

with $C_{k(p),V,W}$ the constant from Lemma 1.2 for $k=k(p)$.

$$\begin{aligned}\mathbf{II} &= \int_{\mathbb{R}^2\setminus B_2(\mathbf{0},\zeta(\alpha))} |\gamma_{\alpha\mathbf{0}\mathbf{1}}(x)|^p\,dx\\ &\le \widetilde{C}^p_{p,V,W,\alpha_0}\alpha^{3p/2-2k(p)p}\tau(\alpha)^{p/2-2k(p)p}\int_{\mathbb{R}^2\setminus B_2(\mathbf{0},\zeta(\alpha))}\frac{dx}{|x|^{2k(p)p}}\\ &\quad \big(\text{by (3) with } k=k(p)\big)\\ &= \widetilde{C}^p_{p,V,W,\alpha_0}\alpha^{3p/2-2k(p)p}\tau(\alpha)^{p/2-2k(p)p}\frac{2\pi\big(\alpha\tau(\alpha)\big)^{2k(p)p-2}}{2k(p)p-2}.\end{aligned}$$

This last step is true since $1-k(p)p<0$. Thus,

$$\left(\int_{\mathbb{R}^2}|\gamma_{\alpha\mathbf{0}\mathbf{1}}(x)|^p\,dx\right)^{1/p} \le C(V,W,\alpha_0,p)\alpha^{3/2-2/p}\tau(\alpha)^{1/2-2/p},$$

where

$$C(V,W,\alpha_0,p) = 2\big(\alpha_0^2-\tfrac{1}{\alpha_0^2}\big)\max\left\{\pi^{1/p}\|W\|_{L_\infty(0,\infty)}\,\|V\|_{L_\infty(\mathbb{R})},\ \frac{(2\pi)^{1/p-2k(p)}C_{k(p),V,W}}{\big(2k(p)p-2\big)^{1/p}}\right\}.$$

After appropriate shift and rotation, estimate (4) is obtained for all $0<p\le 1$, $\alpha>0$, $\beta\in\mathbb{R}^2$ and $\theta\in\mathbb{S}^1$.

The cases $1<p<\infty$ of (4) follow from Hölder's inequality:

$$\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} \le \|\gamma_{\alpha\beta\theta}\|^{1/p}_{L_1(\mathbb{R}^2)}\,\|\gamma_{\alpha\beta\theta}\|^{1-1/p}_{L_\infty(\mathbb{R}^2)} \le C(V,W,\alpha_0,p)\,\alpha^{3/2-2/p}\tau(\alpha)^{1/2-2/p}.$$

□

Other estimates for the L_p–(quasi)norms are possible by direct computation, and posterior interpolation between the new estimates. The following are some useful examples.

Lemma 1.4. *For $2\le p\le\infty$,*

$$\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} \le C(V,W,\alpha_0,p)\,\alpha^{3/2-2/p}\tau(\alpha)^{1/2-1/p}. \tag{5}$$

Proof. Notice first that we have an exact value for the L_2–norm of curvelets:

$$\|\gamma_{\alpha\beta\theta}\|_{L_2(\mathbb{R}^2)} = C(V,W)\alpha^{1/2}, \tag{6}$$

with $C(V,W) = \|V\|_{L_2(\mathbb{R})}\Big(\int_{\mathbb{R}} r^2 W(r)^2\,dr\Big)^{1/2}$. Indeed:

$$\begin{aligned}
\|\gamma_{\alpha\beta\theta}\|^2_{L_2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\gamma_{\alpha\beta\theta}(x)|^2\,dx = \int_{\mathbb{R}^2} |\widehat{\gamma}_{\alpha\beta\theta}(\xi)|^2\,d\xi \qquad \big(\text{by Plancherel's Theorem }\big)\\
&= \int_0^\infty \int_{\mathbb{S}^1} \tfrac{1}{\alpha} W\big(\tfrac{\rho}{\alpha}\big)^2\, \tfrac{1}{\tau(\alpha)} V\big(\tfrac{\arg\omega - \arg\theta}{\tau(\alpha)}\big)^2\,\rho\,d\rho\,d\sigma(\omega)\\
&= \alpha \int_0^\infty \tfrac{\rho}{\alpha} W\big(\tfrac{\rho}{\alpha}\big)^2\,\tfrac{d\rho}{\alpha} \int_{\arg\theta-\pi}^{\arg\theta+\pi} V\big(\tfrac{s-\arg\theta}{\tau(\alpha)}\big)^2\,\tfrac{ds}{\tau(\alpha)}\\
&= \alpha\Big(\int_{\mathbb{R}} r^2\,W(r)^2\,dr\Big)\,\|V\|^2_{L_2(\mathbb{R})}. \qquad \big(\text{with } r=\tfrac{\rho}{\alpha},\ dr=\tfrac{d\rho}{\alpha}\big)
\end{aligned}$$

This gives equation (5) for $p=2$, and further use of Hölder's inequality together with the L_∞–norm obtained in (4), gives (5) for all $2<p<\infty$:

$$\begin{aligned}
\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} &\le \|\gamma_{\alpha\beta\theta}\|^{2/p}_{L_2(\mathbb{R}^2)}\|\gamma_{\alpha\beta\theta}\|^{1-2/p}_{L_\infty(\mathbb{R}^2)}\\
&\le C(V,W,p,\alpha_0)(\alpha^{1/2})^{2/p}\big(\alpha^{3/2}\tau(\alpha)^{1/2}\big)^{1-2/p} \qquad \square
\end{aligned}$$

Lemma 1.5. *For $1\le p\le 2$,*

$$\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} \le C(V,W,\alpha_0,p)\alpha^{3/2-2/p}\tau(\alpha)^{3/2-3/p}. \tag{7}$$

Proof. This is a direct consequence of Hölder's inequality together with estimates (6) and (4) for $p=1$:

$$\begin{aligned}
\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} &\le \|\gamma_{\alpha\beta\theta}\|^{2/p-1}_{L_1(\mathbb{R}^2)}\|\gamma_{\alpha\beta\theta}\|^{2-2/p}_{L_2(\mathbb{R}^2)}\\
&\le C(V,W,\alpha_0,p)\,\big(\alpha^{-1/2}\tau(\alpha)^{-3/2}\big)^{2/p-1}(\alpha^{1/2})^{2-2/p}. \qquad \square
\end{aligned}$$

Lemma 1.6. *For $1\le p\le 2$, and all $m\in\mathbb{N}$,*

$$\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} \le C(m,V,W,\alpha_0,p)\,\alpha^{2/p-1/2}\tau(\alpha)^{1/p-1/2}\big(1+\alpha^{-2m}\tau(\alpha)^{-2m}\big)^{2/p-1}. \tag{8}$$

Proof. Using equation (3) twice (one with $k=0$, one with $k=m$), it is

$$|\gamma_{\alpha\beta\theta}(x)| \le C(m,V,W,\alpha_0)\frac{\alpha^{3/2}\tau(\alpha)^{1/2}\big(1+\alpha^{-2m}\tau(\alpha)^{-2m}\big)}{1+|x-\beta|^{2m}}.$$

Integrating the previous expression, the following bound for the L_1–norm of curvelets is obtained:

$$\begin{aligned}
&\|\gamma_{\alpha\beta\theta}\|_{L_1(\mathbb{R}^2)}\\
&\le C(m,V,W,\alpha_0)\,\underbrace{\Big(\int_{\mathbb{R}^2}\frac{dx}{1+|x-\beta|^{2m}}\Big)}_{\frac{\pi^2}{m}\csc\left(\frac{\pi}{m}\right)}\,\alpha^{3/2}\tau(\alpha)^{1/2}\big(1+\alpha^{-2m}\tau(\alpha)^{-2m}\big).
\end{aligned}$$

This gives (8) for $p=1$. Further interpolation between (6) and this estimate gives (8) for $1<p<2$:

$$\begin{aligned}
&\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} \le \|\gamma_{\alpha\beta\theta}\|^{2/p-1}_{L_1(\mathbb{R}^2)}\,\|\gamma_{\alpha\beta\theta}\|^{2-2/p}_{L_2(\mathbb{R}^2)}\\
&\qquad\le C(m,V,W,\alpha_0,p)\Big(\alpha^{3/2}\tau(\alpha)^{1/2}\big(1+\alpha^{-2m}\tau(\alpha)^{-2m}\big)\Big)^{2/p-1}(\alpha^{1/2})^{2-2/p}. \ \square
\end{aligned}$$

Lemma 1.7. *For $1 \le p \le \infty$, and all $m \in \mathbb{N}$,*

$$\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} \le C(m,V,W,\alpha_0,p)\,\alpha^{3/2}\tau(\alpha)^{1/2}\big(1+\alpha^{-2m}\tau(\alpha)^{-2m}\big)^{1/p}. \tag{9}$$

Proof. Interpolation between (8) for $p=1$, $m \in \mathbb{N}$, and (4) for $p=\infty$ gives estimate (9) directly:

$$\begin{aligned}\|\gamma_{\alpha\beta\theta}\|_{L_p(\mathbb{R}^2)} &\le \|\gamma_{\alpha\beta\theta}\|_{L_1(\mathbb{R}^2)}^{1/p}\|\gamma_{\alpha\beta\theta}\|_{L_\infty(\mathbb{R}^2)}^{1-1/p}\\ &\le C(m,V,W,\alpha_0,p)\Big(\alpha^{3/2}\tau(\alpha)^{1/2}\big(1+\alpha^{-2m}\tau(\alpha)^{-2m}\big)\Big)^{1/p}\big(\alpha^{3/2}\tau(\alpha)^{1/2}\big)^{1-1/p}. \ \square\end{aligned}$$

1.1.3. *Shape of curvelets.* The modulus of the curvelet $\gamma_{\alpha\beta\theta}$ is a smooth function in $\mathbb{R}^2$ with graph presenting a "plateau" effect: its mass is concentrated in a region around the location β. This region appears to the naked eye as the interior of an ellipse with axes being both θ and $\theta^\perp$, and eccentricity proportional to $\tau(\alpha)$. Outside of this region, the graph decreases to zero rapidly. In order to explain this phenomenon, the use of differential geometry is needed.

Lemma 1.8. *For any $\alpha > 0$, $\beta \in \mathbb{R}^2$ and $\theta \in \mathbb{S}^1$, the surface given by*

$$\boldsymbol{\Gamma}_{\alpha\beta\theta} \colon \mathbb{R}^2 \ni (u,v) \mapsto \big(u,v,|\gamma_{\alpha\beta\theta}(u,v)|^2\big) \in \mathbb{R}^3,$$

is regular at the point $\big(\beta,|\gamma_{\alpha\beta\theta}(\beta)|^2\big)$, its tangent plane being horizontal, and the First Fundamental Form being the identity matrix. The coefficients of the Second Fundamental Form are given by

$$\begin{aligned}
\boldsymbol{e} &= \frac{8\pi^2\alpha^5}{\tau(\alpha)}\bigg[\Big\{\int_0^\infty r^2\,W(r)\,dr\Big\}^2\Big\{\int_{-\pi}^{\pi}\cos\omega\,V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\Big\}^2\\
&\qquad -\int_0^\infty r\,W(r)\,dr\int_{-\pi}^{\pi}V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\int_0^\infty r^3\,W(r)\,dr\int_{-\pi}^{\pi}\cos^2\omega\,V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\bigg],\\
\boldsymbol{f} &= -\frac{8\pi^2\alpha^5}{\tau(\alpha)}\bigg[\Big\{\int_0^\infty r^2\,W(r)\,dr\Big\}^2\int_{-\pi}^{\pi}\sin\omega\,V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\int_{-\pi}^{\pi}\cos\omega\,V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\\
&\qquad +\int_0^\infty r\,W(r)\,dr\int_{-\pi}^{\pi}V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\int_0^\infty r^3\,W(r)\,dr\int_{-\pi}^{\pi}\sin\omega\,\cos\omega\,V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\bigg],\\
\boldsymbol{g} &= \frac{8\pi^2\alpha^5}{\tau(\alpha)}\bigg[\Big\{\int_0^\infty r^2\,W(r)\,dr\Big\}^2\Big\{\int_{-\pi}^{\pi}\sin\omega\,V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\Big\}^2\\
&\qquad -\int_0^\infty r\,W(r)\,dr\int_{-\pi}^{\pi}V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\int_0^\infty r^3\,W(r)\,dr\int_{-\pi}^{\pi}\sin^2\omega\,V\big(\tfrac{\omega}{\tau(\alpha)}\big)\,d\omega\bigg].
\end{aligned}$$

Proof. For simplicity, assume $\beta = \mathbf{0}$ and $\theta = \mathbf{1}$; the general case is obtained from this after proper shift and rotation. Notice that

$$|\gamma_{\alpha\mathbf{01}}(u,v)|^2 = \big(\Re\gamma_{\alpha\mathbf{01}}(u,v)\big)^2 + \big(\Im\gamma_{\alpha\mathbf{01}}(u,v)\big)^2,$$

where the real and imaginary parts of the curvelet are given by

$$\begin{aligned}
\Re\gamma_{\alpha\mathbf{01}}(u,v) &= \int_{\mathbb{R}^2}\tfrac{1}{\alpha^{1/2}}\,W\big(\tfrac{|\xi|}{\alpha}\big)\,\tfrac{1}{\tau(\alpha)^{1/2}}\,V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\cos\big(2\pi(u,v)\cdot\xi\big)\,d\xi,\\
\Im\gamma_{\alpha\mathbf{01}}(u,v) &= \int_{\mathbb{R}^2}\tfrac{1}{\alpha^{1/2}}\,W\big(\tfrac{|\xi|}{\alpha}\big)\,\tfrac{1}{\tau(\alpha)^{1/2}}\,V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\sin\big(2\pi(u,v)\cdot\xi\big)\,d\xi.
\end{aligned}$$

In particular,

$$\begin{aligned}
\frac{\partial \Re\gamma_{\alpha\mathbf{01}}}{\partial u}(u,v) &= -\tfrac{2\pi}{\alpha^{1/2}\tau(\alpha)^{1/2}} \int_{\mathbb{R}^2} \xi_1 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, \sin\big(2\pi(u,v)\cdot\xi\big)\, d\xi,\\
\frac{\partial \Re\gamma_{\alpha\mathbf{01}}}{\partial v}(u,v) &= -\tfrac{2\pi}{\alpha^{1/2}\tau(\alpha)^{1/2}} \int_{\mathbb{R}^2} \xi_2 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, \sin\big(2\pi(u,v)\cdot\xi\big)\, d\xi,\\
\frac{\partial \Im\gamma_{\alpha\mathbf{01}}}{\partial u}(u,v) &= \tfrac{2\pi}{\alpha^{1/2}\tau(\alpha)^{1/2}} \int_{\mathbb{R}^2} \xi_1 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, \cos\big(2\pi(u,v)\cdot\xi\big)\, d\xi,\\
\frac{\partial \Im\gamma_{\alpha\mathbf{01}}}{\partial v}(u,v) &= \tfrac{2\pi}{\alpha^{1/2}\tau(\alpha)^{1/2}} \int_{\mathbb{R}^2} \xi_2 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, \cos\big(2\pi(u,v)\cdot\xi\big)\, d\xi,
\end{aligned}$$

and furthermore,

$$\begin{aligned}
\frac{\partial|\gamma_{\alpha\mathbf{01}}|^2}{\partial u}(u,v) &= 2\,\Re\gamma_{\alpha\mathbf{01}}(u,v)\,\frac{\partial\Re\gamma_{\alpha\mathbf{01}}}{\partial u}(u,v) + 2\,\Im\gamma_{\alpha\mathbf{01}}(u,v)\,\frac{\partial\Im\gamma_{\alpha\mathbf{01}}}{\partial u}(u,v)\\
&= \frac{4\pi}{\alpha\tau(\alpha)}\bigg[-\int_{\mathbb{R}^2} W\big(\tfrac{|\zeta|}{\alpha}\big)\, V\big(\tfrac{\arg\zeta}{\tau(\alpha)}\big) \cos\big(2\pi(u,v)\cdot\zeta\big)\, d\zeta\\
&\qquad\qquad \int_{\mathbb{R}^2} \xi_1 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big) \sin\big(2\pi(u,v)\cdot\xi\big)\, d\xi\\
&\qquad + \int_{\mathbb{R}^2} W\big(\tfrac{|\zeta|}{\alpha}\big)\, V\big(\tfrac{\arg\zeta}{\tau(\alpha)}\big) \sin\big(2\pi(u,v)\cdot\zeta\big)\, d\zeta\\
&\qquad\qquad \int_{\mathbb{R}^2} \xi_1 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big) \cos\big(2\pi(u,v)\cdot\xi\big)\, d\xi\bigg],\\
\frac{\partial|\gamma_{\alpha\mathbf{01}}|^2}{\partial v}(u,v) &= \frac{4\pi}{\alpha\tau(\alpha)}\bigg[-\int_{\mathbb{R}^2} W\big(\tfrac{|\zeta|}{\alpha}\big)\, V\big(\tfrac{\arg\zeta}{\tau(\alpha)}\big) \cos\big(2\pi(u,v)\cdot\zeta\big)\, d\zeta\\
&\qquad\qquad \int_{\mathbb{R}^2} \xi_2 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big) \sin\big(2\pi(u,v)\cdot\xi\big)\, d\xi\\
&\qquad + \int_{\mathbb{R}^2} W\big(\tfrac{|\zeta|}{\alpha}\big)\, V\big(\tfrac{\arg\zeta}{\tau(\alpha)}\big) \sin\big(2\pi(u,v)\cdot\zeta\big)\, d\zeta\\
&\qquad\qquad \int_{\mathbb{R}^2} \xi_2 W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big) \cos\big(2\pi(u,v)\cdot\xi\big)\, d\xi\bigg].
\end{aligned}$$

At the origin, $(u,v)=\mathbf{0}$, it is

$$\frac{\partial|\gamma_{\alpha\mathbf{01}}|^2}{\partial u}(0,0) = \frac{\partial|\gamma_{\alpha\mathbf{01}}|^2}{\partial v}(0,0) = 0.$$

So at $\big(0,0,|\gamma_{\alpha\beta\theta}(0,0)|^2\big)$ the tangent plane is spanned by both $(1,0,\frac{\partial|\gamma_{\alpha\mathbf{01}}|^2}{\partial u}\big|_{\mathbf{0}}) = (1,0,0)$ and $(0,1,\frac{\partial|\gamma_{\alpha\mathbf{01}}|^2}{\partial v}\big|_{\mathbf{0}}) = (0,1,0)$. The coefficients of the First Fundamental form are,

$$\begin{aligned}
E &= \big|\tfrac{\partial\mathbf{\Gamma}_{\alpha\mathbf{01}}}{\partial u}(0,0)\big|^2 = \big|\big(1,0,\tfrac{\partial|\gamma_{\alpha\beta\theta}|^2}{\partial u}(0,0)\big)\big|^2 = 1,\\
F &= \tfrac{\partial\mathbf{\Gamma}_{\alpha\mathbf{01}}}{\partial u}(0,0)\cdot\tfrac{\partial\mathbf{\Gamma}_{\alpha\mathbf{01}}}{\partial v}(0,0) = \big(1,0,\tfrac{\partial|\gamma_{\alpha\beta\theta}|^2}{\partial u}(0,0)\big)\cdot\big(0,1,\tfrac{\partial|\gamma_{\alpha\beta\theta}|^2}{\partial v}(0,0)\big) = 0,\\
G &= \big|\tfrac{\partial\mathbf{\Gamma}_{\alpha\mathbf{01}}}{\partial v}(0,0)\big|^2 = \big|\big(0,1,\tfrac{\partial|\gamma_{\alpha\beta\theta}|^2}{\partial v}(0,0)\big)\big|^2 = 1.
\end{aligned}$$

The coefficients of the Second Fundamental Form at $\big(0, |\gamma_{\alpha\mathbf{01}}(0,0)|^2\big)$ are,

$$
\begin{aligned}
\boldsymbol{e} &= \frac{1}{\sqrt{EG-F^2}} \det\Big(\tfrac{\partial^2 \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial u^2}\Big|_{\mathbf{0}} \, \tfrac{\partial \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial u}\Big|_{\mathbf{0}} \, \tfrac{\partial \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial v}\Big|_{\mathbf{0}} \Big) = \begin{vmatrix} 0 & 0 & \frac{\partial^2 |\gamma_{\alpha\mathbf{01}}|^2}{\partial u^2}(0,0) \\ 1 & 0 & \frac{\partial |\gamma_{\alpha\mathbf{01}}|^2}{\partial u}(0,0) \\ 0 & 1 & \frac{\partial |\gamma_{\alpha\mathbf{01}}|^2}{\partial v}(0,0) \end{vmatrix} \\
&= 2\left(\tfrac{\partial}{\partial u} \Re\gamma_{\alpha\mathbf{01}}(0,0)\right)^2 + 2\,\Re\gamma_{\alpha\mathbf{01}}(0,0)\, \tfrac{\partial^2}{\partial u^2} \Re\gamma_{\alpha\mathbf{01}}(0,0) \\
&\quad + 2\left(\tfrac{\partial}{\partial u} \Im\gamma_{\alpha\mathbf{01}}(0,0)\right)^2 + 2\,\Im\gamma_{\alpha\mathbf{01}}(0,0)\, \tfrac{\partial^2}{\partial u^2} \Im\gamma_{\alpha\mathbf{01}}(0,0) \\
&= \frac{8\pi^2}{\alpha\tau(\alpha)} \Bigg[-\int_{\mathbb{R}^2} W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \int_{\mathbb{R}^2} \xi_1^2\, W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \\
&\qquad\qquad + \Big\{ \int_{\mathbb{R}^2} \xi_1\, W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \Big\}^2 \Bigg] \\
&= \frac{8\pi^2}{\alpha\tau(\alpha)} \Bigg[-\int_0^\infty \int_{-\pi}^{\pi} \rho\, W\big(\tfrac{\rho}{\alpha}\big)\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega\, d\rho \\
&\qquad\qquad \int_0^\infty \int_{-\pi}^{\pi} \rho^3 \cos^2\omega\, W\big(\tfrac{\rho}{\alpha}\big)\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega\, d\rho \\
&\qquad\qquad + \Big\{ \int_0^\infty \int_{-\pi}^{\pi} \rho^2 \cos\omega\, W\big(\tfrac{\rho}{\alpha}\big)\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega\, d\rho \Big\}^2 \Bigg] \\
&= \frac{8\pi^2\alpha^5}{\tau(\alpha)} \Bigg[-\int_0^\infty r\, W(r)\, dr \int_{-\pi}^{\pi} V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \\
&\qquad\qquad \int_0^\infty r^3\, W(r)\, dr \int_{-\pi}^{\pi} \cos^2\omega\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \\
&\qquad\qquad + \Big\{ \int_0^\infty r^2\, W(r)\, dr \Big\}^2 \Big\{ \int_{-\pi}^{\pi} \cos\omega\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \Big\}^2 \Bigg], \\
\boldsymbol{f} &= \frac{1}{\sqrt{EG-F^2}} \det\Big(\tfrac{\partial^2 \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial u \partial v}\Big|_{\mathbf{0}} \, \tfrac{\partial \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial u}\Big|_{\mathbf{0}} \, \tfrac{\partial \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial v}\Big|_{\mathbf{0}} \Big) = \frac{\partial^2 |\gamma_{\alpha\mathbf{01}}|^2}{\partial u\, \partial v}(0,0) \\
&= 2\, \tfrac{\partial}{\partial u} \Re\gamma_{\alpha\mathbf{01}}(0,0)\, \tfrac{\partial}{\partial v} \Re\gamma_{\alpha\mathbf{01}}(0,0) + 2\,\Re\gamma_{\alpha\mathbf{01}}(0,0)\, \tfrac{\partial^2}{\partial u\, \partial v} \Re\gamma_{\alpha\mathbf{01}}(0,0) \\
&\quad + 2\, \tfrac{\partial}{\partial u} \Im\gamma_{\alpha\mathbf{01}}(0,0)\, \tfrac{\partial}{\partial v} \Im\gamma_{\alpha\mathbf{01}}(0,0) + 2\,\Im\gamma_{\alpha\mathbf{01}}(0,0)\, \tfrac{\partial^2}{\partial u\, \partial v} \Im\gamma_{\alpha\mathbf{01}}(0,0) \\
&= -\frac{8\pi^2}{\alpha\tau(\alpha)} \Bigg[\int_{\mathbb{R}^2} W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \int_{\mathbb{R}^2} \xi_1 \xi_2\, W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \\
&\qquad\qquad + \int_{\mathbb{R}^2} \xi_1\, W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \int_{\mathbb{R}^2} \xi_2\, W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \Bigg] \\
&= -\frac{8\pi^2\alpha^5}{\tau(\alpha)} \Bigg[\int_0^\infty r\, W(r)\, dr \int_{-\pi}^{\pi} V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \\
&\qquad\qquad \int_0^\infty r^3\, W(r)\, dr \int_{-\pi}^{\pi} \sin\omega\, \cos\omega\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \\
&\qquad\qquad + \Big\{ \int_0^\infty r^2\, W(r)\, dr \Big\}^2 \int_{-\pi}^{\pi} \sin\omega\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \int_{-\pi}^{\pi} \cos\omega\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \Bigg], \\
\boldsymbol{g} &= \frac{1}{\sqrt{EG-F^2}} \det\Big(\tfrac{\partial^2 \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial v^2}\Big|_{\mathbf{0}} \, \tfrac{\partial \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial u}\Big|_{\mathbf{0}} \, \tfrac{\partial \boldsymbol{\Gamma}_{\alpha\mathbf{01}}}{\partial v}\Big|_{\mathbf{0}} \Big) = \frac{\partial^2 |\gamma_{\alpha\mathbf{01}}|^2}{\partial v^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{8\pi^2}{\alpha\tau(\alpha)} \Big[-\int_{\mathbb{R}^2} W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \int_{\mathbb{R}^2} \xi_2^2\, W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \\
&\qquad + \Big\{ \int_{\mathbb{R}^2} \xi_2\, W\big(\tfrac{|\xi|}{\alpha}\big)\, V\big(\tfrac{\arg\xi}{\tau(\alpha)}\big)\, d\xi \Big\}^2 \Big] \\
&= \frac{8\pi^2}{\alpha\tau(\alpha)} \Big[-\int_0^\infty \int_{-\pi}^{\pi} \rho\, W\big(\tfrac{\rho}{\alpha}\big)\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega\, d\rho \\
&\qquad \int_0^\infty \int_{-\pi}^{\pi} \rho^3 \sin^2\omega\, W\big(\tfrac{\rho}{\alpha}\big)\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega\, d\rho \\
&\qquad + \Big\{ \int_0^\infty \int_{-\pi}^{\pi} \rho^2 \sin\omega\, W\big(\tfrac{\rho}{\alpha}\big)\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega\, d\rho \Big\}^2 \Big] \\
&= \frac{8\pi^2\alpha^5}{\tau(\alpha)} \Big[-\int_0^\infty r\, W(r)\, dr \int_{-\pi}^{\pi} V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \\
&\qquad \int_0^\infty r^3\, W(r)\, dr \int_{-\pi}^{\pi} \sin^2\omega\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \\
&\qquad + \Big\{ \int_0^\infty r^2\, W(r)\, dr \Big\}^2 \Big\{ \int_{-\pi}^{\pi} \sin\omega\, V\big(\tfrac{\omega}{\tau(\alpha)}\big)\, d\omega \Big\}^2 \Big]. \qquad \square
\end{aligned}
$$

Remark 1.2. Lemma 1.8 justifies the computation of the Dupin Indicatrix at the center of surfaces $\mathbf{\Gamma}_{\alpha\beta\theta}$ as means of approximating the shape of the highest level curves for the square of moduli of curvelets $\gamma_{\alpha\beta\theta}$ nearby their locations $\beta \in \mathbb{R}^2$.

Remark 1.3. The matrix representation of the Dupin Indicatrix of $\mathbf{\Gamma}_{\alpha\mathbf{01}}$ at the point $\big(\mathbf{0}, |\gamma_{\alpha\mathbf{01}}(\mathbf{0})|^2\big)$ is given by

$$
\begin{pmatrix} 1 \\ x \\ y \end{pmatrix}^{\top} \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \boldsymbol{e} & \boldsymbol{f} \\ 0 & \boldsymbol{f} & \boldsymbol{g} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = 0.
$$

According to the sign of the sub-determinant resulting from removing the first row and first column, the conics given by the Dupin Indicatrix is one of the following:

(i) If $\det\big(\begin{smallmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{f} & \boldsymbol{g} \end{smallmatrix}\big) = 0$, the conic is degenerate. In this case, two parallel straight lines.
(ii) If $\det\big(\begin{smallmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{f} & \boldsymbol{g} \end{smallmatrix}\big) > 0$, the conic is an ellipse.
(iii) Otherwise, it is a pair of hyperbolas.

In the computations below, let $M = \big(8\pi^2\alpha^5\tau(\alpha)^{-1}\big)^{-2}$, and

$$
\begin{aligned}
\boldsymbol{\mathcal{V}}_f\big(\lambda\big) &= \int_{-\pi}^{\pi} V\big(\tfrac{\omega}{\lambda}\big) f(\omega)\, d\omega, \qquad \boldsymbol{\mu}_k(W) = \int_0^\infty r^k W(r)\, dr \quad (k \in \mathbb{N}), \\
\mathbf{I}_{V,\tau(\alpha)} &= \boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big) - \boldsymbol{\mathcal{V}}_{\sin\omega\cos\omega}\big(\tau(\alpha)\big)^2, \\
\mathbf{II}_{V,\tau(\alpha)} &= \boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)^2 + \boldsymbol{\mathcal{V}}_{\sin\omega}\big(\tau(\alpha)\big)^2\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big) \\
&\quad + 2\boldsymbol{\mathcal{V}}_{\sin\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\sin\omega\cos\omega}\big(\tau(\alpha)\big).
\end{aligned}
$$

Then,

$$
M \det\big(\begin{smallmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{f} & \boldsymbol{g} \end{smallmatrix}\big) = \Big(\boldsymbol{\mu}_2(W)^2\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)^2 - \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big)\Big)
$$

$$
\begin{aligned}
&\Big(\boldsymbol{\mu}_2(W)^2\boldsymbol{\mathcal{V}}_{\sin\omega}\big(\tau(\alpha)\big)^2 - \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big)\Big)\\
&- \Big(\boldsymbol{\mu}_2(W)^2\boldsymbol{\mathcal{V}}_{\sin\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)\\
&\qquad + \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\sin\omega\cos\omega}\big(\tau(\alpha)\big)\Big)^2\\
=\;& \boldsymbol{\mu}_1(W)^2\boldsymbol{\mu}_3(W)^2\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)^2\Big(\boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big)\\
&\qquad\qquad - \boldsymbol{\mathcal{V}}_{\sin\omega\cos\omega}\big(\tau(\alpha)\big)^2\Big)\\
&- \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_2(W)^2\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\Big(\boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)^2\\
&\qquad\qquad + \boldsymbol{\mathcal{V}}_{\sin\omega}\big(\tau(\alpha)\big)^2\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big)\\
&\qquad\qquad + 2\boldsymbol{\mathcal{V}}_{\sin\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\sin\omega\cos\omega}\big(\tau(\alpha)\big)\Big)\\
=\;& \boldsymbol{\mu}_1(W)^2\boldsymbol{\mu}_3(W)^2\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)^2\,\mathbf{I}_{V,\tau(\alpha)}\\
&- \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_2(W)^2\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\,\mathbf{II}_{V,\tau(\alpha)}.
\end{aligned}
$$

Definition. A curvelet $\gamma_{\alpha\beta\theta}$ is said to be *elliptic* (resp. *hyperbolic*, *flat*) provided the Dupin Indicatrix at the center of the corresponding surface $\boldsymbol{\Gamma}_{\alpha\beta\theta}$ is an ellipse (resp. pair of hyperbolas, pair of parallel lines). The expression

$$
\operatorname{sign}\Big(\boldsymbol{\mu}_1(W)^2\boldsymbol{\mu}_3(W)^2\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)^2\,\mathbf{I}_{V,\tau(\alpha)} - \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_2(W)^2\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\,\mathbf{II}_{V,\tau(\alpha)}\Big) \tag{10}
$$

is called the *shape discriminant* of the curvelet $\gamma_{\alpha\beta\theta}$.

Lemma 1.9. *If $V \in C_c^\infty(\mathbb{R})$ is a positive function on its support,* $\operatorname{supp} V = [-1,1]$, *then*

$$
\frac{1}{2}\tau(\alpha)^3\int_{-1}^{1}\omega^2V(\omega)\,d\omega \le \boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big) \le \tau(\alpha)^3\int_{-1}^{1}\omega^2V(\omega)\,d\omega. \tag{11}
$$

Proof. This is direct from the inequalities $\zeta^2/2 \le \sin^2\zeta \le \zeta^2$ for all $-1 \le \zeta \le 1$, and the identity

$$
\begin{aligned}
\int_{-\pi}^{\pi} V\big(\tfrac{\omega}{\tau(\alpha)}\big)\sin^2\omega\,d\omega &= \tau(\alpha)\int_{-\pi/\tau(\alpha)}^{\pi/\tau(\alpha)} V(\zeta)\sin^2\tau(\alpha)\zeta\,d\zeta \qquad \big(\zeta = \tfrac{\omega}{\tau(\alpha)}\big)\\
&= \tau(\alpha)\int_{-1}^{1} V(\zeta)\sin^2\tau(\alpha)\zeta\,d\zeta. \qquad \big(\text{since } \tfrac{\pi}{\tau(\alpha)} > 1\big) \ \square
\end{aligned}
$$

Lemma 1.10. *Let $V \in C_c^\infty(\mathbb{R})$ be an even function strictly positive on its support,* $\operatorname{supp} V = [-1,1]$. *Then, for all $\alpha > 0$,*

$$
\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big) = \|V\|_{L_1(\mathbb{R})}\tau(\alpha) > 0, \tag{12}
$$

$$
\frac{\sqrt{2}}{2}\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big) \le \boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big) \le \boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big), \tag{13}
$$

$$
\frac{1}{2}\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big) \le \boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big) \le \boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big), \tag{14}
$$

and in particular,

$$
\frac{1}{2} \le \frac{\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big)}{\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)^2} \le 4 \tag{15}
$$

Proof. Identity (12) is direct by definition. To prove estimate (13), notice first that $0 < \tau(\alpha) < \pi/4$, and hence $\frac{\sqrt{2}}{2} \leq \cos\tau(\alpha) \leq \cos\tau(\alpha)\omega \leq 1$ for $1 \leq \omega \leq 1$. This is used in the identity below

$$\int_{-\pi}^{\pi} V\big(\tfrac{\omega}{\tau(\alpha)}\big)\cos\omega\,d\omega = \tau(\alpha)\int_{-1}^{1} V(\omega)\cos\tau(\alpha)\omega\,d\omega.$$

Estimate (14) follows from a similar argument, and (15) is a direct consequence of the previous estimates. ☐

Lemma 1.11. *Consider the curvelet $\gamma_{\alpha\mathbf{01}}$ with both windows V and W positive in their respective domains. If the phase window is an even function, then $\boldsymbol{f} = 0$, $|\boldsymbol{e}| \asymp \alpha^5\tau(\alpha)$, and $|\boldsymbol{g}| \asymp \alpha^5\tau(\alpha)^3$, where the constants of proportionality depend at most on the choice of windows V and W.*

Proof. As V is an even function, $\boldsymbol{\mathcal{V}}_{\sin\omega}\big(\tau(\alpha)\big) = 0$ as well as $\boldsymbol{\mathcal{V}}_{\sin\omega\cos\omega}\big(\tau(\alpha)\big) = 0$ and furthermore, $\boldsymbol{f} = 0$. Notice that the following are all strictly positive,

$$\begin{aligned} &\boldsymbol{\mu}_1(W) > 0, && \boldsymbol{\mu}_2(W) > 0, && \boldsymbol{\mu}_3(W) > 0,\\ &\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big) > 0, && \boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big) > 0, && \boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big) > 0, \end{aligned}$$

as well as $\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big) > 0$ by Lemma 1.10. It is then

$$\begin{aligned}
\boldsymbol{e} &= \frac{8\pi^2\alpha^5}{\tau(\alpha)}\Big(\boldsymbol{\mu}_2(W)^2\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)^2 - \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big)\Big)\\
&\leq \frac{8\pi^2\alpha^5}{\tau(\alpha)}\Big(\boldsymbol{\mu}_2(W)^2 - \tfrac{1}{2}\boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\Big)\|V\|_{L_1(\mathbb{R})}\tau(\alpha)^2 && \big(\text{by Lemma 1.10}\big)\\
&= 8\pi^2\|V\|_{L_1(\mathbb{R})}\Big(\boldsymbol{\mu}_2(W)^2 - \tfrac{1}{2}\boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\Big)\alpha^5\tau(\alpha),\\
\boldsymbol{e} &\geq \frac{8\pi^2\alpha^5}{\tau(\alpha)}\Big(\tfrac{\sqrt{2}}{2}\boldsymbol{\mu}_2(W)^2 - \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\Big)\|V\|_{L_1(\mathbb{R})}\tau(\alpha)^2 && \big(\text{by Lemma 1.10}\big)\\
&= 8\pi^2\|V\|_{L_1(\mathbb{R})}\Big(\tfrac{\sqrt{2}}{2}\boldsymbol{\mu}_2(W)^2 - \boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\Big)\alpha^5\tau(\alpha),\\
|\boldsymbol{g}| &= \frac{8\pi^2\alpha^5}{\tau(\alpha)}\boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\sin^2\omega}\big(\tau(\alpha)\big)\\
&\asymp \frac{8\pi^2\alpha^5}{\tau(\alpha)}\boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\tau(\alpha)\|V\|_{L_1(\mathbb{R})}\tau(\alpha)^3\boldsymbol{\mu}_2(V) && \big(\text{by Lemma 1.9}\big)\\
&= 8\pi^2\|V\|_{L_1(\mathbb{R})}\boldsymbol{\mu}_2(V)\boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)\alpha^5\tau(\alpha)^3,
\end{aligned}$$

where $\boldsymbol{\mu}_2(V) = \int_{-1}^{1}\omega^2 V(\omega)\,d\omega$ is the second moment of V. ☐

Proposition 1.1. *Consider the curvelet $\gamma_{\alpha\mathbf{01}}$ with both the windows V and W positive in their respective domains. If the phase window is an even function, then the Dupin Indicatrix of the regular surface $\boldsymbol{\Gamma}_{\alpha\mathbf{01}}$ at the point $\big(\mathbf{0}, |\gamma_{\alpha\mathbf{01}}(\mathbf{0})|^2\big)$ is*

(i) *A pair of parallel lines, provided*

$$\frac{\boldsymbol{\mu}_2(W)^2}{\boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)} = \frac{\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big)}{\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)^2}, \tag{16}$$

the distance $\boldsymbol{d}$ between those lines satisfies $\boldsymbol{d} \asymp \alpha^{-5/2}\tau(\alpha)^{-3/2}$, where the constants of proportionality depend at most on V and W;

(ii) *An ellipse with area* $\boldsymbol{A}$ *and semi-axes* $\boldsymbol{a}$, $\boldsymbol{b}$, *satisfying* $\boldsymbol{A} \asymp \alpha^{-5}\tau(\alpha)^{-2}$, $\boldsymbol{a}/\boldsymbol{b} \asymp \tau(\alpha)$, *so long as*

$$\frac{\boldsymbol{\mu}_2(W)^2}{\boldsymbol{\mu}_1(W)\boldsymbol{\mu}_3(W)} < \frac{\boldsymbol{\mathcal{V}}_1\big(\tau(\alpha)\big)\boldsymbol{\mathcal{V}}_{\cos^2\omega}\big(\tau(\alpha)\big)}{\boldsymbol{\mathcal{V}}_{\cos\omega}\big(\tau(\alpha)\big)^2}. \tag{17}$$

(iii) *Otherwise, a pair of hyperbolas having the same set of asymptotes. Their semi-axes* $\boldsymbol{a}$, $\boldsymbol{b}$, *and the angle* $\boldsymbol{\theta}$ *between the two asymptotes satisfy* $\boldsymbol{a}/\boldsymbol{b} \asymp \tau(\alpha)$, *and* $\tan\boldsymbol{\theta} \asymp \tau(\alpha)$.

Proof. The classification follows from direct inspection of the shape discriminant (10):

(i) If (16) holds, then the Dupin Indicatrix is the pair of parallel lines $y^2 = |\boldsymbol{g}|^{-1}$, since $\boldsymbol{e} = 0$. The distance between these lines is given by $\boldsymbol{d} = 2|\boldsymbol{g}|^{-1/2} \asymp \alpha^{-5/2}\tau(\alpha)^{-3/2}$, by Lemma 1.9.

(ii) If (17) is satisfied, the Dupin Indicatrix is the ellipse $|\boldsymbol{e}|x^2 + |\boldsymbol{g}|y^2 = 1$. Its semi-axes have sizes $\boldsymbol{a} = |\boldsymbol{e}|^{-1/2} \asymp \alpha^{-5/2}\tau(\alpha)^{-1/2}$ and $\boldsymbol{b} = |\boldsymbol{g}|^{-1/2} \asymp \alpha^{-5/2}\tau(\alpha)^{-3/2}$, and its area is given by $\boldsymbol{A} = \pi|\boldsymbol{eg}|^{-1/2} \asymp \alpha^{-5}\tau(\alpha)^{-2}$.

(iii) Notice that in this case it is $\boldsymbol{e} > 0$, $\boldsymbol{g} < 0$, and so the Dupin Indicatrix reduces to the pair of hyperbolas $|\boldsymbol{e}|x^2 - |\boldsymbol{g}|y^2 = 1$, $|\boldsymbol{g}|y^2 - |\boldsymbol{e}|x^2 = 1$. The set of asymptotes of both hyperbolas is given by the equations

$$|\boldsymbol{e}|^{1/2}x - |\boldsymbol{g}|^{1/2}y = 0, \qquad |\boldsymbol{e}|^{1/2}x + |\boldsymbol{g}|^{1/2}y = 0.$$

From the values of the slopes of both lines, it is $\tan\boldsymbol{\theta} = 2|\boldsymbol{eg}|^{1/2}\big(|\boldsymbol{e}| - |\boldsymbol{g}|\big) \asymp \tau(\alpha)\big(1-\tau(\alpha)^2\big)^{-1}$. Finally, the semi-axes are given by $\boldsymbol{a} = |\boldsymbol{e}|^{-1/2}$, $\boldsymbol{b} = |\boldsymbol{g}|^{-1/2}$. The rest of the statement follows. □

Remark 1.4. Lemma 1.10 shows that it is possible to obtain a family of elliptical curvelets $\{\gamma_{\alpha\beta\theta}\}$ by choosing wisely the amplitude window. In particular, any admissible function $W \in C_c^\infty(\mathbb{R})$ satisfying $\boldsymbol{\mu}_2(W)^2\boldsymbol{\mu}_1(W)^{-1}\boldsymbol{\mu}_3(W)^{-1} < 1/2$ offers such possibility.

1.2. **The Curvelet Transform in** $L_2(\mathbb{R}^2)$**.**

Proposition 1.2. *Given real-valued smooth functions* $W \in C_c^\infty(0,\infty)$ *with support in the interval* $[\frac{1}{\alpha_0}, \alpha_0]$ $(\alpha_0 > 1)$ *satisfying* $\int_0^\infty W(t)^2\,\frac{dt}{t} = 1$, *and* $V \in C_c^\infty(\mathbb{R})$ *with support in* $[-1,1]$, *satisfying* $\|V\|_{L_2(\mathbb{R})} = 1$; *then the following identity holds for all functions* $f, g \in L_2(\mathbb{R}^2)$:

$$\langle f, g\rangle = \int_0^\infty\int_{\mathbb{S}^1}\int_{\mathbb{R}^2} \langle f, \gamma_{\alpha\beta\theta}\rangle\overline{\langle g, \gamma_{\alpha\beta\theta}\rangle}d\beta\,d\sigma(\theta)\,d\alpha. \tag{18}$$

Proof. Set $\boldsymbol{I}(f,g) = \int_0^\infty\int_{\mathbb{S}^1}\int_{\mathbb{R}^2}\langle f, \gamma_{\alpha\beta\theta}\rangle\overline{\langle g, \gamma_{\alpha\beta\theta}\rangle}d\beta\,d\sigma(\theta)\,d\alpha$. It is then

$$\begin{aligned}
\boldsymbol{I}(f,g) &= \int_0^\infty\int_{\mathbb{S}^1}\int_{\mathbb{R}^2}\langle \widehat{f}, \widehat{\gamma}_{\alpha\beta\theta}\rangle\overline{\langle \widehat{g}, \widehat{\gamma}_{\alpha\beta\theta}\rangle}d\beta\,d\sigma(\theta)\,d\alpha \\
&= \int_0^\infty\int_{\mathbb{S}^1}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\widehat{f}(\xi)W_\alpha(|\xi|)V_{\tau(\alpha),\theta}(\xi/|\xi|)e^{-2\pi i\beta\cdot\xi}d\xi\right) \\
&\qquad\qquad\left(\int_{\mathbb{R}^2}\overline{\widehat{g}(\zeta)}W_\alpha(|\zeta|)V_{\tau(\alpha),\theta}(\zeta/|\zeta|)e^{2\pi i\beta\cdot\zeta}d\zeta\right)d\beta\,d\sigma(\theta)\,d\alpha.
\end{aligned}$$

Consider the auxiliary functions

$$F_{\alpha\theta}(\xi) = \overline{\widehat{f}(\xi)} W_\alpha(|\xi|) V_{\tau(\alpha),\theta}(\xi/|\xi|)$$
$$G_{\alpha\theta}(\xi) = \widehat{g}(\xi) W_\alpha(|\xi|) V_{\tau(\alpha),\theta}(\xi/|\xi|);$$

because the Fourier transform is unitary, it follows that

$$\begin{aligned}
\boldsymbol{I}(f,g) &= \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} \overline{\widehat{F}_{\alpha\theta}(\beta)} \widehat{G}_{\alpha\theta}(\beta) d\beta\, d\sigma(\theta)\, d\alpha \\
&= \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} \overline{F_{\alpha\theta}(\xi)} G_{\alpha\theta}(\xi) d\xi\, d\sigma(\theta)\, d\alpha \\
&= \int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \int_0^\infty W_\alpha(|\xi|)^2 \int_{\mathbb{S}^1} V_{\tau(\alpha),\theta}(\xi/|\xi|)^2\, d\sigma(\theta)\, d\alpha\, d\beta \\
&\qquad \text{(interchange is allowed by Fubini's Theorem)} \\
&= \int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \int_0^\infty W_\alpha(|\xi|)^2 \int_{\omega-\pi}^{\omega+\pi} \tfrac{1}{\tau(\alpha)} V\big(\tfrac{\omega-\omega_0}{\tau(\alpha)}\big)^2 d\omega_0\, d\alpha\, d\xi \\
&= \int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \int_0^\infty W_\alpha(|\xi|)^2 \int_{-\pi/\tau(\alpha)}^{\pi/\tau(\alpha)} V(s)^2\, ds\, d\alpha\, d\xi \\
&\qquad \text{(apply the change of variables } s = \tfrac{\omega-\omega_0}{\tau(\alpha)}\text{)} \\
&= \|V\|^2_{L_2(\mathbb{R})} \int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \int_0^\infty W_\alpha(|\xi|)^2\, d\alpha\, d\xi \\
&= \int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \int_0^\infty \tfrac{1}{\alpha} W\big(\tfrac{|\xi|}{\alpha}\big)^2\, d\alpha\, d\xi \\
&\qquad \big(\text{apply the change of variables } t = \tfrac{|\xi|}{\alpha}\big) \\
&= \int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \int_0^\infty W(t)^2\, \tfrac{dt}{t}\, d\xi = \langle f, g\rangle.
\end{aligned}$$

□

Corollary 1.2.1 (Plancherel's Theorem for the Continuous Curvelet Transform)**.** *For each $f \in L_2(\mathbb{R}^2)$,*

$$\|f\|^2_{L_2(\mathbb{R}^2)} = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} |\langle f, \gamma_{\alpha\beta\theta}\rangle|^2\, d\beta\, d\sigma(\theta)\, d\alpha. \tag{19}$$

Remark 1.5. Use the previous corollary to construct an isometric isomorphism between $L_2(\mathbb{R}^2)$ and a subspace of the square integrable functions over the measure space (Ω,μ), where $\Omega = (0,\infty)\times\mathbb{S}^1\times\mathbb{R}^2$ is the scale/direction/location space, and $d\mu = d\beta\, d\sigma(\theta)\, d\alpha$:

$$\mathcal{T}\colon L_2(\mathbb{R}^2) \to L_2(\Omega,\mu),$$

given by $\mathcal{T}f(\alpha,\theta,\beta) = \langle f, \gamma_{\alpha\beta\theta}\rangle$ for $(\alpha,\theta,\beta)\in\Omega$. The mapping $\mathcal{T}$ is called the *continuous curvelet transform*, or simply the *curvelet transform*. On occasion $\mathcal{T}f(\alpha,\theta,\beta)$ is called the *curvelet coefficient* of f at the scale α, location β and direction θ.

Lemma 1.12. *The operator $\mathcal{T}\colon L_2(\mathbb{R}^2)\to L_2(\Omega,\mu)$ defined above is bounded, linear, one-to-one, and isometric on its range.*

Proof. $\mathcal{T}$ is trivially linear by construction. Let $f\in L_2(\mathbb{R}^2)$ be a function such that $0 = \mathcal{T}f(\alpha,\theta,\beta) = \langle f,\gamma_{\alpha\beta\theta}\rangle = \langle \widehat{f}, \widehat{\gamma}_{\alpha\beta\theta}\rangle$ for all $(\alpha,\theta,\beta)\in\Omega$. As $\widehat{\gamma}_{\alpha0\theta}$ is non-negative for each choice of α and θ, both $\Re\widehat{f}$ and $\Im\widehat{f}$ must be identically zero in the support of the frequency of each curvelet and, therefore, $\widehat{f}$ is identically zero everywhere. It must then be that f is identically zero, which proves $\mathcal{T}$ one-to-one.

Finally, both boundedness and the isometry property of $\mathcal{T}$ are a direct consequence of (19). □

Denote $\mathcal{H} = \mathcal{T}L_2(\mathbb{R}^2)$, and call it the *space of curvelet coefficients* for the continuous curvelet transform in $L_2(\mathbb{R}^2)$. Putting all together we finally obtain the claimed result.

Proposition 1.3. *The operator $\mathcal{T}\colon L_2(\mathbb{R}^2) \to \mathcal{H}$ constructed above is an isometric isomorphism.*

A *Resolution of the Identity* is needed in order to find the inverse of this operator explicitly.

Theorem 1.1 (Calderón Resolution of the Identity for the Continuous Curvelet Transform)**.** *Under the same hypotheses as in Proposition 1.2, the following Calderón Resolution of the Identity holds:*

$$f(x) = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} \langle f, \gamma_{\alpha\beta\theta}\rangle \gamma_{\alpha\beta\theta}(x)\, d\beta\, d\sigma(\theta)\, d\alpha. \tag{20}$$

Proof. Denote vol_d the volume of the d-dimensional unit ball, and A_{d-1} the surface area of the corresponding sphere. Let $0 < \alpha_1 < \alpha_2$, and $\rho > 0$. For functions $f, g \in L_2(\mathbb{R}^2)$, it is

$$\begin{aligned}&\int_{\alpha_1}^{\alpha_2} \int_{\mathbb{S}^1} \int_{|\beta|\leq\rho} \langle f, \gamma_{\alpha\beta\theta}\rangle \overline{\langle g, \gamma_{\alpha\beta\theta}\rangle}\, d\beta\, d\sigma(\theta)\, d\alpha \\ &\quad\leq\ |\alpha_2 - \alpha_1|\, A_1 \rho^2\, \text{vol}_2\, \|f\|_{L_2(\mathbb{R}^2)}\, \|g\|_{L_2(\mathbb{R}^2)}\, \|\gamma_{\alpha\beta\theta}\|^2_{L_2(\mathbb{R}^2)};\end{aligned}$$

therefore, for each $f \in L_2(\mathbb{R}^2)$ the linear forms $T = T(f; \alpha_1, \alpha_2, \rho; \cdot)\colon L_2(\mathbb{R}^2) \to \mathbb{R}$ given by the integrals above are continuous. By the Riesz Representation Theorem for the Hilbert Space $L_2(\mathbb{R}^2)$, the functions

$$x \mapsto \int_{\alpha_1}^{\alpha_2} \int_{\mathbb{S}^1} \int_{|\beta|\leq\rho} \langle f, \gamma_{\alpha\beta\theta}\rangle \gamma_{\alpha\beta\theta}(x)\, d\beta\, d\sigma(\theta)\, d\alpha$$

are well defined and belong to $L_2(\mathbb{R}^2)$. In that case,

$$\begin{aligned}&\left\| f - \int_{\alpha_1}^{\alpha_2} \int_{\mathbb{S}^1} \int_{|\beta|\leq\rho} \langle f, \gamma_{\alpha\beta\theta}\rangle \gamma_{\alpha\beta\theta}(x)\, d\beta\, d\sigma(\theta)\, d\alpha \right\|^2_{L_2(\mathbb{R}^2)} \\ &= \sup_{\substack{g\in L_2(\mathbb{R}^2)\\ \|g\|_{L_2(\mathbb{R}^2)}=1}} \left| \left\langle f - \int_{\alpha_1}^{\alpha_2} \int_{\mathbb{S}^1} \int_{|\beta|\leq\rho} \langle f, \gamma_{\alpha\beta\theta}\rangle \gamma_{\alpha\beta\theta}(x)\, d\beta\ d\alpha, g \right\rangle \right| \\ &= \sup_{\substack{g\in L_2(\mathbb{R}^2)\\ \|g\|_{L_2(\mathbb{R}^2)}=1}} \left| \iiint_{\Omega\setminus[\alpha_1,\alpha_2]\times\mathbb{S}^1\times B_2(0,\rho)} \langle f, \gamma_{\alpha\beta\theta}\rangle \overline{\langle g, \gamma_{\alpha\beta\theta}\rangle}\, d\beta\, d\sigma(\theta)\, d\alpha \right| \\ &\leq \sup_{\substack{g\in L_2(\mathbb{R}^2)\\ \|g\|_{L_2(\mathbb{R}^2)}=1}} \left\{ \left(\iiint_{\Omega\setminus[\alpha_1,\alpha_2]\times\mathbb{S}^1\times B_2(0,\rho)} |\langle f, \gamma_{\alpha\beta\theta}\rangle|^2 d\beta\, d\sigma(\theta)\, d\alpha \right)^{1/2} \right.\end{aligned}$$

$$\left(\iiint_\Omega |\langle g, \gamma_{\alpha\beta\theta}\rangle|^2 d\beta\, d\sigma(\theta)\, d\alpha \right)^{1/2} \Bigg\}$$

$$= \left(\iiint_{\Omega\setminus[\alpha_1,\alpha_2]\times\mathbb{S}^1\times B_2(0,\rho)} |\langle f, \gamma_{\alpha\beta\theta}\rangle|^2 d\beta\, d\sigma(\theta)\, d\alpha \right)^{1/2}.$$

But the last integral converges to zero as $\alpha_1 \to 0$, $\alpha_2, \rho \to \infty$, since the same integral over all of Ω converges by equation (19). The stated result follows. □

Remark 1.6. This last result allows the construction of the *inverse curvelet transform* as follows: Given $F \in \mathcal{H}$, there is a unique $f \in L_2(\mathbb{R}^2)$ such that $F(\alpha,\theta,\beta) = \langle f, \gamma_{\alpha\beta\theta}\rangle$. In that case

$$f(x) = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} F(\alpha,\theta,\beta)\, \gamma_{\alpha\beta\theta}(x)\, d\beta\, d\sigma(\theta)\, d\alpha.$$

Consequently, $\mathcal{T}^{-1} \colon \mathcal{H} \to L_2(\mathbb{R}^2)$ is defined using the integral operator above.

The space $\mathcal{H}$ has an interesting structure.

Proposition 1.4. *$\mathcal{H} = \mathcal{T} L_2(\mathbb{R}^2)$ is a reproducing kernel Hilbert Space: For each $F \in \mathcal{H}$,*

$$F(\alpha',\theta',\beta') = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} F(\alpha,\theta,\beta)\, \mathcal{K}(\alpha',\beta',\theta';\alpha,\beta,\theta)\, d\beta\, d\sigma(\theta)\, d\alpha,$$

where the kernel $\mathcal{K}$ is defined by $\mathcal{K}(\alpha',\beta',\theta';\alpha,\beta,\theta) = \langle \gamma_{\alpha\beta\theta}, \gamma_{\alpha'\beta'\theta'}\rangle$.

Proof. Given $F \in \mathcal{H}$, there exists a unique function $f \in L_2(\mathbb{R}^2)$ such that $F(\alpha,\theta,\beta) = \langle f, \gamma_{\alpha\beta\theta}\rangle$. In that case, for another choice $(\alpha',\theta',\beta') \in \Omega$, it is

$$F(\alpha',\theta',\beta') = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} \langle f, \gamma_{\alpha\beta\theta}\rangle \overline{\langle \gamma_{\alpha'\beta'\theta'}, \gamma_{\alpha\beta\theta}\rangle}\, d\beta\, d\sigma(\theta)\, d\alpha.$$ □

1.3. **The Curvelet Transform in $L_p(\mathbb{R}^2)$ for $1 \le p < 2$.** Plancherel's Theorem can be extended from the exponent 2 to a general exponent p. Throughout this section, denote $p' = p/(p-1)$ and with a similar convention for any other letters.

Proposition 1.5. *Assume there exists a constant $A > 0$ such that $\tau(\alpha) \le A\alpha^{-s}$ for some $s \ge 3$ and all $\alpha \ge m_\tau$; then, the curvelet transform $\mathcal{T} \colon L_2(\mathbb{R}^2) \to L_2(\Omega,\mu)$ has a bounded extension from $L_p(\mathbb{R}^2)$ to $L_{p'}(\Omega,\mu)$, satisfying for all $f \in L_p(\mathbb{R}^2)$,*

$$\|\mathcal{T} f\|_{L_{p'}(\Omega,\mu)} \le C(V,W,\alpha_0,A,m_\tau,s,p)\, \|f\|_{L_p(\mathbb{R}^2)}.$$

Proof. Both linearity and boundedness properties for $\mathcal{T}$ as an operator from $L_2(\mathbb{R}^2)$ to $L_2(\Omega,\mu)$ were already proven in Lemma 1.12.

Given $f \in L_1(\mathbb{R}^2)$, using (4), we have

$$\begin{aligned} |\langle f, \gamma_{\alpha\beta\theta}\rangle| &= \left| \int_{\mathbb{R}^2} f(x)\, \overline{\gamma_{\alpha\beta\theta}(x)}\, dx \right| \le \|\gamma_{\alpha\beta\theta}\|_{L_\infty(\mathbb{R}^2)}\, \|f\|_{L_1(\mathbb{R}^2)} \\ &\le \left(\alpha_0^2 - \tfrac{1}{\alpha_0^2}\right) \|V\|_{L_\infty(\mathbb{R})}\, \|W\|_{L_\infty(0,\infty)}\, \alpha^{3/2} \tau(\alpha)^{1/2} \|f\|_{L_\infty(\mathbb{R}^2)}. \end{aligned}$$

By hypothesis, $\alpha^{3/2}\tau(\alpha)^{1/2} \le A\, m_\tau^{(3-s)/2}$ for all $\alpha > 0$, and therefore

$$\|\mathcal{T} f\|_{L_\infty(\Omega)} \le C(V,W,\alpha_0,A,m_\tau,s)\, \|f\|_{L_1(\mathbb{R}^2)};$$

thus, this gives a bounded and linear operator $\mathcal{T}\colon L_2(\mathbb{R}^2)\cap L_1(\mathbb{R}^2)\to L_2(\Omega,\mu)+L_\infty(\Omega,\mu)$. By the Riesz-Thorin interpolation theorem, it has a bounded extension from $L_{p(t)}(\mathbb{R}^2)$ to $L_{q(t)}(\Omega,\mu)$ satisfying

$$\|\mathcal{T}f\|_{L_{q(t)}(\Omega)}\le C(V,W,\alpha_0,A,m_\tau,s)^{(1-t)}\|f\|_{L_{p(t)}(\mathbb{R}^2)}$$

for any $0<t<1$, where $p(t)=2/(1+t)$, and $q(t)=2/(1-t)$, as stated. □

1.4. Curvelet Frames in $L_2(\mathbb{R}^2)$.

1.4.1. *Semi-Discrete Curvelet Transform.* We proceed to the discretization of the continuous curvelet transform in two steps. In a first step, we consider a "uniform subdivision of the domain $(0,\infty)\times\mathbb{S}^1$ into cubes," and force extra conditions in the window functions so that the integral over each of these cubes equals the value of their sizes multiplied by the evaluation of the functions at one of their points. By "uniform subdivision into cubes" it is implied a partition in the following way

$$\bigcup_{\substack{n\in\mathbb{Z}\\ k=1,\dots,\eta_n}}(b_n,b_{n+1}]\times E_{nk}=(0,\infty)\times\mathbb{S}^1,$$

where $0<b_n<b_{n+1}$, $\mathring{E}_{nk}\cap\mathring{E}_{nj}=\emptyset$ for $k\ne j$, and $\sigma(E_{nk})=\sigma(E_{nj})$ for all possible indices n,k,j.

Such a construction is presented here based on similar ideas for wavelets as developed in Daubechies [6] (see Figure 2): The interval $(0,\infty)$ is partitioned into subintervals of the form $(\alpha_0^n,\alpha_0^{n+1}]$, $n\in\mathbb{Z}$, for the same value $\alpha_0>1$ as in the support of the amplitude window W. For each level n, the circle is divided into equally-sized mutually disjoint sectors with size τ_n (less than, but as close to $\tau(\alpha_0^n)$ as possible). This divides the circle uniformly into $\eta_n:=2\pi/\tau_n$ sectors, denoted here by $\{E_{nk}:k=1,\dots,\eta_n\}$. The set of endpoints of these sectors is given by $\{\theta_{nk}:k=1,\dots,\eta_n\}$, where $\theta_{nk}=e^{ik\tau_n}$.

$$\tau_n=\sup\{2\pi/k\le\tau(\alpha_0^n):k\in\mathbb{N}\},\qquad \eta_n=2\pi/\tau_n=\left\lceil\frac{2\pi}{\tau(\alpha_0^n)}\right\rceil,\tag{21}$$

$$E_{nk}=\{e^{is}:k\tau_n\le s<(k+1)\tau_n\},\qquad \theta_{nk}=e^{ik\tau_n}.\tag{22}$$

For each function $f\in L_2(\mathbb{R}^2)$,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{S}^1}\int_{\mathbb{R}^2}\langle f,\gamma_{\alpha\beta\theta}\rangle\gamma_{\alpha\beta\theta}(x)\,d\beta\,d\sigma(\theta)d\alpha\\ &=\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\eta_n}\int_{\alpha_0^n}^{\alpha_0^{n+1}}\int_{E_{nk}}\int_{\mathbb{R}^2}\langle f,\gamma_{\alpha\beta\theta}\rangle\gamma_{\alpha\beta\theta}(x)\,d\beta\,d\sigma(\theta)\,d\alpha.\end{aligned}$$

Further conditions are imposed upon the functions defining our curvelets so that each term in the previous sum satisfies

$$\int_{\alpha_0^n}^{\alpha_0^{n+1}}\int_{E_{nk}}\int_{\mathbb{R}^2}\langle f,\gamma_{\alpha\beta\theta}\rangle\gamma_{\alpha\beta\theta}(x)\,d\alpha\,d\sigma(\theta)\,d\beta=C_{\alpha_0,n}\int_{\mathbb{R}^2}\langle f,\gamma_{\alpha_0^n\beta\theta_{nk}}\rangle\gamma_{\alpha_0^n\beta\theta_{nk}}(x)\,d\beta,$$

where $C_{\alpha_0,n}>0$ is a positive constant that depends on the size of the integration domain, which in this case is $|\alpha_0^{n+1}-\alpha_0^n|\,\sigma(E_{nk})=\alpha_0^n(\alpha_0-1)\tau_n$.

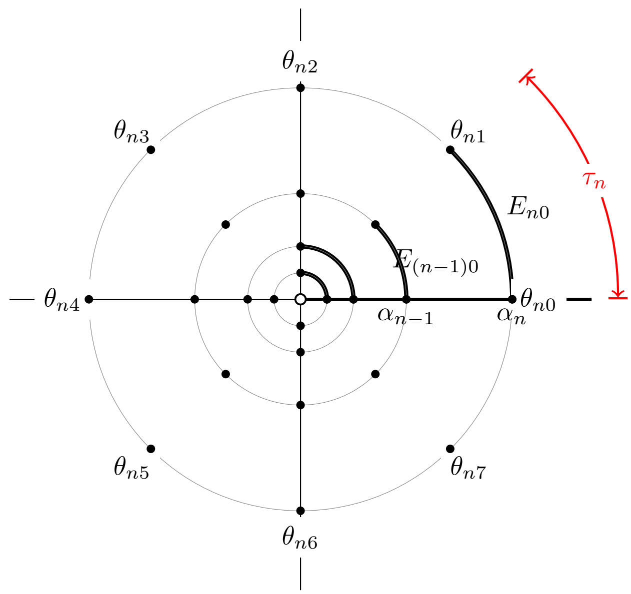

FIGURE 2. Representation of admissible set of pairs $\{(\alpha_n, \theta_{nk})\}$ in $(0, \infty) \times \mathbb{S}^1$ and corresponding sectors $\{E_{nk}\}$ on each circle $\mathbb{S}^1$ used for discretization of the CCT.

Lemma 1.13. *If there exists a constant $C > 0$ such that the function W used to generate the amplitude windows satisfies $W(\rho)^2 + W(\alpha_0 \rho)^2 = C$ for all $\rho \in (1/\alpha_0, 1]$, then necessarily $C \log \alpha_0 = 1$, and*

$$\sum_{n \in \mathbb{Z}} \alpha_0^n W_{\alpha_0^n}(\rho)^2 = \tfrac{1}{\log \alpha_0} \text{ for all } \rho > 0. \tag{23}$$

Similarly, if there exists a constant $C > 0$ such that the function V used above to generate the phase windows satisfies $V(\omega)^2 + V(\omega - 1)^2 = C$ for all $\omega \in [0, 1)$, then it must be $C = \|V\|^2_{L_2(\mathbb{R})} = 1$, and

$$\sum_{k=1}^{\eta_n} V_{\tau_n, \theta_{nk}}(\xi/|\xi|)^2 = \tfrac{1}{\tau_n} \text{ for all } \xi \neq \mathbf{0}. \tag{24}$$

Proof. Given $\rho > 0$, there exists a unique integer $n \in \mathbb{Z}$ such that $\alpha_0^n < \rho \leq \alpha_0^{n+1}$. For this ρ, the sum in (23) reduces to

$$W(\alpha_0^{1-n}\rho)^2 + W(\alpha_0^{-n}\rho)^2 = C.$$

But notice

$$\begin{aligned} 1 &= \int_0^\infty W(\rho)^2 \tfrac{d\rho}{\rho} = \int_{1/\alpha_0}^1 W(\rho)^2 \tfrac{d\rho}{\rho} + \int_1^{\alpha_0} W(\rho)^2 \tfrac{d\rho}{\rho} \\ &= \int_{1/\alpha_0}^1 \big(C - W(\alpha_0\rho)^2\big) \tfrac{d\rho}{\rho} + \int_1^{\alpha_0} W(\rho)^2 \tfrac{d\rho}{\rho} \\ &= \int_1^{\alpha_0} \big(C - W(\rho)^2\big) \tfrac{d\rho}{\rho} + \int_1^{\alpha_0} W(\rho)^2 \tfrac{d\rho}{\rho} = C \log \alpha_0. \end{aligned}$$

This proves (23). Now, given $\xi \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and an integer $n \in \mathbb{Z}$, there exists a unique $k \in \{1, \dots, \eta_n\}$ such that $\arg(\theta_{nk}) < \arg(\xi) \leq \arg(\theta_{n(k+1)}) = \arg(\theta_{nk}) + \tau_n$. For this point ξ, it is

$$\begin{aligned}\sum_{k=1}^{\eta_n} V_{\tau_n,\theta_{nk}}(\xi/|\xi|)^2 &= \tfrac{1}{\tau_n} V\Big(\tfrac{\arg(\xi)-\arg(\theta_{n(k-1)})}{\tau_n}\Big)^2 + \tfrac{1}{\tau_n} V\Big(\tfrac{\arg(\xi)-\arg(\theta_{nk})}{\tau_n}\Big)^2 \\ &= \tfrac{1}{\tau_n}\left[V\Big(\tfrac{\arg(\xi)-\arg(\theta_{nk})}{\tau_n} - 1\Big)^2 + V\Big(\tfrac{\arg(\xi)-\arg(\theta_{nk})}{\tau_n}\Big)^2 \right] = \tfrac{1}{\tau_n} C.\end{aligned}$$

But notice

$$\begin{aligned}1 = \|V\|^2_{L_2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |V(t)|^2\, dt = \int_{-1}^0 V(t)^2\, dt + \int_0^1 V(t)^2\, dt \\ &= \int_0^1 V(t-1)^2\, dt + \int_0^1 V(t)^2\, dt = \int_0^1 \big(C - V(t)^2\big)\, dt + \int_0^1 V(t)^2\, dt = C.\end{aligned}$$

This gives (24). □

Proposition 1.6. *Let $\Phi_{n\beta k}(x) = \gamma_{\alpha\beta\theta}(x)$, where $\alpha = \alpha_0^n$ and $\theta = \theta_{nk}$. If the smooth functions $W \in C_c^\infty(0,\infty)$, $V \in C_c^\infty(\mathbb{R})$ used in the construction of the curvelets $\gamma_{\alpha\beta\theta}$ satisfy the additional admissibility conditions given by Lemma 1.13, then the following identities hold for all $f \in L_2(\mathbb{R}^2)$:*

$$f = (\log \alpha_0) \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n \int_{\mathbb{R}^2} \langle f, \Phi_{n\beta k}\rangle \Phi_{n\beta k}(x)\, d\beta, \tag{25}$$

$$\sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n \int_{\mathbb{R}^2} |\langle f, \Phi_{n\beta k}\rangle|^2\, d\beta = \tfrac{1}{\log \alpha_0} \|f\|^2_{L_2(\mathbb{R}^2)}. \tag{26}$$

Proof. Notice that the integral

$$g_{nk}(x) = \int_{\mathbb{R}^2} \langle f, \Phi_{n\beta k}\rangle \Phi_{n\beta k}(x)\, d\beta$$

is the convolution $\Phi_{n\mathbf{0}k} * \widetilde{\Phi}_{n\mathbf{0}k} * f$, where $\widetilde{\Phi}_{n\mathbf{0}k}(x) = \Phi_{n\mathbf{0}k}(-x)$; therefore, it is a $L_2(\mathbb{R}^2)$ function. Its Fourier transform is given by $\widehat{g}_{nk}(\xi) = |\widehat{\Phi}_{n\mathbf{0}k}(\xi)|^2 \widehat{f}(\xi)$.

Consider for each $m \in \mathbb{N}$ the sequence of functions $\big\{G_m(x)\big\}_m$ defined by

$$G_m(x) = \sum_{|n|\leq m} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n g_{nk}(x).$$

These are also square integrable, with Fourier transforms given by

$$\begin{aligned}\widehat{G}_m(\xi) &= \sum_{|n|\leq m} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n \widehat{g}_{nk}(\xi) \\ &= \widehat{f}(\xi) \sum_{|n|\leq m} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n |\widehat{\Phi}_{n\mathbf{0}k}(\xi)|^2 \\ &= \widehat{f}(\xi) \sum_{|n|\leq m} \alpha_0^n W_{\alpha_0^n}\big(|\xi|\big)^2 \sum_{k=1}^{\eta_n} \tau_n V_{\tau_n,\theta_{nk}}(\xi/|\xi|)^2 \\ &= \widehat{f}(\xi) \sum_{|n|\leq m} \alpha_0^n W_{\alpha_0^n}\big(|\xi|\big)^2,\end{aligned}$$

as a consequence of equation (24).

Observe that this sequence of Fourier transforms converges pointwise to the function $\frac{1}{\log \alpha_0}\widehat{f}(\xi)$ by virtue of equation (23); the convergence of the series in (25) follows then from the Dominated Convergence Theorem.

To prove (26), observe that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n \int_{\mathbb{R}^2} \lvert\langle f, \Phi_{n\beta k}\rangle\rvert^2 \, d\beta \\
&\quad= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n \int_{\mathbb{R}^2} \lvert\widehat{f}(\xi)\rvert^2 W_{\alpha_0^n}\big(\lvert\xi\rvert\big)^2 V_{\alpha_0^n,\tau_n}\big(\xi/\lvert\xi\rvert\big)^2 \, d\xi \\
&\quad= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^2} \lvert\widehat{f}(\xi)\rvert^2 \, \alpha_0^n W_{\alpha_0^n}\big(\lvert\xi\rvert\big)^2 \left(\sum_{k=1}^{\eta_n} \tau_n V_{\alpha_0^n,\tau_n}\big(\xi/\lvert\xi\rvert\big)^2\right) d\xi \\
&\quad= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^2} \lvert\widehat{f}(\xi)\rvert^2 \, \alpha_0^n W_{\alpha_0^n}\big(\lvert\xi\rvert\big)^2 \, d\xi.
\end{aligned}$$

The first equality is obtained as in the proof of Proposition 1.2. Interchanging sum and integral in the last expression is now allowed by the Monotone Convergence Theorem, and our statement follows. □

1.5. **Discrete Curvelet Transform.** In order to fully discretize the transform, we need the following lemma.

Lemma 1.14. *Suppose that $h \in L_2(\mathbb{R}^d)$ is a bandlimited function with $\operatorname{supp}\widehat{h} \subset [-M,M]^d$ for some $M > 0$. Consider the Fourier multiplier $Q\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ given by $\widehat{Qf}(\xi) = \lvert\widehat{h}(\xi)\rvert^2 \widehat{f}(\xi)$ for all $f \in L_2(\mathbb{R}^d)$. Then,*

$$\langle Qf, g\rangle = \sum_{\boldsymbol{z}\in\mathbb{Z}^d} \langle f, h_{\boldsymbol{z}}\rangle \, \langle h_{\boldsymbol{z}}, g\rangle \tag{27}$$

for all $f, g \in L_2(\mathbb{R}^d)$, where $h_{\boldsymbol{z}}(x) = h\big(x - \frac{\pi}{M}\boldsymbol{z}\big)$; also,

$$\lim_n \Big\lVert Qf - \sum_{\lvert\boldsymbol{z}\rvert<n} \langle f, h_{\boldsymbol{z}}\rangle h_{\boldsymbol{z}} \Big\rVert_{L_2(\mathbb{R}^d)} = 0. \tag{28}$$

Proof. The identity (27) follows directly from the decomposition of both $\widehat{f}(\xi)\overline{\widehat{h}(\xi)}$ and $\widehat{g}(\xi)\overline{\widehat{h}(\xi)}$ with respect to the following orthonormal basis of $L_2\big([-M,M]^d\big)$,

$$\big\{(2M)^{-d/2} e^{-i(\pi\boldsymbol{z}/M)\cdot\xi} : \boldsymbol{z} \in \mathbb{Z}^d\big\}.$$

The convergence in (28) is proven applying the identity (27) to the last line below:

$$\begin{aligned}
&\left\| Qf - \sum_{|\boldsymbol{z}|<n} \langle f, h_{\boldsymbol{z}}\rangle h_{\boldsymbol{z}} \right\|_{L_2(\mathbb{R}^d)} \\
&= \sup_{\substack{g\in L_2(\mathbb{R}^d)\\ \|g\|_{L_2(\mathbb{R}^d)}=1}} \left| \left\langle Qf - \sum_{|\boldsymbol{z}|<n} \langle f, h_{\boldsymbol{z}}\rangle h_{\boldsymbol{z}},\, g \right\rangle \right| \\
&= \sup_{\substack{g\in L_2(\mathbb{R}^d)\\ \|g\|_{L_2(\mathbb{R}^d)}=1}} \left| \langle Qf, g\rangle - \sum_{|\boldsymbol{z}|<n} \langle f, h_{\boldsymbol{z}}\rangle \langle h_{\boldsymbol{z}}, g\rangle \right|. \qquad \square
\end{aligned}$$

Theorem 1.2. *If the smooth functions $W \in C_c^\infty(0,\infty)$, $V \in C_c^\infty(\mathbb{R})$ used in the construction of the curvelets $\gamma_{\alpha\beta\theta}$ satisfy the admissibility conditions given in Lemma 1.13, then the family $\{\phi_{n\boldsymbol{z}k}(x) : n \in \mathbb{Z}; k = 1,\dots,\eta_n; \boldsymbol{z} \in \mathbb{Z}^2\}$ of functions*

$$\phi_{n\boldsymbol{z}k}(x) = \alpha_0^{n/2}(\log\alpha_0)^{1/2}\tau_n^{1/2}\Phi_{n\beta k}(x) \text{ (with } \beta = \pi\alpha_0^{-n-1}\boldsymbol{z}\text{)},$$

is a tight frame in $L_2(\mathbb{R}^2)$ with frame bound 1: For $f \in L_2(\mathbb{R}^2)$,

$$\sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^2 = \|f\|^2_{L_2(\mathbb{R}^2)}. \tag{29}$$

Proof. This is a direct consequence of identities (26) and (27), since

$$\int_{\mathbb{R}^2} |\langle f, \Phi_{n\beta k}\rangle|^2 d\beta = \langle g_{nk}, f\rangle.$$

It is then:

$$\begin{aligned}
\tfrac{1}{\log\alpha_0}\|f\|^2_{L_2(\mathbb{R}^2)} &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n \int_{\mathbb{R}^2} |\langle f, \Phi_{n\beta k}\rangle|^2\, d\beta \\
&= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \alpha_0^n \tau_n \,\langle g_{nk}, f\rangle \\
&= \sum_{n\in\mathbb{Z}} \alpha_0^n \tau_n \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \Phi_{n\boldsymbol{0}k}(\cdot - \pi\alpha_0^{-n-1}\boldsymbol{z})\rangle|^2. \qquad \square
\end{aligned}$$

Associated to this frame (see Christensen [5]), the set of indices and corresponding space of square-summable sequences over those indices are denoted respectively by

$$\mathbb{F} = \{(n, \boldsymbol{z}, k) \in \mathbb{Z}^4 : n \in \mathbb{Z}, 1 \le k \le \eta_n, \boldsymbol{z} \in \mathbb{Z}^2\},$$

$$\ell_2(\mathbb{F}) = \Big\{ (c_{n,k,\boldsymbol{z}})_{(n,k,\boldsymbol{z})\in\mathbb{F}} : c_{n,k,\boldsymbol{z}} \in \mathbb{C}, \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |c_{n,k,\boldsymbol{z}}|^2 < \infty \Big\}.$$

For this frame, the *synthesis operator* $T\colon \ell_2(\mathbb{F}) \to L_2(\mathbb{R}^2)$ is given by

$$T\{c_{n\boldsymbol{z}k}\}_{(n,\boldsymbol{z},k)\in\mathbb{F}} = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} c_{n\boldsymbol{z}k}\phi_{n\boldsymbol{z}k}.$$

Its adjoint, the *analysis operator* $T^*\colon L_2(\mathbb{R}^2) \to \ell_2(\mathbb{F})$ is given by

$$T^* f = \{\langle f, \phi_{n\boldsymbol{z}k}\rangle\}_{(n,\boldsymbol{z},k)\in\mathbb{F}},$$

and thus the *frame operator* $S\colon L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ is given by

$$Sf = TT^* f = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} \langle f, \phi_{n\boldsymbol{z}k}\rangle \phi_{n\boldsymbol{z}k}.$$

Notice that, as with all tight frames with frame bound 1, the frame operator is simply the identity.

Corollary 1.2.1. *Under the same hypotheses as in Theorem 1.2, the following series converges for all $f \in L_2(\mathbb{R}^2)$, and all permutation $\varsigma\colon \mathbb{F} \to \mathbb{F}$:*

$$f = \sum_{(n,\boldsymbol{z},k)\in\mathbb{F}} \langle f, \phi_{\varsigma(n,\boldsymbol{z},k)}\rangle\, \phi_{\varsigma(n,\boldsymbol{z},k)}.$$

In particular,

$$f = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} \langle f, \phi_{n\boldsymbol{z}k}\rangle \phi_{n\boldsymbol{z}k}, \tag{30}$$

Proof. This is consequence of Theorem 5.1.6 in Christensen [5]. □

Proposition 1.7. *The tight frame $\{\phi_{n\boldsymbol{z}k}\}_{(n,\boldsymbol{z},k)\in\mathbb{F}}$ is not a Riesz basis of $L_2(\mathbb{R}^2)$.*

Proof. For any index $(n_0, \boldsymbol{z}_0, k_0) \in \mathbb{F}$, it is $\langle \phi_{n_0\boldsymbol{z}_0k_0}, \phi_{n\boldsymbol{z}k}\rangle = 0$ for indices $(n, \boldsymbol{z}, k) \in \mathbb{F}$ with $n < n_0 - 1$ or $n > n_0 + 1$, since the supports of the Fourier transform of the respective curvelets are disjoint. By (30),

$$\phi_{n_0\boldsymbol{z}_0k_0} = \sum_{n=n_0-1}^{n_0+1} \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} \langle \phi_{n_0\boldsymbol{z}_0k_0}, \phi_{n\boldsymbol{z}k}\rangle \phi_{n\boldsymbol{z}k}.$$

Notice that in particular,

$$\begin{aligned}\langle \phi_{n_0\boldsymbol{z}_0k_0}, \phi_{n_0\boldsymbol{z}_0k_0}\rangle &= \alpha_0^{n_0}(\log\alpha_0)\tau_{n_0} \big\| \gamma_{\alpha_0^{n_0}(\pi\alpha_0^{-n_0-1}\boldsymbol{z})\theta_{n_0,k_0}} \big\|_{L_2(\mathbb{R}^2)}^2 \\ &= C_{V,W}\,(\log\alpha_0)\,\alpha_0^{2n_0}\tau_{n_0} \le C_{V,W}\,(\log\alpha_0)\alpha_0^{2n}\tau(\alpha_0^n)\end{aligned}$$

As $\alpha_0 > 1$ and $\lim_{\alpha\to 0}\tau(\alpha) = 0$, there exists $N \in \mathbb{N}$ such that $C_{V,W}(\log\alpha_0)\alpha_0^n\tau_n < 1$ for all $n \le -N$. In this case, choose $n_0 \le -N$, and consider the sum

$$\sum_{n=n_0-1}^{n_0+1} \sum_{k=1}^{\eta_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} c_{n\boldsymbol{z}k}\phi_{n\boldsymbol{z}k},$$

with $c_{n_0\boldsymbol{z}_0k_0} = \|\phi_{n_0\boldsymbol{z}_0k_0}\|_{L_2(\mathbb{R}^2)}^2 - 1$, $c_{n\boldsymbol{z}k} = \langle \phi_{n_0\boldsymbol{z}_0k_0}, \phi_{n\boldsymbol{z}k}\rangle$ otherwise. Notice this sum converges to zero, but $c_{n_0\boldsymbol{z}_0k_0} \ne 0$; therefore, $\{\phi_{n\boldsymbol{z}k}\}_{(n,\boldsymbol{z},k)\in\mathbb{F}}$ is not an ω-independent sequence in $L_2(\mathbb{R}^2)$, and by Theorem 6.1.1 in Christensen [5], it is not a Riesz basis. □

2. APPROXIMATION PROPERTIES OF CURVELET FRAMES

2.1. **Characterization of Regularity.**

2.1.1. *Lipschitz Regularity.* Holschneider and Tchamitchian show in [9] how to analyze the regularity $0 < s \le 1$ of a Hölder function $f \in \mathrm{Lip}\big(s, \mathbb{R}\big)$, by means of the decreasing rate of its wavelet coefficients. Propositions 2.1 and 2.2 and below present similar results for functions $f \in \mathrm{Lip}\big(s, \mathbb{R}^2\big)$, employing curvelets instead.

Lemma 2.1. *For $0 < s \le 1/2$, and any curvelet $\gamma_{\alpha\beta\theta}$, there exists a constant $C = C(V, W, \alpha_0) > 0$ such that the following estimate holds for all $h \in \mathbb{R}^2$ with $|h| \le 1$:*

$$\int_{\mathbb{R}^2} |\gamma_{\alpha\beta\theta}(x+h) - \gamma_{\alpha\beta\theta}(x)| \, dx \le C_{V,W,\alpha_0} \alpha^{-5/2} \tau(\alpha)^{-7/2} |h|^s. \tag{31}$$

Proof. Given $h \in \mathbb{R}^2$, $D_h \gamma_{\alpha\mathbf{01}}(x) = \int_{\mathbb{R}^2} -2\pi i (h \cdot \xi) \, \widehat{\gamma}_{\alpha\mathbf{01}}(\xi) \, e^{2\pi i x \cdot \xi} \, d\xi$; hence,

$$\begin{aligned}
|D_h \gamma_{\alpha\mathbf{01}}(x)| &\le 2\pi \int_{\mathbb{R}^2} |h \cdot \xi| \, |\widehat{\gamma}_{\alpha\mathbf{01}}(\xi)| \, d\xi \\
&= 2\pi \int_0^\infty \int_{-\pi}^{\pi} \rho |h \cdot (\cos\omega, \sin\omega)| \, \tfrac{1}{\alpha^{1/2}} W\big(\tfrac{\rho}{\alpha}\big) \, \tfrac{1}{\tau(\alpha)^{1/2}} V\big(\tfrac{\omega}{\tau(\alpha)}\big) \, d\omega \, \rho d\rho \\
&\le 2\pi |h| \alpha^{5/2} \, \tau(\alpha)^{1/2} \Big(\int_0^\infty r^2 \, W(r) \, dr \Big) \Big(\int_{-1}^{1} V(\zeta) \, d\zeta \Big), \qquad \big(r = \tfrac{\rho}{\alpha}, \zeta = \tfrac{\omega}{\tau(\alpha)}\big)
\end{aligned}$$

thus proving $\gamma_{\alpha\mathbf{01}} \in \mathrm{Lip}\big(1, \mathbb{R}^2\big)$, with

$$|\gamma_{\alpha\mathbf{01}}(x+h) - \gamma_{\alpha\mathbf{01}}(x)| \le C_{V,W} \, \alpha^{5/2} \, \tau(\alpha)^{1/2} \, |h|. \tag{32}$$

The same estimate holds for the general curvelet, since

$$\gamma_{\alpha\beta\theta}(x+h) - \gamma_{\alpha\beta\theta}(x) = \gamma_{\alpha\mathbf{01}}\big(R_\theta(x-\beta) - R_\theta h\big) - \gamma_{\alpha\mathbf{01}}\big(R_\theta(x-\beta)\big).$$

For a vector $h \in \mathbb{R}^2$ with $|h| < 1$, and any value $0 < s \le 1/2$,

$$\int_{\mathbb{R}^2} |\gamma_{\alpha\beta\theta}(x+h) - \gamma_{\alpha\beta\theta}(x)| \, dx = \int_{\mathbb{R}^2} |\gamma_{\alpha\mathbf{01}}(y + R_\theta h) - \gamma_{\alpha\mathbf{01}}(y)| \, dy.$$

Set $\zeta(h) = |h|^{-s/2} + |h|$, and

$$\begin{aligned}
\mathbf{I} &= \int_{|y| \le \zeta(h)} |\gamma_{\alpha\mathbf{01}}(y + R_\theta h) - \gamma_{\alpha\mathbf{01}}(y)| \, dy, \\
\mathbf{II} &= \int_{|y| > \zeta(h)} |\gamma_{\alpha\mathbf{01}}(y + R_\theta h) - \gamma_{\alpha\mathbf{01}}(y)| \, dy.
\end{aligned}$$

It is then $\int_{\mathbb{R}^2} |\gamma_{\alpha\mathbf{01}}(y + R_\theta h) - \gamma_{\alpha\mathbf{01}}(y)| \, dy = \mathbf{I} + \mathbf{II}$, and

$$\begin{aligned}
\mathbf{I} &\le C_{V,W} \alpha^{5/2} \tau(\alpha)^{1/2} \, |h| \big(|h|^{-s/2} + |h|\big)^2 && \big(\text{by } (32)\big) \\
&= C_{V,W} \alpha^{5/2} \, \tau(\alpha)^{1/2} \, \big(|h|^{1-s} + |h|^3 + 2|h|^{2-s/2}\big) \\
&\le C_{V,W} \alpha^{5/2} \tau(\alpha)^{1/2} |h|^s && \big(\text{since } |h| \le 1, s \le 1/2\big)
\end{aligned}$$

$$\begin{aligned}
\mathbf{II} &\le \int_{|y| > \zeta(h)} |\gamma_{\alpha\mathbf{01}}(y + R_\theta h)| \, dy + \int_{|y| > \zeta(h)} |\gamma_{\alpha\mathbf{01}}(y)| \, dy \\
&\le C_{V,W,\alpha_0} \, \alpha^{-5/2} \tau(\alpha)^{-7/2} \int_{|y| > \zeta(h)} \Big(\frac{1}{|y+h|^4} + \frac{1}{|y|^4} \Big) \, dy \quad \big(\text{by } (3) \text{ with } k = 2\big) \\
&\le C_{V,W,\alpha_0} \, \alpha^{-5/2} \tau(\alpha)^{-7/2} \int_{|y| > |h|^{-s/2}} \frac{dy}{|y|^4} \\
&\le C_{V,W,\alpha_0} \, \alpha^{-5/2} \tau(\alpha)^{-7/2} |h|^s.
\end{aligned}$$

Finally, notice that $\alpha^{-5/2}\tau(\alpha)^{-7/2} \geq 1 \geq \alpha^{5/2}\tau(\alpha)^{1/2}$ for all $\alpha > 0$. This proves the statement. $\square$

Proposition 2.1. *Given $0 < s \leq 1$, if $f \in \operatorname{Lip}\big(s, \mathbb{R}^2\big)$, then its curvelet coefficients satisfy $|\langle f, \gamma_{\alpha\beta\theta}\rangle| \leq C(V, W, \alpha_0, f)\, \alpha^{-s-1/2}\tau(\alpha)^{-s-3/2}$ for all $\alpha > 0$, $\beta \in \mathbb{R}^2$ and $\theta \in \mathbb{S}^1$.*

Proof. Since $\int_{\mathbb{R}^2} \gamma_{\alpha\beta\theta}(x)\, dx = 0$, it is

$$\begin{aligned}
|\langle f, \gamma_{\alpha\beta\theta}\rangle| &= \Big| \int_{\mathbb{R}^2} \big(f(x) - f(\beta)\big)\, \overline{\gamma_{\alpha\beta\theta}(x)}\, dx \Big| \\
&\leq \int_{\mathbb{R}^2} |f(x) - f(\beta)|\, |\gamma_{\alpha\beta\theta}(x)|\, dx \\
&\leq C_f \int_{\mathbb{R}^2} |x - \beta|^s |\gamma_{\alpha\beta\theta}(x)|\, dx && \big(f \in \operatorname{Lip}\big(s, \mathbb{R}^2\big)\big) \\
&= C_f \int_{\mathbb{R}^2} |y|^s |\gamma_{\alpha\mathbf{01}}(y)|\, dy. && \big(y = x - \beta\big)
\end{aligned}$$

Set $\zeta(\alpha) = \alpha^{-1}\tau(\alpha)^{-1}$, and

$$\mathbf{I} = \int_{|y| \leq \zeta(\alpha)} |y|^s |\gamma_{\alpha\mathbf{01}}(y)|\, dy, \qquad \mathbf{II} = \int_{|y| > \zeta(\alpha)} |y|^s |\gamma_{\alpha\mathbf{01}}(y)|\, dy.$$

Notice that $\int_{\mathbb{R}^2} |y|^s |\gamma_{\alpha\mathbf{01}}(y)|\, dy = \mathbf{I} + \mathbf{II}$, and

$$\begin{aligned}
\mathbf{I} &\leq \zeta(\alpha)^s \|\gamma_{\alpha\mathbf{01}}\|_{L_\infty(\mathbb{R}^2)} \pi \zeta(\alpha)^2 \\
&\leq C_{V,W,\alpha_0}\, \alpha^{-s-1/2}\tau(\alpha)^{-s-3/2}, && \big(\text{by } (4)\big) \\
\mathbf{II} &\leq C_{V,W,\alpha_0}\, \alpha^{-5/2}\tau(\alpha)^{-7/2} \int_{|y| > \zeta(\alpha)} |y|^{s-4}\, dy && \big(\text{by } (3) \text{ with } k = 2\big) \\
&\leq \frac{C_{V,W,\alpha_0}}{2 - s} \alpha^{-s-1/2}\tau(\alpha)^{-s-3/2} \\
&\leq C_{V,W,\alpha_0} \alpha^{-s-1/2}\tau(\alpha)^{-s-3/2}. && \big(\text{since } 0 < s \leq 1\big)
\end{aligned}$$

$\square$

Proposition 2.2. *Let $\{\gamma_{\alpha\beta\theta}(x) : \alpha > 0, \beta \in \mathbb{R}^2, \theta \in \mathbb{S}^1\}$ be a family of curvelets for which the aspect-ratio weight function satisfies the following condition: there exists $A > 0$ and $0 < r < 1/7$ such that $\tau(\alpha) \geq A\alpha^{-r}$ for $\alpha > m_\tau$. Let $f \in L_2(\mathbb{R}^2)$, and let f_{LG} denote the* "large curvelet scales" *of f:*

$$f_{LG}(x) = \int_{m_\tau}^{\infty} \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} \langle f, \gamma_{\alpha\beta\theta}\rangle \gamma_{\alpha\beta\theta}(x)\, d\beta\, d\sigma(\theta)\, d\alpha.$$

If for some $3r/(1-r) < s \leq 1/2$ and $M > 0$ the curvelet coefficients of a function $f \in L_2(\mathbb{R}^2)$ satisfy

$$|\langle f, \gamma_{\alpha\beta\theta}\rangle| \leq M\alpha^{-s-1/2}\tau(\alpha)^{-s-3/2}, \tag{33}$$

then $f_{LG} \in \operatorname{Lip}\big(s, \mathbb{R}^2\big)$.

Proof. Notice f_{LG} is uniformly bounded in x:

$$\begin{aligned}
|f_{LG}(x)| &\leq \int_{m_\tau}^{\infty} \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} |\langle f, \gamma_{\alpha\beta\theta}\rangle|\, |\gamma_{\alpha\beta\theta}(x)|\, d\beta\, d\sigma(\theta)\, d\alpha \\
&\leq \int_{m_\tau}^{\infty} M\, \alpha^{-s-1/2}\tau(\alpha)^{-s-3/2} \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} |\gamma_{\alpha\mathbf{0}\theta}(x - \beta)|\, d\beta\, d\sigma(\theta)\, d\alpha \quad \big(\text{by } (33)\big)
\end{aligned}$$

$$\le M\, C_{V,W,\alpha_0} \int_{m_\tau}^{\infty} \alpha^{-s-1/2}\tau(\alpha)^{-s-3/2}\alpha^{-1/2}\tau(\alpha)^{-3/2}\, d\alpha \qquad \big(\text{by (4), } p=1\big)$$

$$\le M\, C_{V,W,\alpha_0} \int_{m_\tau}^{\infty} \alpha^{-s-1}\tau(\alpha)^{-s-3}\, d\alpha.$$

But by hypothesis, $\alpha^{-s-1}\tau(\alpha)^{-s-3} \le A^{-s-3}\alpha^{r(s+3)-s-1}$ for all $\alpha > m_\tau$; thus

$$\int_{m_\tau}^{\infty} \alpha^{-s-1}\tau(\alpha)^{-s-3}\, d\alpha \le A^{-s-3}\int_{m_\tau}^{\infty} \alpha^{r(s+3)-s-1}\, d\alpha = A^{-s-3}\frac{m_\tau^{r(s+3)-s}}{s-r(s+3)} < \infty,$$

since $r(s+3)-s<0$. This implies f_{LG} is uniformly bounded, and the constant depends at most on V, W, α_0, m_τ, M, A, r and s.

Now, given $h\in\mathbb{R}^2$ with $|h|<1$,

$$|f_{LG}(x+h)-f_{LG}(x)| \le \int_{m_\tau}^{\infty}\int_{\mathbb{S}^1}\int_{\mathbb{R}^2} |\langle f,\gamma_{\alpha\beta\theta}\rangle||\gamma_{\alpha\beta\theta}(x+h)-\gamma_{\alpha\beta\theta}(x)|\, d\beta d\sigma(\theta)\, d\alpha$$

$$\le M\, C_{V,W,\alpha_0}|h|^s \int_{m_\tau}^{\infty} \alpha^{-s-1/2}\tau(\alpha)^{-s-3/2}\alpha^{-5/2}\tau(\alpha)^{-7/2}\, d\alpha, \tag{34}$$

$$= C(V,W,\alpha_0,M)|h|^s \int_{m_\tau}^{\infty} \alpha^{-s-3}\tau(\alpha)^{-s-5}\, d\alpha$$

$$\le C(V,W,\alpha_0,M,A,s)|h|^s \int_{m_\tau}^{\infty} \alpha^{r(s+5)-s-3}\, d\alpha \tag{35}$$

$$\le C(V,W,\alpha_0,M,A,s,r,m_\tau)|h|^s. \tag{36}$$

Inequality (34) followed from both (33) on $|\langle f,\gamma_{\alpha\beta\theta}\rangle|$, and (31) on the inner-most integral. The step (35) is direct by the hypothesis $\tau(\alpha)\ge A\alpha^{-r}$. Finally, (36) follows from the fact that $r(s+5)-s-2<0$ as a consequence of the hypothesis $3r/(1-r)<s\le 1$. Indeed, for $0<r<1/7$, $3r>5r-2$, and furthermore, $s>3r/(1-r)>(5r-2)/(1-r)$.

This holds for all $|h|<1$; together with the bound on $|f_{LG}|$ computed above, it follows that $|f_{LG}(x+h)-f_{LG}(x)|\le C|h|^s$ for all h, uniformly in x. □

2.1.2. *Besov Regularity.* The ideas for this section come from Borup and Nielsen [1], in which the authors develop a new construction of tight frames for $L_2(\mathbb{R}^2)$ with flexible time-frequency localization, and adapt those frames to form atomic decompositions for several smoothness spaces on $\mathbb{R}^d$.

Lemma 2.2. *Let* $\alpha_0>1$*;* V,W *admissible windows in the sense of Lemma 1.13, and assume there exist constants* $0<M_1\le M_2$ *such that the aspect-ratio weight function* $\tau\colon(0,\infty)\to(0,\frac{\pi}{4})$ *satisfies for all* $m\in\mathbb{Z}$*,*

$$M_1 \le \frac{\tau(\alpha_0^m)}{\tau(\alpha_0^{m+1})}\le M_2, \qquad\qquad \alpha_0^m\tau_m\le 1.$$

Consider the corresponding tight frame $\{\phi_{n\boldsymbol{z}k} : (n,\boldsymbol{z},k)\in\mathbb{F}\}$*.*

For each pair of indices (m,k) *with* $m\in\mathbb{Z}$ *and* $k=1,\dots,\eta_m$*, set* $A_m=\begin{pmatrix}\alpha_0^m & 0\\ 0 & 4\tau_n/\pi\end{pmatrix}$*,* $b_{mk}=\big(0,(k-1)\tau_m\big)$*, and* $T_{mk}\colon\mathbb{R}^2\to\mathbb{R}^2$ *the affine map given by* $T_{mk}\boldsymbol{x}=A_m\boldsymbol{x}+b_{mk}$*. Let* $\mathfrak{T}=\{T_{mk}: m\in\mathbb{Z}; k=1,\dots,\eta_m\}$*. Then* $\mathfrak{Q}=\{Q_{mk}=\operatorname{supp}\widehat{\phi}_{m\mathbf{0}k} : m\in\mathbb{Z}; k=1,\dots,\eta_m\}$ *is an admissible covering of* $\mathbb{R}^2\setminus\{\mathbf{0}\}$ *structured by* $\mathfrak{T}$*, and the family* $\mathfrak{F}=\{|\widehat{\phi}_{m\mathbf{0}k}|^2 : m\in\mathbb{Z}; k=1,\dots,\eta_m\}$ *is a bounded admissible partition of unity associated to* $\mathfrak{Q}$*.*

Proof. Let $\Lambda = \{(m,k) : m \in \mathbb{Z}; k = 1, \dots, \eta_m\}$, and consider in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ the open bounded set $Q = \{\xi \in \mathbb{R}^2 : \alpha_0^{-1} < |\xi| < \alpha_0, 0 < \arg \xi < \pi/2\}$. Notice that for each $m \in \mathbb{Z}$; $k = 1, \dots, \eta_m$:

$$\begin{aligned} T_{mk}Q &= \{A_m\left(\begin{smallmatrix}\rho\\ \omega\end{smallmatrix}\right) + b_{mk} : \alpha_0^{-1} < \rho < \alpha_0, 0 < \omega < \pi/2\} \\ &= \{\left(\alpha_0^m \rho, \tfrac{4\tau_n}{\pi}\omega + (k-1)\tau_m\right) : \alpha_0^{-1} < \rho < \alpha_0, 0 < \omega < \pi/2\} \\ &= \{(\rho, \omega) : \alpha_0^{m-1} < \rho < \alpha_0^{m+1}, (k-1)\tau_m < \omega < (k+1)\tau_m\} \\ &= \operatorname{supp} \phi_{m\mathbf{0}k} = Q_{mk}. \end{aligned}$$

Abusing notation, the size of the cubes Q_{mk} is computed with respect to the polar coordinate system, and not the rectangular one; thus, $|Q_{mk}| = 2(\alpha_0 - \alpha_0^{-1})\alpha_0^n \tau_n$.

Given $\xi \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, it is $\xi \in Q_{mk}$, with $m = \lfloor \log_{\alpha_0} |\xi| \rfloor$, and $k = \lfloor (\arg \xi)\eta_m/(2\pi) \rfloor$; therefore, $\mathbb{R}^2 \setminus \{\mathbf{0}\} \subset \cup_{(m,k)\in\Lambda} Q_{mk}$.

Given $m_0 \in \mathbb{Z}$, $k_0 \in \{1, \dots, \eta_{m_0}\}$, if $|m - m_0| > 2$, then trivially $Q_{m_0k_0} \cap Q_{mk} = \emptyset$ for all $k \in \{1, \dots, \eta_m\}$. Also, $Q_{m_0k} \cap Q_{m_0k'} = \emptyset$ provided $[k']_{\eta_{m_0}} \notin \{[k-1]_{\eta_{m_0}}, [k]_{\eta_{m_0}}, [k+1]_{\eta_{m_0}}\}$. It is then

$$\begin{aligned} &|\{(m,k) \in \Lambda : Q_{mk} \cap Q_{m_0k_0} \neq \emptyset\}| \\ &\qquad = 3 + |\{1 \le k \le \eta_{m_0+1} : Q_{(m_0+1)k} \cap Q_{m_0k_0} \neq \emptyset\}| \\ &\qquad\quad + |\{1 \le k \le \eta_{m_0-1} : Q_{(m_0-1)k} \cap Q_{m_0k_0} \neq \emptyset\}| \\ &\qquad \le 3 + \left\lceil \tfrac{\eta_{m_0}}{\eta_{m_0+1}} \right\rceil + \left\lceil \tfrac{\eta_{m_0}}{\eta_{m_0-1}} \right\rceil \\ &\qquad = 3 + \left\lceil \frac{\lceil 2\pi\tau(\alpha_0^{m_0})^{-1} \rceil}{\lceil 2\pi\tau(\alpha_0^{m_0+1})^{-1} \rceil} \right\rceil + \left\lceil \frac{\lceil 2\pi\tau(\alpha_0^{m_0})^{-1} \rceil}{\lceil 2\pi\tau(\alpha_0^{m_0-1})^{-1} \rceil} \right\rceil \\ &\qquad \le 5 + \frac{\lceil 2\pi\tau(\alpha_0^{m_0})^{-1} \rceil}{\lceil 2\pi\tau(\alpha_0^{m_0+1})^{-1} \rceil} + \frac{\lceil 2\pi\tau(\alpha_0^{m_0})^{-1} \rceil}{\lceil 2\pi\tau(\alpha_0^{m_0-1})^{-1} \rceil} \\ &\qquad \le 5 + \left\lceil \frac{2\pi\tau(\alpha_0^{m_0})^{-1}}{2\pi\tau(\alpha_0^{m_0+1})^{-1}} \right\rceil + \left\lceil \frac{2\pi\tau(\alpha_0^{m_0})^{-1}}{2\pi\tau(\alpha_0^{m_0-1})^{-1}} \right\rceil \\ &\qquad = 5 + \left\lceil \frac{\tau(\alpha_0^{m_0+1})}{\tau(\alpha_0^{m_0})} \right\rceil + \left\lceil \frac{\tau(\alpha_0^{m_0-1})}{\tau(\alpha_0^{m_0})} \right\rceil \le 5 + M_1 + M_2, \end{aligned}$$

and $\mathfrak{Q} = \{T_{mk}Q : (m,k) \in \Lambda\}$ is indeed an admissible covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, with $N(\mathfrak{Q}) \le 5 + M_1 + M_2$.

Let $0 < \varepsilon < \pi/4$, and consider the open bounded set $Q_\varepsilon^\star = \{\xi \in \mathbb{R}^2 : \alpha_0^{-1} < |\xi| < \alpha_0, 0 < \arg \xi < \pi/4 + \varepsilon\}$, which is compactly contained in Q, also satisfies $\cup_{(m,k)\in\Lambda} T_{mk} Q_\varepsilon^\star = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and for any $(m_0, k_0) \in \Lambda$, $|\{(m,k) \in \Lambda : Q_{mk} \cap Q_{m_0k_0} = \emptyset\}| \le 5 + M_1 + M_2$ (the proof of these facts are identical to the previous and are left to the reader). It is thus $\{T_{mk}Q_\varepsilon^\star : (m,k) \in \Lambda\}$ an admissible covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

To show that $\mathfrak{Q}$ is an admissible covering structured by $\mathfrak{T}$, it only remains to prove that there exists a constant $M > 0$ such that $\left\| A_{m'}^{-1} A_m \right\|_{\ell_\infty} \le M$ for indices $(m,k), (m',k') \in \Lambda$ for which $T_{mk}Q \cap T_{m'k'}Q \neq \emptyset$ holds. But notice that for $|m' - m| \le 1$ it is

$$\begin{aligned} A_{m'}^{-1} A_m &= \begin{pmatrix} \alpha_0^{-m'} & 0 \\ 0 & \tau_{m'}^{-1}\pi/4 \end{pmatrix} \begin{pmatrix} \alpha_0^m & 0 \\ 0 & 4\tau_m/\pi \end{pmatrix} = \begin{pmatrix} \alpha_0^{m-m'} & 0 \\ 0 & \tau_m \tau_{m'}^{-1} \end{pmatrix}. \\ \left\| A_{m'}^{-1} A_m \right\|_{\ell_\infty} &= \max\{\alpha_0^{m-m'}, \tau_m \tau_{m'}^{-1}\} \le \max\{\alpha_0, \lceil \tau(\alpha_0^m)\tau(\alpha_0^{m'})^{-1} \rceil\} \\ &\le \max\{\alpha_0, M_2\}. \end{aligned}$$

For each $(m,k) \in \Lambda$, set $\psi_{nk}\colon \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ given by $\psi_{nk}(\xi) = |\widehat{\phi}_{m\mathbf{0}k}|^2$. The family $\mathfrak{F} = \{\psi_{mk} : (m,k) \in \Lambda\}$ satisfies the first condition to be a bounded admissible partition of unity associated to $\mathfrak{Q}$ ($\operatorname{supp}\psi_{nk} \subset Q_{mk}$ for all $(m,k) \in \Lambda$) by definition of $\widehat{\phi}_{m\mathbf{0}k}$. The second condition ($\mathfrak{F}$ is a partition of unity) is a direct consequence of Lemma 1.13: Given $\xi \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, let $m_0 = \lfloor \log_{\alpha_0}|\xi| \rfloor$, and $k_0(m,\xi) = \lfloor (\arg\xi) 2\pi/\eta_m \rfloor$. It is then

$$\sum_{m\in\mathbb{Z}} \sum_{k=1}^{\eta_m} \psi_{mk}(\xi) = \log\alpha_0 \sum_{m=m_0}^{m_0+1} \sum_{k=k_0(m,\xi)}^{k_0(m,\xi)+1} W\big(\tfrac{|\xi|}{\alpha_0^m}\big)^2 V\big(\tfrac{\arg\xi - \theta_{mk}}{\tau_m}\big)^2 = 1.$$

The third condition is also satisfied. To prove it, a few estimates are needed:

(i) For all $0 < p \le \infty$, $(m,k) \in \Lambda$ and $\boldsymbol{z} \in \mathbb{Z}^2$, there exists $C = C(V,W,\alpha_0,p) > 0$ such that

$$\|\phi_{n\boldsymbol{z}k}\|_{L_p(\mathbb{R}^2)} = C(V,W,\alpha_0)\big(\alpha_0^n\tau_n\big)^{3/2-1/p}. \tag{37}$$

This follows from the definition of $\phi_{n\boldsymbol{z}k}$, and the fact that $\widehat{\phi}_{n\mathbf{0}k}(\xi) = \widehat{\phi}_{\mathbf{0}\mathbf{0}\mathbf{1}}(T_{nk}\xi)$:

$$\begin{aligned}\|\phi_{n\boldsymbol{z}k}\|_{L_p(\mathbb{R}^2)} &= |\det T_{nk}|^{1-1/p}\|\phi_{\mathbf{0}\mathbf{0}\mathbf{1}}\|_{L_p(\mathbb{R}^2)} && \big(\text{by [1, Lemma 1]}\big)\\ &= \big(4\alpha_0^n\tau_n/\pi\big)^{1-1/p}\big(\alpha_0^{n/2}(\log\alpha_0)^{1/2}\tau_n^{1/2}\big)\|\gamma_{\mathbf{1}\mathbf{0}\mathbf{1}}\|_{L_p(\mathbb{R}^2)}. && \big(\text{by Theorem 1.2}\big)\end{aligned}$$

(ii) For all $0 < p \le \infty$ and $(n,k) \in \Lambda$,

$$\|\mathcal{F}^{-1}\psi_{nk}\|_{L_p(\mathbb{R}^2)} = \|\phi_{n\mathbf{0}k} * \phi_{n\mathbf{0}k}\|_{L_p(\mathbb{R}^2)}. \tag{38}$$

(iii) The space $L_p(\mathbb{R}^2)^K$ of functions in $L_p(\mathbb{R}^2)$ with frequencies compactly supported on a given compact set K is a quasi-normed convolution algebra; therefore, as $\operatorname{supp}\psi_{nk} = Q_{nk}$, by [10, Proposition 1.5.3], there exists $C = C_p > 0$ such that

$$\begin{aligned}\|\phi_{n\mathbf{0}k} * \phi_{n\mathbf{0}k}\|_{L_p(\mathbb{R}^2)} &\le C_p\,|Q_{nk}|^{1/p-1}\|\phi_{n\mathbf{0}k}\|^2_{L_p(\mathbb{R}^2)}\\ &= C(\alpha_0,p)\big(\alpha_0^n\tau_n\big)^{1/p-1}\|\phi_{n\mathbf{0}k}\|^2_{L_p(\mathbb{R}^2)}\end{aligned} \tag{39}$$

It is then, for $0 < p \le 1$, $(n,k) \in \Lambda$,

$$\begin{aligned}|Q_{nk}|^{1/p-1}\|\mathcal{F}^{-1}\psi_{nk}\|_{L_p(\mathbb{R}^2)} &\le C_p\,|Q_{nk}|^{2(1/p-1)}\|\phi_{n\mathbf{0}k}\|^2_{L_p(\mathbb{R}^2)} && \big(\text{by (38) and (39)}\big)\\ &= C(V,W,\alpha_0,p)\big(\alpha_0^n\tau_n\big)^{1/p-1/2}, && \big(\text{by (37)}\big)\end{aligned}$$

and the statement follows from the hypothesis that $\alpha_0^n\tau_n \le 1$ for $n \in \mathbb{Z}$, since $1/p - 1/2 > 1/2$: For $0 < p \le 1$,

$$\sup_{(n,k)\in\Lambda} |Q_{nk}|^{1/p-1}\|\mathcal{F}^{-1}\psi_{nk}\|_{L_p(\mathbb{R}^2)} < \infty. \qquad \square$$

Remark 2.1. Consider a family of curvelets satisfying the hypotheses of Lemma 2.2, and with $\alpha_0 = 2$. Set $\psi_n = \sum_{k=1}^{\eta_n}|\widehat{\phi}_{n\mathbf{0}k}|^2$ for all $n \in \mathbb{Z}$. Notice:

(i) Since $\operatorname{supp}\widehat{\phi}_{n\mathbf{0}k} = \{\xi \in \mathbb{R}^2 : 2^{n-1} \le |\xi| \le 2^{n+1}, |\arg\xi - \theta_{nk}| \le \tau_n\}$, it is

$$\operatorname{supp}\psi_n = \{\xi \in \mathbb{R}^2 : 2^{n-1} \le |x| \le 2^{n+1}\};$$

therefore, the family of the interiors of the previous annuli, $\mathfrak{Q}'$, is an admissible covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ with $N(\mathfrak{Q}') = 2$.

$$\mathfrak{Q}' = \big\{Q'_n = B_2\big(\mathbf{0}, 2^{n+1}\big) \setminus \overline{B_2\big(\mathbf{0}, 2^{n-1}\big)} : n \in \mathbb{Z}\big\}.$$

(ii) $\sum_{n\in\mathbb{Z}}\psi_n = 1$, by Lemma 1.13, and thus $\mathfrak{F}' = \{\psi_n : n \in \mathbb{Z}\}$ gives a bounded admissible partition of unity associated to $\mathfrak{Q}'$.

(iii) $\sup_{n\in\mathbb{Z}}|Q'_n|^{1/p-1}\|\mathcal{F}^{-1}\psi_n\|_{L_p(\mathbb{R}^2)} < \infty$ for all $0 < p < 1$. The proof of this fact follows the same lines as the end of the proof of Lemma 2.2.

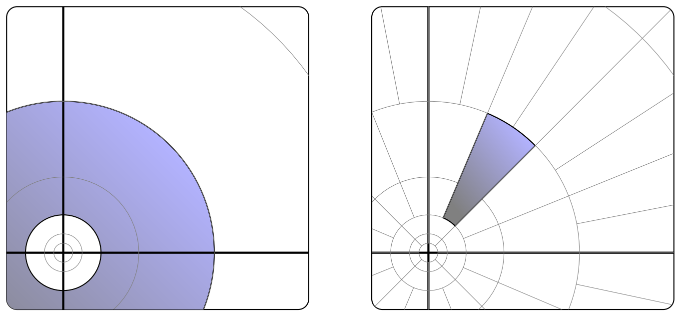

FIGURE 3. **Left:** One of the cubes Q'_n in the covering $\mathfrak{Q}'$ for the description of Besov spaces $B^s_q\big(L_p(\mathbb{R}^2)\big)$ as a decomposition space. **Right:** One of the cubes Q_{nk} in the covering $\mathfrak{Q}$ for the description of spaces of decomposition $\mathfrak{G}^s_q\big(L_p(\mathbb{R}^2)\big)$ associated to curvelet tight frames.

The spaces $B^s_q\big(L_p(\mathbb{R}^2)\big)$ may be realized as the decomposition spaces

$$D\big(\mathfrak{Q}', \mathfrak{F}'\big)^{\ell_q(\mathbb{Z},\boldsymbol{\omega}')}_{L_p(\mathbb{R}^2)},$$

where $\boldsymbol{\omega}' = \{\omega'_n = 2^{sn} : n \in \mathbb{Z}\}$, and $\mathcal{D}_n f$ are the Fourier multipliers given by $\widehat{\mathcal{D}_n f} = \psi_n \widehat{f}$. The choice of symbols of those multipliers, constructed from a frame of curvelets, is precisely the link needed to measure Besov regularity of a function f by means of suitable expressions on its curvelet coefficients $\langle f, \phi_{n\boldsymbol{z}k}\rangle$. This is accomplished by means of two results: Proposition 2.3 gives an atomic decomposition of the decomposition spaces generated from curvelet frames, and Proposition 2.4 uses this atomic decomposition to find suitable embeddings of decomposition spaces between Besov spaces. In the rest of this section, it is assumed $\alpha_0 = 2$.

Lemma 2.3. *Given a tight frame of curvelets $\{\phi_{n\boldsymbol{z}k}\}$, and $0 < p < \infty$, the sequences $\{\phi_{n\boldsymbol{z}k}(\boldsymbol{x})\}_{\boldsymbol{z}\in\mathbb{Z}^2}$ belong to $\ell_p(\mathbb{Z}^2)$ for all $\boldsymbol{x} \in \mathbb{R}^2$ and $(n,k) \in \Lambda$, with*

$$\big\|\{\phi_{n\boldsymbol{z}k}(x)\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_p(\mathbb{Z}^2)} \le C_{V,W,p}\,\big(2^{2n}\tau_n\big)\tau_n^{-2\lceil 1/p\rceil}. \tag{40}$$

Proof. Recall the definition of $\Phi_{n\beta k}$ for $(n,k) \in \Lambda$, $\beta = \pi 2^{-n-1}\boldsymbol{z}$, $\boldsymbol{z} \in \mathbb{Z}^2$, and its relation to $\phi_{n\boldsymbol{z}k}$ from Theorem 1.2.

$$\begin{aligned}
|\phi_{n\boldsymbol{z}k}(\mathbf{0})| &= 2^{n/2}(\log 2)^{1/2}\tau_n^{1/2}|\Phi_{n\beta k}(\mathbf{0})| && \big(\text{by Theorem1.2}\big)\\
&\le C_{V,W}\,2^{n/2}\tau_n^{1/2}(2^n)^{3/2}\tau_n^{1/2}; && \big(\text{by (3) with exponent } 0\big)\\
&\le C_{V,W}\,2^{2n}\tau_n. && (41)
\end{aligned}$$

For $\boldsymbol{z} \in \mathbb{Z}^2$, $\beta = \pi 2^{-n-1}\boldsymbol{z}$, by virtue of (3) with exponent $j \in \mathbb{N}$,

$$\begin{aligned}
|\boldsymbol{z}|^{2j}|\Phi_{n\beta k}(\mathbf{0})| &= \pi^{-2j}2^{2j(n+1)}|\beta|^{2j}|\Phi_{n\beta k}(\mathbf{0})|\\
&\le C_{V,W,j}\,2^{2j(n+1)}(2^n)^{3/2-2j}\tau_n^{1/2-2j}
\end{aligned}$$

$$= C_{V,W,j}\, 2^{3n/2} \tau_n^{1/2-2j};$$

thus,

$$|\boldsymbol{z}|^{2j} |\phi_{n\boldsymbol{z}k}(\mathbf{0})| = 2^{n/2} (\log 2)^{1/2} \tau_n^{1/2} |\boldsymbol{z}|^{2j} |\Phi_{n\beta k}(\mathbf{0})| \le C_{V,W,j}\, 2^{2n} \tau_n^{1-2j},$$

what gives

$$|\tau_n \boldsymbol{z}|^{2j} |\phi_{n\boldsymbol{z}k}(\mathbf{0})| \le C_{V,W,j}\, 2^{2n} \tau_n. \tag{42}$$

Estimates (41) and (42) give for any $0 < p < \infty$,

$$\big(1 + |\tau_n \boldsymbol{z}|^{2jp}\big) |\phi_{n\boldsymbol{z}k}(\mathbf{0})|^p \le \big(C_{V,W,j}\, 2^{2n} \tau_n\big)^p,$$

and therefore,

$$\sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\phi_{n\boldsymbol{z}k}(\mathbf{0})|^p \le \big(C_{V,W,j}\, 2^{2n} \tau_n\big)^p \sum_{\boldsymbol{z}\in\mathbb{Z}^2} \frac{1}{1 + |\tau_n \boldsymbol{z}|^{2jp}}.$$

Let $j > 1/p$, and consider for each $m \in \mathbb{N}$ the set $\square_m$ of points $\boldsymbol{z} \in \mathbb{Z}^2$ located on the border of the cube $[-m, m]^2$. There are exactly $8m$ such indices on this set, and for each of them, it is $m \le |z| \le m\sqrt{2}$. Therefore,

$$\begin{aligned}\sum_{\boldsymbol{z}\in\mathbb{Z}^2} \frac{1}{1 + |\tau_n \boldsymbol{z}|^{2jp}} &= 1 + \sum_{m=1}^{\infty} \sum_{\boldsymbol{z}\in\square_m} \frac{1}{1 + |\tau_n \boldsymbol{z}|^{2jp}} \\ &\le 1 + \sum_{m=1}^{\infty} \sum_{\boldsymbol{z}\in\square_m} \frac{1}{|\tau_n \boldsymbol{z}|^{2jp}} \\ &\le 1 + 8\tau_n^{-2jp} \sum_{m=1}^{\infty} \frac{1}{|\boldsymbol{z}|^{2jp-1}}.\end{aligned}$$

The sum in the right-hand side of the previous expression is finite, since $2jp-1 > 1$ by the choice of j. It is then

$$\sum_{\boldsymbol{z}\in\mathbb{Z}^2} \frac{1}{1 + |\tau_n \boldsymbol{z}|^{2jp}} \le C_p \tau_n^{-2jp},$$

and (40) follows for $\boldsymbol{x} = \mathbf{0}$ with $j = \lceil 1/p \rceil$. The result is also true for any other $\boldsymbol{x} \in \mathbb{R}^2$ by a simple shifting argument. □

Lemma 2.4. *Consider in* Λ *the relation* $\sim$ *given by* $(n_1, k_1) \sim (n_2, k_2)$ *if and only if* $\operatorname{supp} \phi_{n_1\mathbf{0}k_1} \cap \operatorname{supp} \phi_{n_2\mathbf{0}k_2} \ne \emptyset$. *For all* $n_0 \in \mathbb{Z}$, $k_0 \in \{1, \dots, \eta_{n_0}\}$, *let* $\widetilde{\mathcal{D}}_{n_0k_0} \colon L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ *denote the Fourier multiplier with symbol*

$$\sum_{(n,k)\sim(n_0,k_0)} |\widehat{\phi}_{n\mathbf{0}k}|^2.$$

Assume the aspect-ratio weight function τ *satisfies the hypotheses of Lemma 2.2. Then, for all* $0 < p \le \infty$, $(n_0, k_0) \in \Lambda$ *and* $f \in \mathcal{S}(\mathbb{R}^2)$, *it is*

$$\big\| \{\langle f, \phi_{n_0\boldsymbol{z}k_0}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2} \big\|_{\ell_p(\mathbb{Z}^2)} \le C_{V,W,p}\, 2^{2n_0/p} \tau_{n_0}^{1-2\lceil 1/p\rceil} \|\widetilde{\mathcal{D}}_{n_0k_0} f\|_{L_p(\mathbb{R}^2)}, \tag{43}$$

$$\|\mathcal{D}_{n_0k_0} f\|_{L_p(\mathbb{R}^2)} \le C_{V,W,p} \sum_{(n,k)\sim(n_0,k_0)} 2^{2n(1-1/p)} \tau_n^{\zeta(p)} \big\| \{\langle f, \phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2} \big\|_{\ell_p(\mathbb{Z}^2)}, \tag{44}$$

where

$$\zeta(p) = \begin{cases} 1 - 2/p & \text{if } 0 < p \le 1, \\ -1 & \text{if } 1 < p \le \infty. \end{cases}$$

Proof. For all $f \in \mathcal{S}'(\mathbb{R}^2)$, $(n_0, k_0) \in \Lambda$, and $\boldsymbol{z} \in \mathbb{Z}^2$:

$$\begin{aligned}\langle f, \phi_{n_0\boldsymbol{z}k_0}\rangle &= \int_{\mathbb{R}^2} \widehat{f}(\xi)\,\overline{\widehat{\phi}_{n_0\boldsymbol{z}k_0}(\xi)}\,d\xi \\ &\leq \int_{\mathbb{R}^2} \Big(\textstyle\sum_{(n,k)\sim(n_0,k_0)} \psi_{nk}(\xi)\Big)\widehat{f}(\xi)\,\overline{\widehat{\phi}_{n_0\boldsymbol{z}k_0}}\,d\xi,\end{aligned}$$

since $\sum_{(n,k)\sim(n_0,k_0)} \psi_{nk}(\xi)$ is identically one in the support of $\widehat{\phi}_{n_0\boldsymbol{z}k_0}$. It is then

$$\langle f, \phi_{n_0\boldsymbol{z}k_0}\rangle = \langle \widetilde{\mathcal{D}}_{n_0k_0} f, \phi_{n_0\boldsymbol{z}k_0}\rangle.$$

Suppose $p \leq 1$. Given $(n_0, k_0) \in \Lambda$,

$$\begin{aligned}|\langle f,\phi_{n_0\boldsymbol{z}k_0}\rangle|^p &= |\langle \widetilde{\mathcal{D}}_{n_0k_0} f, \phi_{n_0\boldsymbol{z}k_0}\rangle|^p \leq \|\phi_{n_0\boldsymbol{z}k_0}\widetilde{\mathcal{D}}_{n_0k_0} f\|^p_{L_1(\mathbb{R}^2)} \\ &\leq C_p\, 2^{2n_0(1-p)}\|\phi_{n_0\boldsymbol{z}k_0}\widetilde{\mathcal{D}}_{n_0k_0} f\|^p_{L_p(\mathbb{R}^2)}.\end{aligned}$$

The last step followed from Nikolski's inequality (see Triebel [10], for example), since the support of $\widehat{\phi}_{n_0\boldsymbol{z}k_0}$ is contained in the cube $[-2^{n_0+1}, 2^{n_0+1}]^2$, and $p \leq 1$. It is then

$$\begin{aligned}\sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f,\phi_{n_0\boldsymbol{z}k_0}\rangle|^p &\leq C_p\, 2^{2n_0(1-p)} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} \|\phi_{n_0\boldsymbol{z}k_0}\widetilde{\mathcal{D}}_{n_0k_0} f\|^p_{L_p(\mathbb{R}^2)} \\ &= C_p\, 2^{2n_0(1-p)} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} \int_{\mathbb{R}^2} |\phi_{n_0\boldsymbol{z}k_0}(x)\widetilde{\mathcal{D}}_{n_0k_0} f(x)|^p\,dx \\ &= C_p\, 2^{2n_0(1-p)} \int_{\mathbb{R}^2} |\widetilde{\mathcal{D}}_{n_0k_0} f(x)|^p \big\|\{\phi_{n_0\boldsymbol{z}k_0}(\boldsymbol{x})\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|^p_{\ell_p(\mathbb{Z}^2)} dx \quad \big(\text{by DCT}\big) \\ &\leq C_{V,W,p}\, 2^{2n_0(1-p)} \big(2^{2n_0}\tau_{n_0}\big)^p \tau_{n_0}^{-2p\lceil 1/p\rceil} \|\widetilde{\mathcal{D}}_{n_0k_0} f\|^p_{L_p(\mathbb{R}^2)} \quad \big(\text{by (40)}\big) \\ &= C_{V,W,p}\, 2^{2n_0} \tau_{n_0}^{p(1-2\lceil 1/p\rceil)} \|\widetilde{\mathcal{D}}_{n_0k_0} f\|^p_{L_p(\mathbb{R}^2)}.\end{aligned}$$

This proves (43) for $0 < p \leq 1$. The estimate also holds for $p = \infty$, where obviously the sum over $\mathbb{Z}^2$ is changed to the supremum:

$$\begin{aligned}|\langle f,\phi_{n_0\boldsymbol{z}k_0}\rangle| &= |\langle \widetilde{\mathcal{D}}_{n_0k_0} f, \phi_{n_0\boldsymbol{z}k_0}\rangle| \\ &\leq \|\phi_{n_0\boldsymbol{z}k_0}\|_{L_1(\mathbb{R}^2)} \|\widetilde{\mathcal{D}}_{n_0k_0} f\|_{L_\infty(\mathbb{R}^2)} \\ &= 2^{n_0/2}(\log 2)^{1/2}\tau_{n_0}^{1/2} \|\Phi_{n_0\beta k_0}\|_{L_1(\mathbb{R}^2)} \|\widetilde{\mathcal{D}}_{n_0k_0} f\|_{L_\infty(\mathbb{R}^2)} \quad \big(\beta = \pi 2^{-n_0-1}\boldsymbol{z}\big) \\ &\leq C_{V,W} 2^{n_0/2}\tau_{n_0}^{1/2}\big(2^{-n_0/2}\tau_{n_0}^{-3/2}\big) \|\widetilde{\mathcal{D}}_{n_0k_0} f\|_{L_\infty(\mathbb{R}^2)} \quad \big(\text{by (4) for } p = 1\big) \\ &\leq C_{V,W}\tau_{n_0}^{-1} \|\widetilde{\mathcal{D}}_{n_0k_0} f\|_{L_\infty(\mathbb{R}^2)}.\end{aligned}$$

Application of the Riesz-Thorin Interpolation Theorem gives (43) for $1 < p < \infty$.

To obtain (44), notice first that $\|\mathcal{D}_{n_0k_0} f\|_{L_p(\mathbb{R}^2)} \leq \sum_{(n,k)\sim(n_0,k_0)} \|\mathcal{D}_{n,k} f\|_{L_p(\mathbb{R}^2)}$. For $0 < p \leq 1$, it is

$$\begin{aligned}\|\mathcal{D}_{nk} f\|^p_{L_p(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \Big|\sum_{\boldsymbol{z}\in\mathbb{Z}^2} \langle f, \phi_{n\boldsymbol{z}k}\rangle \phi_{n\boldsymbol{z}k}(x)\Big|^p dx \quad \big(\text{by Lemma 1.14}\big) \\ &\leq \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^p \|\phi_{n\boldsymbol{z}k}\|^p_{L_p(\mathbb{R}^2)} \quad \big(\text{by MCT}\big) \\ &\leq C^p_{V,W} 2^{2n(p-1)}\tau_n^{p-2} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^p. \quad \big(\text{by (4)}\big)\end{aligned}$$

As a consequence,

$$\|\mathcal{D}_{n_0k_0}f\|_{L_p(\mathbb{R}^2)} \le C_{V,W,p} \sum_{(n,k)\sim(n_0,k_0)} 2^{2n(1-1/p)}\tau_n^{1-2/p}\Big(\sum_{\boldsymbol{z}\in\mathbb{Z}^2}|\langle f,\phi_{n\boldsymbol{z}k}\rangle|^p\Big)^{1/p}.$$

To estimate the same norms for $p=\infty$ notice that for all $\boldsymbol{x}\in\mathbb{R}^2$,

$$\begin{aligned}|\mathcal{D}_{nk}f(x)| &= \Big|\sum_{\boldsymbol{z}\in\mathbb{Z}^2}\langle f,\phi_{n\boldsymbol{z}k}\rangle\phi_{n\boldsymbol{z}k}(x)\Big|\\ &\le \big\|\{\langle f,\phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_\infty(\mathbb{Z}^2)}\,\big\|\{\phi_{n\boldsymbol{z}k}(x)\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_1(\mathbb{Z}^2)}\\ &\le C_{V,W}\,2^{2n}\tau_n^{-1}\big\|\{\langle f,\phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_\infty(\mathbb{Z}^2)} \qquad \big(\text{by } (40)\big)\end{aligned}$$

This gives

$$\|\mathcal{D}_{n_0k_0}f\|_{L_\infty(\mathbb{R}^2)} \le C_{V,W}\sum_{(n,k)\sim(n_0,k_0)} 2^{2n}\tau_n^{-1}\big\|\{\langle f,\phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_\infty(\mathbb{Z}^2)}$$

For each $(n,k)\sim(n_0,k_0)$, estimates for each norm $\|\mathcal{D}_{nk}f\|_{L_p(\mathbb{R}^2)}$ follow now for $1<p<\infty$ by interpolation via the Riesz-Thorin interpolation theorem.

$$\begin{aligned}\|\mathcal{D}_{nk}f\|_{L_p(\mathbb{R}^2)} &\le C_{V,W}\big(\tau_n^{-1}\big)^{1/p}\big(2^{2n}\tau_n^{-1}\big)^{1-1/p}\big\|\{\langle f,\phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_p(\mathbb{Z}^2)}\\ &\le C_{V,W}2^{2n(1-1/p)}\tau_n^{-1}\big\|\{\langle f,\phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_p(\mathbb{Z}^2)}.\end{aligned}$$

Adding all those estimates in the expression on the left hand side of (44), the result follows. ☐

Proposition 2.3. *Let $\{\phi_{n\boldsymbol{z}k}\}$ be a tight frame of curvelets satisfying the hypotheses in Lemma 2.2, and let $\mathfrak{G}^s_q\big(L_p(\mathbb{R}^2)\big)=\mathcal{D}\big(\mathfrak{Q},\mathfrak{F}\big)^{\ell_q(\Lambda,\boldsymbol{\omega})}_{L_p(\mathbb{R}^2}$ be the corresponding decomposition space, with associated $\mathfrak{Q}$-moderate weight $\boldsymbol{\omega}=\{\omega_{nk}=2^{ns}:(n,k)\in\Lambda\}$.*

(i) *Let $\zeta(p)$ defined as in Lemma 2.4. If the curvelet coefficients of $f\in\mathcal{S}'(\mathbb{R}^2)$ satisfy*

$$\Big(\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\eta_n}\Big(2^{ns}2^{2n(1-1/p)}\tau_n^{\zeta(p)}\big\|\{\langle f,\phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_p(\mathbb{Z}^2)}\Big)^q\Big)^{1/q}<\infty. \tag{45}$$

then $f\in\mathfrak{G}^s_q\big(L_p(\mathbb{R}^2)\big)$.

(ii) *If $f\in\mathfrak{G}^s_q\big(L_p(\mathbb{R}^2)\big)$, then the curvelet coefficients $\langle f,\phi_{n\boldsymbol{z}k}\rangle$ satisfy*

$$\Big(\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\eta_n}\Big(2^{ns}2^{-2n/p}\tau_n^{2\lceil 1/p\rceil-1}\big\|\{\langle f,\phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_p(\mathbb{Z}^2)}\Big)^q\Big)^{1/q}<\infty.$$

Proof. Both statements follow from Lemma 2.4:

(i) Assume $f\in\mathcal{S}'(\mathbb{R}^2)$, and let $\mathbf{I}_s(f)=\Big(\sum_{(n,k)\in\Lambda}2^{nqs}\|\mathcal{D}_{nk}f\|^q_{L_p(\mathbb{R}^2)}\Big)^{1/q}$. By (44), it is

$$\mathbf{I}_s(f)\le\Big(\sum_{(n,k)\in\Lambda}\Big(2^{ns}C_{V,W,p}\sum_{(n',k')\sim(n,k)}2^{2n'(1-1/p)}\tau_{n'}^{\zeta(p)}\big\|\{\langle f,\phi_{n'\boldsymbol{z}k'}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2}\big\|_{\ell_p(\mathbb{Z}^2)}\Big)^q\Big)^{1/q}.$$

But $(n',k')\sim(n,k)$ implies $|n-n'|\le1$. As $M_1\le\tau(2^n)/\tau(2^{n-1})\le M_2$ by hypothesis, the following two simple estimates hold:

$$\tau_n\ge\max(M_2,M_1^{-1})\tau_{n'}, \tag{46}$$

$$2^{2n'(1-1/p)} \le \underbrace{\big(\max_{\epsilon=-1,0,1} 2^{\epsilon(1-1/p)}\big)}_{C_p} 2^{2n(1-1/p)}. \tag{47}$$

By virtue of these inequalities, and the fact that $\zeta(p) < 0$ for all $0 < p \le \infty$, it is

$$\mathbf{I}_s(f) \le C_{V,W,p} \left(\sum_{(n,k)\in\Lambda} \left(2^{ns} 2^{2n(1-1/p)} \tau_n^{\zeta(p)} \sum_{(n',k')\sim(n,k)} \big\| \{\langle f, \phi_{n'\boldsymbol{z}k'}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2} \big\|_{\ell_p(\mathbb{Z}^2)} \right)^q \right)^{1/q}.$$

For each $(n,k) \in \Lambda$, the expression $\left(\sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^p \right)^{1/p}$ appears at most $1 + N(\mathfrak{Q}) = 6 + M_1 + M_2$ times in the right hand side of the previous expression, each of them with one of the coefficients $2^{2n(1-1/p)}\tau_n^{\zeta(p)}$ in front, where $(n', k') \sim (n, k)$. Therefore, using estimates (46) and (47) again, it is

$$\mathbf{I}_s(f) \le C_{V,W,p,M_1,M_2} \left(\sum_{(n,k)\in\Lambda} \left(2^{ns} 2^{2n(1-1/p)} \tau_n^{\zeta(p)} \big\| \{\langle f, \phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2} \big\|_{\ell_p(\mathbb{Z}^2)} \right)^q \right)^{1/q}$$

Thus, if f satisfies (45), then $f \in \mathfrak{G}_q^s\big(L_p(\mathbb{R}^2)\big)$.

(ii) Assume $f \in \mathfrak{G}_q^s\big(L_p(\mathbb{R}^2)\big)$.

$$\begin{aligned}
\mathbf{II}_s(f) &:= \left(\sum_{(n,k)\in\Lambda} \left(2^{ns} 2^{-2n/p} \tau_n^{2\lceil 1/p\rceil -1} \big\| \{\langle f, \phi_{n\boldsymbol{z}k}\rangle\}_{\boldsymbol{z}\in\mathbb{Z}^2} \big\|_{\ell_p(\mathbb{Z}^2)} \right)^q \right)^{1/q} \\
&\le C_{V,W,p} \left(\sum_{(n,k)\in\Lambda} \left(2^{ns} 2^{-2n/p} \tau_n^{2\lceil 1/p\rceil -1} \big(2^{2n/p} \tau_n^{1-2\lceil 1/p\rceil} \big) \|\widetilde{\mathcal{D}}_{nk} f\|_{L_p(\mathbb{R}^2)} \right)^q \right)^{1/q} \\
&= C_{V,W,p} \left(\sum_{(n,k)\in\Lambda} 2^{ns} \|\widetilde{\mathcal{D}}_{nk} f\|^q_{L_p(\mathbb{R}^2)} \right)^{1/q}.
\end{aligned}$$

The second estimate followed from (43). The previous expression is equivalent to $|f|_{D(\mathfrak{Q},\mathfrak{F})^{\ell_q(\Lambda,\boldsymbol{\omega})}_{L_p(\mathbb{R}^2)}} = |f|_{\mathfrak{G}_q^s(L_p(\mathbb{R}^2))}$, which is finite by hypothesis. Thus, $\mathbf{II}_s(f) < \infty$ and the statement is proved. □

Proposition 2.4. *Let $\{\phi_{n\boldsymbol{z}k}\}$ be a tight frame of curvelets satisfying the hypotheses in Lemma 2.2. For $0 < p \le \infty$, $0 < s, q < \infty$ and $s' = \frac{1}{2}\big(\max(1, 1/p) - \min(1, 1/q)\big)$, the following embeddings hold:*

$$B_q^{s+1/(2q)}\big(L_p(\mathbb{R}^2)\big) \hookrightarrow \mathfrak{G}_q^s\big(L_p(\mathbb{R}^2)\big) \hookrightarrow B_q^{s-s'}\big(L_p(\mathbb{R}^2)\big).$$

Proof. The notation in this proof comes form remark 2.1. Recall that for each $n \in \mathbb{Z}$, $\operatorname{supp}\psi_n = \cup_{k=1}^{\eta_n} \operatorname{supp}\psi_{nk}$, with $\eta_n = 2\pi/\tau_n$; hence, for all $0 < p \le \infty$, $n \in \mathbb{Z}$ and $k = 1, \dots, \eta_n$, there exists a constant $c > 0$ independent of the size of the support of either ψ_n or ψ_{nk}, such that (see [10, Proposition 1.5.1 and Theorem 1.5.2, Remark 3]).

$$\|\mathcal{F}^{-1}\psi_n\|_{L_p(\mathbb{R}^2)} \le c\|f\|_{L_p(\mathbb{R}^2)}, \tag{48}$$

$$\|D_{nk} f\|_{L_p(\mathbb{R}^2)} \le c\|f\|_{L_p(\mathbb{R}^2)}. \tag{49}$$

For the first embedding, notice that

$$\sum_{(n,k)\in\Lambda}\Big(2^n s\|D_{nk}f\|_{L_p(\mathbb{R}^2)}\Big)^q = \sum_{(n,k)\in\Lambda}\Big(2^{ns}\|D_{nk}\mathcal{F}^{-1}\big(\textstyle\sum_{n'=n-1}^{n+1}\psi_{n'}\widehat{f}\big)\|_{L_p(\mathbb{R}^2)}\Big)^q$$

$$\leq c\sum_{(n,k)\in\Lambda}\Big(2^{ns}\big\|\mathcal{F}^{-1}\big(\textstyle\sum_{n'=n-1}^{n+1}\psi_{n'}\widehat{f}\big)\big\|_{L_p(\mathbb{R}^2)}\Big)^q$$

$$\leq c\sum_{n\in\mathbb{Z}}\Big(2^{ns}\eta_n\big\|\mathcal{F}^{-1}\big(\textstyle\sum_{n'=n-1}^{n+1}\psi_{n'}\widehat{f}\big)\big\|_{L_p(\mathbb{R}^2)}\Big)^q.$$

For the second embedding, for $p \geq 1$ and $q < 1$, it is

$$\sum_{n\in\mathbb{Z}}\Big(2^{ns}\|\mathcal{F}^{-1}\psi_n\widehat{f}\|_{L_p(\mathbb{R}^2)}\Big)^q = \sum_{n\in\mathbb{Z}}\Big(2^{ns}\big\|\mathcal{F}^{-1}\psi_n\mathcal{F}\big(\textstyle\sum_{k=1}^{\eta_n}\widetilde{\mathcal{D}}_{nk}f\big)\big\|_{L_p(\mathbb{R}^2)}\Big)^q$$

$$\leq c\sum_{n\in\mathbb{Z}}\Big(2^{ns}\big\|\big(\textstyle\sum_{k=1}^{\eta_n}\widetilde{\mathcal{D}}_{nk}f\big)\big\|_{L_p(\mathbb{R}^2)}\Big)^q$$

$$\leq c\sum_{n\in\mathbb{Z}}\Big(2^{ns}\sum_{k=1}^{\eta_n}\|\widetilde{\mathcal{D}}_{nk}f\|_{L_p(\mathbb{R}^2)}\Big)^q$$

$$\leq c\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\eta_n}\Big(2^{ns}\|\widetilde{\mathcal{D}}_{nk}f\|_{L_p(\mathbb{R}^2)}\Big)^q.$$

The other cases are obtained by applying the Riesz-Thorin interpolation theorem, and using the bound on the sum over k (which is η_n) for a given level $n \in \mathbb{Z}$. □

2.2. **Linear Approximation.** Consider for each positive integer $m \in \mathbb{N}$ the finite-dimensional linear sub-spaces $X_m = \operatorname{span}\{\phi_{n\boldsymbol{z}k}\}_{\mathbb{F}_m} \subset L_2(\mathbb{R}^2)$, where $\mathbb{F}_m$ is a set of indices defined by $\mathbb{F}_m = \{(n,\boldsymbol{z},k) \in \mathbb{F} : \max(|z_1|,|z_2|) \leq m\alpha_0^{n+1}/\pi, |n| \leq m\}$. These are finite sets of indices associated with curvelets $\phi_{n\boldsymbol{z}k}$ satisfying:

(i) *Bounded set of Scales*: Only scales $-m \leq n \leq m$; therefore, the support of the frequencies of the curvelets are contained in the annulus $\{2^{-m-1} \leq |\xi| \leq 2^{m+1}\}$.

(ii) *Bounded set of Locations*: For each scale n in the range, the set of locations for the cuvelets indexed by $\mathbb{F}_m$ are precisely those inside the cube centered at the origin with side length m.

Lemma 2.5. *The sets of indices $\{\mathbb{F}_m\}_{m\in\mathbb{N}}$ satisfy the following properties:*

(i) $\mathbb{F}_1 \subset \mathbb{F}_2 \subset \cdots \subset \mathbb{F}_m \nearrow \mathbb{F}$.

(ii) $|\mathbb{F}_m| = \displaystyle\sum_{n=-m}^{m}\eta_n\big(1 + 2\lfloor m\alpha_0^{n+1}/\pi\rfloor\big)$.

Proof. Both assertions are trivial. □

Proposition 2.5. *The families $\{\phi_{n\boldsymbol{z}k}\}_{(n,\boldsymbol{z},k)\in\mathbb{F}_m}$ are frames for X_m.*

Proof. The upper bound is trivial: Given $f \in X_m$,

$$\sum_{\mathbb{F}_m}|\langle f,\phi_{n\boldsymbol{z}k}\rangle|^2 \leq \sum_{\mathbb{F}}|\langle f,\phi_{n\boldsymbol{z}k}\rangle|^2 = \|f\|_{L_2(\mathbb{R}^2)}.$$

For the lower bound, set $A_m = \inf\big\{\sum_{\mathbb{F}_m}|\langle f,\phi_{n\boldsymbol{z}k}\rangle|^2 : f \in X_m, \|f\|_{L_2(\mathbb{R}^2)} = 1\big\}$. As the intersection of the unit sphere with X_m is a compact set, there exists $g \in X_m$

with $\|g\|_{L_2(\mathbb{R}^2)} = 1$ such that $A_m = \sum_{\mathbb{F}_m} |\langle g, \phi_{\boldsymbol{nzk}}\rangle|^2$, and thus $A_m > 0$. It is then, for all $f \in X_m$,

$$\sum_{\mathbb{F}_m} |\langle f, \phi_{\boldsymbol{nzk}}\rangle|^2 = \|f\|_{L_2(\mathbb{R}^2)} \sum_{\mathbb{F}_m} |\langle f/\|f\|_{L_2(\mathbb{R}^2)}, \phi_{\boldsymbol{nzk}}\rangle|^2 \geq A_m \|f\|_{L_2(\mathbb{R}^2)}. \qquad \square$$

The corresponding frame operators are denoted by $S_m \colon X_m \to X_m$, and the orthogonal projections by $\Pi_m \colon X \to X_m$. By Proposition 5.3.5 in Christensen [5], for all $f \in L_2(\mathbb{R}^2)$, it is

$$\Pi_m f = \textstyle\sum_{\mathbb{F}_m} \langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle \phi_{\boldsymbol{nzk}}.$$

Notice that for all $m \in \mathbb{N}$, the errors of approximation by elements of the spaces X_m satisfy the following property:

$$E(f, X_m)^2_{L_2(\mathbb{R}^2)} = \|f - \Pi_m f\|^2_{L_2(\mathbb{R}^2)} = \|f\|^2_{L_2(\mathbb{R}^2)} - \|\Pi_m f\|^2_{L_2(\mathbb{R}^2)}.$$

The following results state equivalent conditions for a function $f \in L_2(\mathbb{R}^2)$ to belong to approximation spaces $\mathcal{A}^s_q\big(L_2(\mathbb{R}^2), (X_m)_{m\in\mathbb{N}}\big)$ for $0 < s < \infty$, $0 < q \leq \infty$:

Theorem 2.1. *Let $f \in L_2(\mathbb{R}^2)$ and $0 < s < \infty$. Then $f \in \mathcal{A}^s_\infty\big(L_2(\mathbb{R}^2), (X_m)_{m\in\mathbb{N}}\big)$ if and only if there exists $M > 0$ such that for all $m \in \mathbb{N}$,*

$$\|f\|^2_{L_2(\mathbb{R}^2)} - Mm^{-2s} \leq \frac{1}{2} \sum_{\mathbb{F}_m} \sum_{\mathbb{F}_m} \left[{\langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle \atop \langle f, S_m^{-1}\phi_{\boldsymbol{n'z'k'}}\rangle\langle \phi_{\boldsymbol{n'z'k'}}, \phi_{\boldsymbol{nzk}}\rangle}\right] \leq \|f\|^2_{L_2(\mathbb{R}^2)},$$

where $\left[{a \atop b}\right] = a\overline{b} + \overline{a}b$ for $a, b \in \mathbb{C}$.

Proof. By Lemma 5.4.2 in Christensen [5], for each $m \in \mathbb{N}$,

$$\begin{aligned}
\|\Pi_m f\|^2_{L_2(\mathbb{R}^2)} &= \sum_{\mathbb{F}} |\langle \Pi_m f, \phi_{\boldsymbol{nzk}}\rangle|^2 \\
&= \sum_{\mathbb{F}_m} |\langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle|^2 - \sum_{\mathbb{F}_m} |\langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle - \langle \Pi_m f, \phi_{\boldsymbol{nzk}}\rangle|^2 \\
&\quad - \sum_{\mathbb{F}\setminus\mathbb{F}_m} |\langle \Pi_m f, \phi_{\boldsymbol{nzk}}\rangle|^2 \\
&= \sum_{\mathbb{F}_m} |\langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle|^2 - \sum_{\mathbb{F}_m} |\langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle|^2 - \sum_{\mathbb{F}_m} |\langle \Pi_m f, \phi_{\boldsymbol{nzk}}\rangle|^2 \\
&\quad + \sum_{\mathbb{F}_m} \left[{\langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle \atop \langle \Pi_m f, \phi_{\boldsymbol{nzk}}\rangle}\right] - \sum_{\mathbb{F}\setminus\mathbb{F}_m} |\langle \Pi_m f, \phi_{\boldsymbol{nzk}}\rangle|^2 \\
&= \sum_{\mathbb{F}_m} \sum_{\mathbb{F}_m} \left[{\langle f, S_m^{-1}\phi_{\boldsymbol{nzk}}\rangle \atop \langle f, S_m^{-1}\phi_{\boldsymbol{n'z'k'}}\rangle\langle \phi_{\boldsymbol{n'z'k'}}, \phi_{\boldsymbol{nzk}}\rangle}\right] - \|\Pi_m f\|^2_{L_2(\mathbb{R}^2)}.
\end{aligned}$$

The statement follows. $\square$

Theorem 2.2. *Let $f \in L_2(\mathbb{R}^2)$, $0 < s < \infty$, and for each $m \in \mathbb{N}$, $j \geq m$, set*

$$\zeta_{m,j} = \sum_{\mathbb{F}_{j+1}\setminus\mathbb{F}_j} |\langle f, \phi_{\boldsymbol{nzk}}\rangle|^2 - \sum_{\mathbb{F}_{j+1}\setminus\mathbb{F}_j} |\langle \Pi_m f, \phi_{\boldsymbol{nzk}}\rangle|^2.$$

Then,

(i) $f \in \mathcal{A}^s_\infty\big(L_2(\mathbb{R}^2), (X_m)_{m\in\mathbb{N}}\big)$ *if and only if there exists* $M > 0$ *such that for all* $m \in \mathbb{N}$,

$$m^{2s} \sum_{j=m}^{\infty} \zeta_{m,j} \le M.$$

(ii) $f \in \mathcal{A}^s_q\big(L_2(\mathbb{R}^2), (X_m)_{m\in\mathbb{N}}\big)$ *if and only if*

$$\sum_{m=0}^{\infty} m^{qs-1}\Big(\sum_{j=m}^{\infty} \zeta_{m,j}\Big)^{q/2} < \infty.$$

Proof. Notice that for each $f \in L_2(\mathbb{R}^2)$ and any $m \in \mathbb{N}$,

$$\begin{aligned} E(f,X_m)^2_{L_2(\mathbb{R}^2)} &= \|f\|^2_{L_2(\mathbb{R}^2)} - \|\Pi_m f\|^2_{L_2(\mathbb{R}^2)} \\ &= \textstyle\sum_{\mathbb{F}\backslash\mathbb{F}_m} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^2 - \sum_{\mathbb{F}\backslash\mathbb{F}_m} |\langle \Pi_m f, \phi_{n\boldsymbol{z}k}\rangle|^2 \\ &= \sum_{j=m}^{\infty} \Bigg(\underbrace{\sum_{\mathbb{F}_{j+1}\backslash\mathbb{F}_j} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^2 - \sum_{\mathbb{F}_{j+1}\backslash\mathbb{F}_j} |\langle \Pi_m f, \phi_{n\boldsymbol{z}k}\rangle|^2}_{\zeta_{m,j}}\Bigg). \end{aligned}$$

The statements follow from the definition of approximation spaces. □

Lemma 2.6. *For all* $f, g \in L_2(\mathbb{R}^2)$, $\lim_m \|f - S_m f\|_{L_2(\mathbb{R}^2)} = 0$.

Proof. For all $g \in \mathbb{R}^2$ with $\|g\|_{L_2(\mathbb{R}^2)} = 1$,

$$\begin{aligned} |\langle f - S_m f, g\rangle| &= \Big|\sum_{\mathbb{F}\backslash\mathbb{F}_m} \langle g, \phi_{n\boldsymbol{z}k}\rangle\,\langle f, \phi_{n\boldsymbol{z}k}\rangle\Big| \\ &\le \Big(\sum_{\mathbb{F}\backslash\mathbb{F}_m} |\langle g, \phi_{n\boldsymbol{z}k}|^2\Big)^{1/2}\Big(\sum_{\mathbb{F}\backslash\mathbb{F}_m} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^2\Big)^{1/2} \\ &\le \Big(\sum_{\mathbb{F}\backslash\mathbb{F}_m} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^2\Big)^{1/2} \to 0 \text{ as } m \to \infty. \end{aligned}$$ □

Lemma 2.7. *For each* $\lambda \in (0, 1)$ *and for each* $m \in \mathbb{N}$, *there exists a non-negative integer* $\mu(\lambda, m) \ge 0$ *such that for all* $f \in X_m$,

$$\lambda\, \|f\|^2_{L_2(\mathbb{R}^2)} \le \textstyle\sum_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle f, \phi_{n\boldsymbol{z}k}\rangle|^2 \le \|f\|^2_{L_2(\mathbb{R}^2)}. \tag{50}$$

Furthermore, $\{\Pi_m \phi_{n\boldsymbol{z}k} : (n, \boldsymbol{z}, k) \in \mathbb{F}_{m+\mu(\lambda,m)}\}$ *is a frame for* X_m *with frame bounds* λ, 1, *and associated frame operator* $\mathfrak{S}_{\lambda,m} = \Pi_m S_{m+\mu(\lambda,m)}$ *satisfying* $\|\mathfrak{S}_{\lambda,m}\| \le 1$, $\|\mathfrak{S}^{-1}_{\lambda,m}\| \le \lambda^{-1}$.

Proof. Consider a finite set of functions $\{f_1, \dots, f_J\}$ in X_m satisfying $\|f_j\|_{L_2(\mathbb{R}^2)} = 1$ for all $j = 1, \dots, J$, and such that

$$\{f \in X_m : \|f\|_{L_2(\mathbb{R}^2)} = 1\} \subset \bigcup_{j=1}^{J} B\big(f_j, (1 - \lambda^{1/2})/2\big).$$

There exists $\mu(\lambda, m) \in \mathbb{N}$ such that for all $j = 1, \dots, J$,

$$\Big(\frac{1 + \lambda^{1/2}}{2}\Big)^2 \le \textstyle\sum_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle f_j, \phi_{n\boldsymbol{z}k}\rangle|^2.$$

Given $f \in X_m \setminus \{0\}$, set $g = f/\|f\|_{L_2(\mathbb{R}^2)}$, and let $j_0 \in \{1, \dots, J\}$ such that $\|g - f_{j_0}\|_{L_2(\mathbb{R}^2)} \le (1-\lambda^{1/2})/2$. Then,

$$\begin{aligned}\Big(\sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle f, \phi_{\boldsymbol{nzk}}\rangle|^2\Big)^{1/2} &= \|f\|_{L_2(\mathbb{R}^2)} \Big(\sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle g, \phi_{\boldsymbol{nzk}}\rangle|^2\Big)^{1/2} \\ &\ge \|f\|_{L_2(\mathbb{R}^2)} \Big\{ \Big(\sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle f_{j_0}, \phi_{\boldsymbol{nzk}}\rangle|^2\Big)^{1/2} \\ &\qquad - \Big(\sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle f_{j_0} - g, \phi_{\boldsymbol{nzk}}\rangle|^2\Big)^{1/2} \Big\} \\ &\ge \|f\|_{L_2(\mathbb{R}^2)} \Big(\frac{1+\lambda^{1/2}}{2} - \frac{1-\lambda^{1/2}}{2} \Big) = \lambda^{1/2} \|f\|_{L_2(\mathbb{R}^2)},\end{aligned}$$

thus proving estimate (50).

Now, for each $f \in X_m$,

$$\lambda \|f\|^2_{L_2(\mathbb{R}^2)} \le \sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle f, \phi_{\boldsymbol{nzk}}\rangle|^2 = \sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} |\langle f, \Pi_m \phi_{\boldsymbol{nzk}}\rangle|^2 \le \|f\|^2_{L_2(\mathbb{R}^2)};$$

therefore, $\{\Pi_m \phi_{\boldsymbol{nzk}}\}_{\mathbb{F}_{m+\mu(\lambda,m)}}$ is a frame for X_m with the claimed frame bounds. In order to find the corresponding frame operators $\mathfrak{S}_{\lambda,m}$, notice that for $f \in X_m$,

$$\sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} \langle f, \Pi_m \phi_{\boldsymbol{nzk}}\rangle \, \Pi_m \phi_{\boldsymbol{nzk}} = \Pi_m \Big(\sum\nolimits_{\mathbb{F}_{m+\mu(\lambda,m)}} \langle f, \phi_{\boldsymbol{nzk}}\rangle \phi_{\boldsymbol{nzk}} \Big) = \Pi_m S_{m+\mu(\lambda,m)} f.$$

The norm estimates for $\mathfrak{S}_{\lambda,m} = \Pi_m S_{m+\mu(\lambda,m)}$ and $\mathfrak{S}^{-1}_{\lambda,m} = \big(\Pi_m S_{m+\mu(\lambda,m)}\big)^{-1}$ follow from Proposition 5.4.4. in Christensen [5]. □

Proposition 2.6. *For all $\lambda \in (0,1)$, the sequence of operators $\{\mathfrak{S}^{-1}_{\lambda,m}\Pi_m\}_{m\in\mathbb{N}}$ converges weakly to the identity: For all $f, g \in L_2(\mathbb{R}^2)$, $\lim_m \langle f - \mathfrak{S}^{-1}_{\lambda,m}\Pi_m f, g\rangle = 0$.*

Proof. For $f \in L_2(\mathbb{R}^2)$,

$$\begin{aligned} f - \mathfrak{S}^{-1}_{\lambda,m}\Pi_m f &= \big(f - \Pi_m f\big) + \big(\Pi_m f - \mathfrak{S}^{-1}_{\lambda,m}\Pi_m f\big) \\ &= \big(f - \Pi_m f\big) + \mathfrak{S}^{-1}_{\lambda,m}\big(\mathfrak{S}_{\lambda,m}\Pi_m f - \Pi_m f\big) \\ &= \big(f - \Pi_m f\big) + \mathfrak{S}^{-1}_{\lambda,m}\big(\Pi_m S_{m+\mu(\lambda,m)}\Pi_m f - \Pi_m f\big) \\ &= \big(f - \Pi_m f\big) + \mathfrak{S}^{-1}_{\lambda,m}\Pi_m\big(S_{m+\mu(\lambda,m)}\Pi_m f - f\big); \end{aligned}$$

and so, using both the boundedness of Π_m and the norm estimates for $\mathfrak{S}^{-1}_{\lambda,m}$ in Lemma 2.7,

$$\begin{aligned} |\langle f - \mathfrak{S}^{-1}_{\lambda,m}\Pi_m f, g\rangle| &\le |\langle f - \Pi_m f, g\rangle| + |\langle \mathfrak{S}^{-1}_{\lambda,m}\Pi_m\big(S_{m+\mu(\lambda,m)}\Pi_m f - f\big), g\rangle| \\ &\le |\langle f - \Pi_m f, g\rangle| + \|\mathfrak{S}^{-1}_{\lambda,m}\Pi_m\big(S_{m+\mu(\lambda,m)}\Pi_m f - f\big)\|_{L_2(\mathbb{R}^2)} \|g\|_{L_2(\mathbb{R}^2)} \\ &\le |\langle f - \Pi_m f, g\rangle| + \lambda^{-1} \|\big(S_{m+\mu(\lambda,m)}\Pi_m f - f\big)\|_{L_2(\mathbb{R}^2)} \|g\|_{L_2(\mathbb{R}^2)}. \end{aligned}$$

As $\lim_m \langle f - \Pi_m f, g\rangle = 0$ for all $f, g \in L_2(\mathbb{R}^2)$, it is enough to show that the term $\|S_{m+\mu(\lambda,m)}\Pi_m f - f\|_{L_2(\mathbb{R}^2)}$ also tends to zero as $m \to \infty$. Notice that

$$\begin{aligned} \|S_{m+\mu(\lambda,m)}\Pi_m f - f\|_{L_2(\mathbb{R}^2)} &\le \|S_{m+\mu(\lambda,m)}\Pi_m f - S_{m+\mu(\lambda,m)} f\|_{L_2(\mathbb{R}^2)} \\ &\quad + \|S_{m+\mu(\lambda,m)} f - f\|_{L_2(\mathbb{R}^2)} \\ &\le \|\Pi_m f - f\|_{L_2(\mathbb{R}^2)} + \|S_{m+\mu(\lambda,m)} f - f\|_{L_2(\mathbb{R}^2)}, \end{aligned}$$

which tends to zero, since $\lim_m \|S_m f - f\|_{L_2(\mathbb{R}^2)} = 0$. □

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