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Abstract

The main goal of this paper is to understand which properties of a basis are important for certain direct and inverse theorems in nonlinear approximation. We study greedy approximation with regard to bases with different properties. We consider bases that are tensor products of univariate greedy bases. Some results known for unconditional bases are extended to the case of quasi-greedy bases.

Keywords. Greedy algorithm, m-term approximation, Greedy basis, Quasi-greedy basis

1 Introduction

We study the efficiency of greedy algorithms for m-term nonlinear approximations with regard to bases. Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_n\}_{n=1}^{\infty}$ be a normalized basis for X ($\|\psi_n\| = 1, n \in N$). All bases considered in our paper are assumed to be normalized. For a given $f \in X$ we define the *best m-term* approximation with regard to Ψ as follows:

$$\sigma_m(f,\Psi)_X := \inf_{b_k,\Lambda} \|f - \sum_{k\in\Lambda} b_k\psi_k\|_X,$$

where the inf is taken over coefficients b_k and sets Λ of indices with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f,\Psi)\psi_k.$$

We call a permutation ρ, $\rho(j) = k_j, j = 1, 2, ...,$ of the positive integers *decreasing* and write $\rho \in D(f)$ if

$$|c_{k_1}(f,\Psi)| \geq |c_{k_2}(f,\Psi)| \geq \dots .$$

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In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by formula

$$G_m(f) := G_m(f,\Psi) := G_m^X(f,\Psi) := G_m(f,\Psi,\rho) := \sum_{j=1}^{m} c_{k_j}(f,\Psi)\psi_{k_j}.$$

This algorithm is known in the theory of nonlinear approximation under the name of Thresholding Greedy Algorithm (TGA).

The best we can achieve with the algorithm G_m is

$$\|f - G_m(f,\Psi)\|_X = \sigma_m(f,\Psi)_X,$$

or a little weaker

$$\|f - G_m(f,\Psi)\|_X \le C\sigma_m(f,\Psi)_X \tag{1.1}$$

for all $f \in X$ with a constant C independent of f and m. Bases satisfying (1.1) are of special interest in nonlinear approximation. The following concept of a greedy basis has been introduced in [6].

Definition We call a basis Ψ a greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that

$$\|f - G_m(f,\Psi)\| \le C\sigma_m(f,\Psi)_X$$

with a constant C independent of f and m.

It is clear that an orthonormal basis is a greedy basis of a Hilbert space. It was proved in [12] that the Haar basis $\mathcal{H}_p$ is a greedy basis of $L_p([0,1))$, $1<p<\infty$.

We recall the definition of the Haar basis. Denote the univariate Haar system by $\mathcal{H} := \{H_I\}_I$, where I are dyadic intervals of the form $I = [(j-1)2^{-n}, j2^{-n})$, $j = 1,\dots,2^n$; $n = 0,1,\dots$ and $I = [0,1]$ with

$$H_{[0,1]}(x) = 1 \quad \text{for} \quad x \in [0,1) \quad ,$$

$$H_{[(j-1)2^{-n},j2^{-n})} = \begin{cases} 2^{n/2}, & x \in [(j-1)2^{-n},(j-1/2)2^{-n}) \\ -2^{n/2}, & x \in [(j-1/2)2^{-n},j2^{-n}) \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{H}_p$ the Haar basis $\mathcal{H}$ renormalized in $L_p([0,1))$. We define the multivariate Haar basis $\mathcal{H}_p^d := \mathcal{H}_p \times \cdots \times \mathcal{H}_p$ as the tensor product of the univariate Haar bases. It consists of functions

$$H_{I,p}(x) = \prod_{j=1}^{d} H_{I_j,p}(x_j), \qquad I = I_1 \times \cdots \times I_d, \quad x = (x_1,\dots,x_d).$$

The main goal of this paper is to understand which properties of a basis are important for certain direct and inverse theorems in nonlinear approximation. The problem of direct and inverse theorems in nonlinear approximation has a rich history (see [2], [16]). We refer the reader for a detailed historical discussion to [16], pp. 288–293. The general direction of previous results can be briefly expressed in the following way. 1. Establish a result for the $\mathcal{H}_p$. 2. Establish a similar result for a greedy basis. 3. Establish a similar result for a quasi-greedy basis with special properties. 4. Establish a result for the multivariate Haar basis. 5. Establish a similar result for a basis that is a tensor product of d univariate bases. Results of Section 2 of this paper fall into the group 5. We refer the reader to Section 2 for a detailed discussion of known results from groups 1 - 4. In Section 2 we extend results known for the Haar basis $\mathcal{H}_p^d$ onto the case $\Psi^d = \Psi \times \cdots \times \Psi$ with Ψ a greedy basis. We note that it is known that $\mathcal{H}_p^d$ is not a greedy basis of $L_p([0,1]^d)$ if $p \neq 2$. However, it is known that $\mathcal{H}_p^d$ is an unconditional basis of $L_p([0,1]^d)$, $1 < p < \infty$. In Section 2 we discuss some properties of a basis that are important for direct and inverse theorems. We prove equivalence of these properties under the assumption that the basis is quasi-greedy (this assumption is weaker than the unconditionality assumption).

A typical problem from approximation theory is to find a decay of errors of an approximation method for a given function class. In Section 3 we apply results of Section 2 to study errors of greedy approximation for some smoothness classes. In Section 4 we discuss greedy approximation with regard to a quasi-greedy basis in a Hilbert space or in the L_p space. We observe that a special structure of Hilbert spaces allows us to obtain the following inequality for any $f \in H$

$$\|f - G_{\lambda m}(f,\Psi)\|_H \leq C(\lambda)\sigma_m(f,\Psi)_H, \quad \lambda > 1, \tag{1.2}$$

for a quasi-greedy basis Ψ. We note that if $G_{\lambda m}$ can be replaced by G_m in (1.2) then Ψ is a greedy basis. It is known ([17], p. 301) that for a separable, infinite dimensional Hilbert space H there exists a quasi-greedy basis that is not an unconditional basis. Therefore, by [6] this basis is not a greedy basis. Thus, one cannot replace the restriction $\lambda > 1$ by $\lambda \geq 1$ in (1.2). We present a related discussion in Section 4. In Section 4 we also consider quasi-greedy bases of the L_p space, $1 < p < \infty$. We prove the following inequality for each $f \in L_p$

$$\|f - G_m(f,\Psi)\|_{L_p} \leq C(p)m^{|1/2-1/p|}\sigma_m(f,\Psi)_{L_p}.$$

This inequality was known (see [17]) in the case of unconditional bases Ψ.

Let us agree to denote by C various positive absolute constants and by C with arguments or indices ($C(q,p), C_r$ and so on) positive numbers which depend on the arguments indicated. For two nonnegative sequences $a = \{a_n\}_{n=1}^\infty$ and $b = \{b_n\}_{n=1}^\infty$ the relation (order inequality) $a_n \ll b_n$ means that there is a number $C(a,b)$ such that $a_n \leq C(a,b)b_n$ for all n; and the relation $a_n \asymp b_n$ means $a_n \ll b_n$ and $b_n \ll a_n$.

2 Some direct and inverse theorems

The direct theorems of approximation theory provide bounds of approximation error (in our case $\sigma_m(f,\Psi)$) in terms of smoothness properties of a function f. These theorems are also known under the name of Jackson-type inequalities. The inverse theorems of approximation theory (also known as Bernstein-type inequalities) provide some smoothness properties of a function f from the sequence of approximation errors (in our case $\{\sigma_m(f,\Psi)\}$).

In the case, when we study best m-term approximation with regard to bases that are L_p-equivalent to the Haar basis $\mathcal{H}_p$, the theory of Jackson and Bernstein inequalities has been developed in [1]. It was used in [1] for a description of approximation spaces defined in terms of $\{\sigma_m(f,\Psi)\}$.

It was pointed out in [13] that in the special case of bases that are L_p-equivalent to the Haar basis there exists a simple direct way to describe the approximation spaces defined in terms of $\{\sigma_m(f,\Psi)\}$. Further investigations in [5] and [7] showed that the above direct way of description of approximation spaces can be extended to some more general bases. In this section we continue these investigations.

We begin with the greedy bases in $L_p([0,1]^d)$. In the case $d=1$ the Haar basis is a greedy basis for L_p, $1<p<\infty$. The following characterization theorem has been established in [13] (for the case $p=2$ see [10], [3]). We will use the notation

$$a_n(f,p) := |c_{k_n}(f,\mathcal{H}_p^d)|$$

for the decreasing rearrangement of the coefficients of f.

Theorem 2.1 *Let $d=1$, $1<p<\infty$ and $0<q<\infty$. Then, for any positive r we have the equivalence relation*

$$\sum_{m=1}^{\infty} \sigma_m(f,\mathcal{H})_p^q m^{rq-1} < \infty \iff \quad \sum_{n=1}^{\infty} a_n(f,p)^q n^{rq-1+q/p} < \infty.$$

Let us recall the definition of the Lorentz spaces of sequences and the definition of spaces which provide finer (logarithmic) scale. Let for a sequence $\{x_k\}_{k=1}^\infty$ a sequence $\{x_{\rho(k)}\}_{k=1}^\infty$ be a decreasing rearrangement

$$|x_{\rho(1)}| \ge |x_{\rho(2)}| \ge \dots \quad .$$

For $r>0$, $0<q<\infty$ denote

$$\ell_q^r := \{\{x_k\}_{k=1}^\infty : \sum_{k=1}^{\infty} |x_{\rho(k)}|^q k^{rq-1} < \infty\}$$

or, equivalently,

$$\ell_q^r := \{\{x_k\}_{k=1}^\infty : \sum_{s=0}^{\infty} |x_{\rho(2^s)}|^q 2^{rqs} < \infty\}.$$

For $r>0$, $b\in\mathbb{R}$, $0<q<\infty$ define

$$\ell_q^{r,b} := \{\{x_k\}_{k=1}^\infty : \sum_{s=1}^{\infty} \big(|x_{\rho(2^s)}| 2^{rs} s^b\big)^q < \infty\}.$$

It is clear that $\ell_q^{r,0} = \ell_q^r$.

The proof of Theorem 2.1 was based on the following two lemmas.

Lemma 2.2 *For any two positive integers $N < M$ we have*

$$a_M(f,p) \le C(p)\sigma_N(f,\mathcal{H})_p(M-N)^{-1/p}.$$

Lemma 2.3 *For any sequence $m_0 < m_1 < \dots$ of nonnegative integers we have*

$$\sigma_{m_s}(f,\mathcal{H})_p \le C(p)\sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}.$$

The following multivariate analogues of the above lemmas have been proved in [5].

Lemma 2.4 *For any two positive integers $N < M$ we have*

$$a_M(f,p) \le C(p,d)\sigma_N(f,\mathcal{H}^d)_p(M-N)^{-1/p}, \quad 2 \le p < \infty;$$

$$a_M(f,p) \le C(p,d)\sigma_N(f,\mathcal{H}^d)_p(M-N)^{-1/p}(\log M)^{h(p,d)}, \quad 1 < p \le 2$$

with $h(p,d) := (d-1)|1/2-1/p|$.

Lemma 2.5 *For any sequence $m_0 < m_1 < \dots$ of nonnegative integers we have*

$$\sigma_{m_s}(f,\mathcal{H}^d)_p \le C(p,d)\sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}(\log m_{i+1})^{h(p,d)}, \quad 2 \le p < \infty;$$

$$\sigma_{m_s}(f,\mathcal{H}^d)_p \le C(p,d)\sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}, \quad 1 < p \le 2.$$

It was pointed out in [5] that by using Lemmas 2.4 and 2.5 one can establish the following embedding theorem in the same way as Theorem 2.1 was deduced from Lemmas 2.2 and 2.3 in [13].

Theorem 2.6 *Let $1 < p < \infty$. Denote*

$$\sigma(f)_p := \{\sigma_m(f,\mathcal{H}^d)_p\}_{m=1}^{\infty} \quad \textit{and} \quad a(f,p) := \{a_n(f,p)\}_{n=1}^{\infty}.$$

Then we have the implications:

$$\sigma(f)_p \in \ell_q^{r,b} \quad \Rightarrow \quad a(f,p) \in \ell_q^{r+1/p,b}, \quad 2 \le p < \infty;$$

$$\sigma(f)_p \in \ell_q^{r,b} \quad \Rightarrow \quad a(f,p) \in \ell_q^{r+1/p,b-h(p,d)}, \quad 1 < p \le 2;$$

$$a(f,p) \in \ell_q^{r+1/p,b} \quad \Rightarrow \quad \sigma(f)_p \in \ell_q^{r,b-h(p,d)}, \quad 2 \le p < \infty;$$

$$a(f,p) \in \ell_q^{r+1/p,b} \quad \Rightarrow \quad \sigma(f)_p \in \ell_q^{r,b}, \quad 1 < p \le 2.$$

In this section we will establish an analogue of Theorem 2.6 for a basis Ψ^d that is a tensor product of d greedy bases Ψ.

Theorem 2.7 *Let $1 < p < \infty$ and let Ψ be a normalized greedy basis for L_p and $\Psi^d := \Psi \times \cdots \times \Psi$. Denote as above $\sigma(f)_p := \{\sigma_m(f, \Psi^d)_p\}_{m=1}^\infty$ and $a(f,p) := \{a_n(f,p)\}_{n=1}^\infty$, where $a_n(f,p) := |c_{k_n}(f, \Psi^d)|$. Then we have the implications:*

$$\sigma(f)_p \in \ell_q^{r,b} \quad \Rightarrow \quad a(f,p) \in \ell_q^{r+1/p,b}, \quad 2 \le p < \infty;$$

$$\sigma(f)_p \in \ell_q^{r,b} \quad \Rightarrow \quad a(f,p) \in \ell_q^{r+1/p,b-h(p,d)}, \quad 1 < p \le 2;$$

$$a(f,p) \in \ell_q^{r+1/p,b} \quad \Rightarrow \quad \sigma(f)_p \in \ell_q^{r,b-h(p,d)}, \quad 2 \le p < \infty;$$

$$a(f,p) \in \ell_q^{r+1/p,b} \quad \Rightarrow \quad \sigma(f)_p \in \ell_q^{r,b}, \quad 1 < p \le 2.$$

The proof of this theorem is similar to the proof of Theorem 2.6. We point out the key difference in the proofs. The proof of Theorem 2.6 was based on the following known result for the multivariate Haar system.

Theorem A. *Let $1 < p < \infty$. Then for any Λ, $|\Lambda| = m$, we have*

$$C_{p,d}^1 m^{1/p} \min_{I \in \Lambda} |c_I| \le \left\| \sum_{I \in \Lambda} c_I H_{I,p} \right\|_p \le C_{p,d}^2 m^{1/p} (\log m)^{h(p,d)} \max_{I \in \Lambda} |c_I|, \quad 2 \le p < \infty;$$

$$C_{p,d}^3 m^{1/p} (\log m)^{-h(p,d)} \min_{I \in \Lambda} |c_I| \le \left\| \sum_{I \in \Lambda} c_I H_{I,p} \right\|_p \le C_{p,d}^4 m^{1/p} \max_{I \in \Lambda} |c_I|, \quad 1 < p \le 2.$$

Theorem A for $d = 1$, $1 < p < \infty$ has been proved in [12]. In the case $d = 2$, $4/3 \le p \le 4$, it has been proved in [13]. Theorem A in the general case has been proved in [17]. It is known ([14]) that the extra logarithmic in m factors in Theorem A are sharp.

Let Ψ be a normalized basis for $L_p([0,1))$. For the space $L_p([0,1)^d)$ we define $\Psi^d := \Psi \times \cdots \times \Psi$($d$ times); $\psi_{\mathbf{n}}(x) := \psi_{n_1}(x_1) \cdots \psi_{n_d}(x_d)$, $x = (x_1, \dots, x_d)$, $\mathbf{n} = (n_1, \dots, n_d)$. The following theorem has been proved in [8].

Theorem B. *Let $1 < p < \infty$ and let Ψ be a greedy basis for $L_p([0,1))$. Then for any Λ, $|\Lambda| = m$, we have*

$$C_{p,d}^5 m^{1/p} \min_{\mathbf{n} \in \Lambda} |c_{\mathbf{n}}| \le \left\| \sum_{\mathbf{n} \in \Lambda} c_{\mathbf{n}} \psi_{\mathbf{n}} \right\|_p \le C_{p,d}^6 m^{1/p} (\log m)^{h(p,d)} \max_{\mathbf{n} \in \Lambda} |c_{\mathbf{n}}|, \quad 2 \le p < \infty;$$

$$C_{p,d}^7 m^{1/p} (\log m)^{-h(p,d)} \min_{\mathbf{n} \in \Lambda} |c_{\mathbf{n}}| \le \left\| \sum_{\mathbf{n} \in \Lambda} c_{\mathbf{n}} \psi_{\mathbf{n}} \right\|_p \le C_{p,d}^8 m^{1/p} \max_{\mathbf{n} \in \Lambda} |c_{\mathbf{n}}|, \quad 1 < p \le 2.$$

It is clear that Theorem B is a full generalization of Theorem A to the case of tensor product of greedy bases. This allows us to use Theorem B in the proof of Theorem 2.7 in the same way as Theorem A was used in the proof of Theorem 2.6. In order to illustrate how Theorem B can be used we give a sketch of the proof of the following theorem.

Theorem 2.8 *Let $1 < p < \infty$ and let Ψ be a greedy basis for $L_p([0,1))$. The following relations hold for $\Psi^d = \Psi \times \cdots \times \Psi$*

$$\sigma_m(f)_p \ll (m+1)^{-r}(\log(m+1))^{-b} \quad \Rightarrow$$

$$a_n(f,p) \ll n^{-r-1/p}(\log(n+1))^{-b}, \quad 2 \le p < \infty;$$

$$a_n(f,p) \ll n^{-r-1/p}(\log(n+1))^{-b} \quad \Rightarrow$$

$$\sigma_m(f)_p \ll (m+1)^{-r}(\log(m+1))^{-b+h(p,d)}, \quad 2 \le p < \infty.$$

$$\sigma_m(f)_p \ll (m+1)^{-r}(\log(m+1))^{-b} \quad \Rightarrow$$

$$a_n(f,p) \ll n^{-r-1/p}(\log(n+1))^{-b+h(p,d)}, \quad 1 < p \le 2;$$

$$a_n(f,p) \ll n^{-r-1/p}(\log(n+1))^{-b} \quad \Rightarrow$$

$$\sigma_m(f)_p \ll (m+1)^{-r}(\log(m+1))^{-b}, \quad 1 < p \le 2.$$

Proof: Theorem 2.8 follows from the following analogues of Lemmas 2.4 and 2.5. $\square$

Lemma 2.9 *Let $1 < p < \infty$ and let Ψ be a normalized greedy basis for L_p and $\Psi^d := \Psi \times \cdots \times \Psi$. For any two positive integers $N < M$ we have*

$$a_M(f,p) \le C(p,d)\sigma_N(f,\Psi^d)_p(M-N)^{-1/p}, \quad 2 \le p < \infty;$$

$$a_M(f,p) \le C(p,d)\sigma_N(f,\Psi^d)_p(M-N)^{-1/p}(\log M)^{h(p,d)}, \quad 1 < p \le 2 \tag{2.3}$$

with $h(p,d) := (d-1)|1/2 - 1/p|$.

Lemma 2.10 *Let $1 < p < \infty$ and let Ψ be a normalized greedy basis for L_p and $\Psi^d := \Psi \times \cdots \times \Psi$. For any sequence $m_0 < m_1 < \dots$ of nonnegative integers we have*

$$\sigma_{m_s}(f,\Psi^d)_p \le C(p,d)\sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}(\log m_{i+1})^{h(p,d)}, \quad 2 \le p < \infty; \tag{2.4}$$

$$\sigma_{m_s}(f,\Psi^d)_p \le C(p,d)\sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}, \quad 1 < p \le 2.$$

The proofs are similar in both cases $1 < p \le 2$ and $2 \le p < \infty$. We will give a proof in the case $2 \le p < \infty$. Lemma 2.10 follows directly from Theorem B.

To prove Lemma 2.9, for given $f = \sum_{\mathbf{n}} c_{\mathbf{n}}\psi_{\mathbf{n},p}$, let Λ_N and $\{u_{\mathbf{n}}, \mathbf{n} \in \Lambda_N\}$ be the set of indices with $\#\Lambda_N = N$ and coefficients such that

$$\left\| f - \sum_{\mathbf{n}\in\Lambda_N} u_{\mathbf{n}}\psi_{\mathbf{n},p} \right\|_p \le 2\sigma_N(f,\Psi^d)_p.$$

Moreover, let $G_M = \{\mathbf{n}_1, \dots, \mathbf{n}_M\}$, where $\{\mathbf{n}_k\}$ are defined as $a_k(f,p) = |c_{\mathbf{n}_k}(f, \Psi^d)|$. By unconditionality of Ψ^d we have

$$\left\| f - \sum_{\mathbf{n}\in\Lambda_N} c_{\mathbf{n}} \psi_{\mathbf{n},p} \right\|_p \le C \left\| f - \sum_{\mathbf{n}\in\Lambda_N} u_{\mathbf{n}} \psi_{\mathbf{n},p} \right\|_p \le 2C\sigma_N(f, \Psi^d)_p$$

and

$$\left\| \sum_{\mathbf{n}\in G_M\setminus\Lambda_N} c_{\mathbf{n}} \psi_{\mathbf{n},p} \right\|_p \le C \left\| f - \sum_{\mathbf{n}\in\Lambda_N} c_{\mathbf{n}} \psi_{\mathbf{n},p} \right\|_p .$$

As $\#(G_M \setminus \Lambda_N) \ge M - N$, Lemma 2.9 follows now from Theorem B.

Remark 2.11 *In Theorems 2.1, 2.6, 2.7, and 2.8 the best m-term approximation $\sigma_m(f)_p$ can be replaced by the m-term greedy approximation $\|f - G_m(f)\|_p$.*

Statement of the above remark is obvious in one direction, when we bound $\{a_n(f,p)\}$ from conditions on $\{\sigma_m(f)_p\}$. In the other direction it follows from the proofs of those theorems.

The inequalities of the type of (2.3) and (2.4) play an important role in the above investigations. We now present some necessary and sufficient conditions for having inequalities (2.3) and (2.4) for a basis Ψ. We will prove some results under weaker conditions on Ψ than the above assumption that Ψ is a tensor product of d greedy bases. We begin with the case of quasi-greedy basis.

In [6] the concept of quasi-greedy basis was introduced.

Definition The basis Ψ is called quasi-greedy if there exists some constant C such that for all $f \in X$,

$$\sup_m \|G_m(f, \Psi)\| \le C\|f\|.$$

Subsequently, Wojtaszczyk [17] proved that these are precisely the bases for which the TGA merely converges, i.e.,

$$\lim_{n\to\infty} G_n(f) = f.$$

It will be convenient to define the quasi-greedy constant K to be the least constant such that

$$\|G_m(f)\| \le K\|f\| \quad \text{and} \quad \|f - G_m(f)\| \le K\|f\|, \quad f \in X.$$

We will need the following known lemma (see, for instance, [16], p. 269).

Lemma 2.12 *Suppose Ψ is a quasi-greedy basis with a quasi-greedy constant K. Then, for any real numbers a_j and any finite set of indices P, we have*

$$(4K^2)^{-1} \min_{j\in P} |a_j| \|\sum_{j\in P} \psi_j\| \le \|\sum_{j\in P} a_j\psi_j\| \le 2K \max_{j\in P} |a_j| \|\sum_{j\in P} \psi_j\|.$$

We will use the notation

$$a_k(f) := |c_{n_k}(f, \Psi)|$$

for the decreasing rearrangement of the coefficients of f. We will also introduce the m-th greedy remainder

$$H_m(f) := f - G_m(f).$$

Theorem 2.13 *Let Ψ be a quasi-greedy basis of Banach space X. Then for $a > 0$ and $b \in \mathbb{R}$, the following three statements are equivalent.*

i) For any sequence $m_0 < m_1 < ...$ of non-negative integers we have

$$\sigma_{m_s}(f, \Psi)_X \ll \sum_{i=s}^{\infty} a_{m_i}(f)(m_{i+1} - m_i)^a (\log(m_{i+1} + 1))^b.$$

ii) For any finite set Λ of indices

$$\left\| \sum_{k\in\Lambda} \psi_k \right\|_X \ll |\Lambda|^a (\log(|\Lambda| + 1))^b.$$

iii) For any sequence $m_0 < m_1 < ...$ of non-negative integers we have

$$\|H_{m_s}(f, \Psi)\| \ll \sum_{i=s}^{\infty} a_{m_i}(f)(m_{i+1} - m_i)^a (\log(m_{i+1} + 1))^b.$$

Proof: For $f = \sum_{k\in\Lambda} \psi_k$, we set $m_0 = 0$, $m_1 = |\Lambda|$. Then $\sigma_0(f, \Psi) = \|f\|$ and ii) follows directly from i).

Next we prove ii) $\Rightarrow$ iii). Let X^* denote the dual space of X. By Hahn-Banach theorem there exists a $F \in X^*$ such that

$$F\left(\sum_{k\in\Lambda} c_k\psi_k\right) = \left\|\sum_{k\in\Lambda} c_k\psi_k\right\|$$

and $\|F\| = 1$. Note that

$$F\left(\sum_{k\in\Lambda} c_k\psi_k\right) = \sum_{k\in\Lambda} c_k F(\psi_k) \le \sum_{k\in\Lambda} |c_k||F(\psi_k)|.$$

We write $|F(\psi_k)|$ as $\varepsilon_k F(\psi_k)$, where $\varepsilon_k = 1$ or -1. Then we have

$$\sum_{k\in\Lambda} |c_k||F(\psi_k)| \le \max_{k\in\Lambda} |c_k| F\left(\sum_{k\in\Lambda} \varepsilon_k\psi_k\right) \le \max_{k\in\Lambda} |c_k| \left\|\sum_{k\in\Lambda} \varepsilon_k\psi_k\right\|.$$

Then by Lemma 2.12 we obtain

$$\left\|\sum_{k\in\Lambda} c_k\psi_k\right\|_X \le 2K \max_{k\in\Lambda} |c_k| \left\|\sum_{k\in\Lambda} \psi_k\right\|.$$

Based on this inequality it is easy to derive iii) from ii). Note that iii) $\Rightarrow$ i) is trivial. We complete the proof of Theorem 2.13. $\square$

Theorem 2.14 *Let Ψ be an unconditional basis of Banach space X. Then for $a > 0$ and $b \in \mathbb{R}$, the following three statements are equivalent.*

i) For any finite set Λ of indices

$$\left\| \sum_{k \in \Lambda} \psi_k \right\| \geq c|\Lambda|^a (\log |\Lambda|)^{-b}.$$

ii) For any two positive integers $N < M$ we have

$$a_M(f) \ll \sigma_N(f)(M-N)^{-a}(\log(M-N))^b.$$

iii) For any two positive integers $N < M$ we have

$$a_M(f) \ll \|H_N(f)\|(M-N)^{-a}(\log(M-N))^b.$$

Proof: We first prove i) $\Rightarrow$ ii). For given $f = \sum_k c_k \psi_k$, let Λ_N and $\{u_k, k \in \Lambda_N\}$ be the set of indices with $|\Lambda_N| = N$ and coefficients such that

$$\left\| f - \sum_{k \in \Lambda_N} u_k \psi_k \right\| \leq 2\sigma_N(f, \Psi).$$

Moreover, let $G_M = \{k_1, \ldots, k_M\}$, and $a_n(f) = |c_{k_n}(f, \Psi)|$. By unconditionality of Ψ we have

$$\left\| f - \sum_{k \in \Lambda_N} c_k \psi_k \right\| \leq C \left\| f - \sum_{k \in \Lambda_N} u_k \psi_k \right\| \leq 2C\sigma_N(f, \Psi)$$

and

$$\left\| \sum_{k \in G_M \setminus \Lambda_N} c_k \psi_k \right\| \leq C \left\| f - \sum_{k \in \Lambda_N} c_k \psi_k \right\|.$$

Any unconditional basis is a quasi-greedy basis. Therefore, by Lemma 2.12 we get

$$a_M(f) \left\| \sum_{k \in G_M \setminus \Lambda_N} \psi_k \right\| \leq C \left\| \sum_{k \in G_M \setminus \Lambda_N} c_k \psi_k \right\|.$$

As $|G_M \setminus \Lambda_N| \geq M - N$, ii) follows now from i).
Clearly ii) implies iii).
Next, we show that iii) implies i). For any finite set Λ of indices, let $N = 1$ and $M = |\Lambda| + 1$. Take any index $n \notin \Lambda$ and define $f := \sum_{k \in \Lambda} \psi_k + 2\psi_n$. Then by iii), we have

$$1 \ll \left\| \sum_{k \in \Lambda} \psi_k \right\| |\Lambda|^{-a} (\log |\Lambda|)^b.$$

Thus we get the desired result. $\square$

Proposition 2.15 *Let Ψ be a quasi-greedy basis of Banach space X. Then for $a > 0$ and $b \in \mathbb{R}$, the following two statements are equivalent.*

i) For any finite set Λ of indices

$$\left\| \sum_{k \in \Lambda} \psi_k \right\| \geq c|\Lambda|^a (\log |\Lambda|)^{-b}.$$

ii) For any two positive integers $N < M$ we have

$$a_M(f) \ll \|H_N(f)\| (M-N)^{-a} (\log(M-N))^b.$$

Proof: It is clear that ii) implies i) from the proof of Theorem 2.14. The other direction can be easily derived by Lemma 2.12. So we complete the proof. $\square$

3 Some results on approximation of classes

In this section we demonstrate how results of Section 2 can be applied in studying greedy approximation of smoothness classes. We consider here the following classes. For $r > 0$ we define

$$F^r(\Psi) := F^r_\infty(\Psi) := \{f : |c_{n_k}(f, \Psi)| \leq k^{-r}, \ k = 1, 2, \ldots\}.$$

Similar to Section 2 we define $\Psi^d := \Psi \times \ldots \times \Psi (d \text{ times})$ as a tensor product of univariate bases Ψ. It is known (see [15]) that the tensor product structure of multivariate wavelet bases makes them universal for approximation of anisotropic smoothness classes with different anisotropy. Theorem 2.8 and Remark 2.11 imply the following theorem.

Theorem 3.1 *Let Ψ be a greedy basis of $L_p([0,1])$.*
Then for $2 \leq p < \infty$, $r > \frac{1}{p}$,

$$\sup_{f \in F^r(\Psi^d)} \|f - G_m(f, \Psi^d)\|_p \ll m^{1/p - r} (\log m)^{h(p,d)},$$

where $h(p,d) = (d-1)(\frac{1}{2} - \frac{1}{p})$.
For $1 < p \leq 2$, $r > \frac{1}{p}$, one has

$$\sup_{f \in F^r(\Psi^d)} \|f - G_m(f, \Psi^d)\|_p \ll m^{1/p - r}.$$

We want to point out that the bounds in Theorem 3.1 are sharp. Let us consider the tensor product of Haar basis. We define

$$\sigma_m(F, \Psi)_X := \sup_{f \in F} \sigma_m(f, \Psi)_X,$$

and

$$G_m(F, \Psi)_X := \sup_{f \in F} \|f - G_m(f, \Psi)\|_X.$$

Theorem 3.2 *Let $\Psi_p^d = \mathcal{H}_p^d$ be the tensor product of Haar basis. Then for $2 < p < \infty$, $r > \frac{1}{p}$,*

$$\sigma_m(F^r, \mathcal{H}_p^d)_p \asymp G_m(F^r, \mathcal{H}_p^d)_p \asymp m^{1/p-r}(\log m)^{h(p,d)};$$

for $1 < p \leq 2$, $r > \frac{1}{p}$, we have

$$\sigma_m(F^r, \mathcal{H}_p^d)_p \asymp G_m(F^r, \mathcal{H}_p^d)_p \asymp m^{1/p-r}.$$

Proof: The upper bounds follow from Theorem 3.1. We only need to prove the lower bounds. To prove lower bounds we need the following lemma.

Lemma 3.3 *Let $\Lambda(n) := \{I : |I| = 2^{-n}\}$. For $k \geq [|\Lambda(n)|/2] + 1$ consider two sets of indices $\Lambda_1 \subset \Lambda(n)$ with $|\Lambda_1| = k$, and Λ_2, a set of k disjoint intervals I. Define for $1 < p < \infty$*

$$g_i := \sum_{I \in \Lambda_i} H_{I,p}, \quad i = 1, 2.$$

Then

$$\|g_1\|_p \asymp 2^{n/p} n^{(d-1)/2}, \quad \|g_2\|_p \asymp 2^{n/p} n^{(d-1)/p}.$$

Proof: The estimate for g_2 is trivial because of the assumption that intervals from Λ_2 are disjoint:

$$\|g_2\|_p = \left(\sum_{I \in \Lambda_2} \|H_{I,p}\|_p^p \right)^{1/p}.$$

We prove the estimate for g_1. By the Littlewood-Paley theorem

$$\|g_1\|_p \asymp \left\| \left(\sum_{I \in \Lambda_1} H_{I,p}^2 \right)^{1/2} \right\|_p. \tag{3.5}$$

As $H_{I,p}$ is normalized in L_p, we have $\|H_{I,p}\|_\infty = |I|^{-1/p} = 2^{n/p}$. Denote for $s = (s_1, \dots, s_d)$

$$J_s := \{I \in \Lambda_1 : I = I_1 \times \cdots \times I_d, |I_j| = 2^{-s_j}, j = 1, \dots, d\}, \quad A_s := \cup_{I \in J_s} I.$$

Then we obtain from (3.5)

$$\|g_1\|_p \asymp 2^{n/p} \left\| \left(\sum_s \chi_{A_s} \right)^{1/2} \right\|_p. \tag{3.6}$$

Using the following two inequalities

$$\sum_s \chi_{A_s} \leq |\{s : \|s\|_1 = n\}|, \quad \int_{[0,1]^d} \sum_s \chi_{A_s} dx = k2^{-n} \geq c|\{s : \|s\|_1 = n\}|,$$

we get from (3.6)

$$\|g_1\|_p \asymp 2^{n/p} n^{(d-1)/2}.$$

$\square$

Now let us return to the proof of Theorem 3.2. We begin with the case $2 \le p < \infty$. For a given m find an n in such a way that it is a minimal natural number satisfying $m < [|\Lambda(n)|/2]$, where $\Lambda(n)$ is defined in Lemma 3.3. Note that $m \asymp 2^n n^{d-1}$ and $n \asymp \log m$. We consider the function $g_m = m^{-r} g_1$. It is clear that $g_m \in F^r(\mathcal{H}_p^d)$. For

$$f = \sum_I^\infty c_I(f, \mathcal{H}_p^d) H_{I,p}$$

we define the following expansional best m-term approximation of f

$$\tilde{\sigma}_m(f) := \inf_{|\Lambda|=m} \left\| f - \sum_{I\in\Lambda} c_I(f, \mathcal{H}_p^d) H_{I,p} \right\|.$$

It follows from Lemma 3.3 that

$$\tilde{\sigma}_m(g_m)_p \ge c(\log m)^{h(p,d)} m^{1/p-r}.$$

It is known that for an unconditional basis Ψ we have

$$\tilde{\sigma}_m(f) \asymp \sigma_m(f).$$

Therefore we complete the proof in the case $2 \le p < \infty$. The case of $1 < p \le 2$ can be proved in a similar way by setting $g_m = m^{-r} g_2$. $\square$

In Theorem 3.1 we assume that a basis Ψ^d has a special structure, namely, Ψ^d is a tensor product of greedy bases. Also, Theorem 3.1 holds for a special Banach space $L_p([0,1]^d)$, $1 < p < \infty$. We now discuss a question of under which other assumptions on a basis and a Banach space we can obtain results similar to Theorem 3.1. We first recall the definition of bases which are called unconditional for constant coefficients, cf. [17].

Definition A basis Ψ is called unconditional for constant coefficients (UCC) if there exist constants C_1 and C_2 such that for each finite subset $A \subset \mathbb{N}$ and for each choice of signs $\varepsilon_i = \pm 1$ we have

$$C_1 \left\| \sum_{i\in A} \psi_i \right\| \le \left\| \sum_{i\in A} \varepsilon_i \psi_i \right\| \le C_2 \left\| \sum_{i\in A} \psi_i \right\|.$$

To formulate our results we need some of the basic concepts of the Banach space theory from [9]. First, let us recall the definition of type and cotype. Let $\{\varepsilon_i\}$ be a sequence of independent Rademacher variables. We say that a Banach space X has type p if there exists a universal constant C_3 such that for $f_k \in X$

$$\left(\mathrm{Ave}_{\varepsilon_k=\pm1} \left\| \sum_{k=1}^n \varepsilon_k f_k \right\|^p \right)^{1/p} \le C_3 \left(\sum_{k=1}^n \|f_k\|^p \right)^{1/p},$$

and X is of cotype q if there exists a universal constant C_4 such that for $f_k \in X$

$$\left(\mathrm{Ave}_{\varepsilon_k=\pm1}\left\|\sum_{k=1}^{n}\varepsilon_k f_k\right\|^q\right)^{1/q} \geq C_4\left(\sum_{k=1}^{n}\|f_k\|^q\right)^{1/q}.$$

Theorem 3.4 *Let X be a Banach space with type $1 < p \leq 2$. If a basis Ψ of X is UCC, then for $r > 1/p$*

$$G_m(F^r, \Psi) \ll m^{1/p-r}.$$

Proof: Since X has type p, we have

$$\left(\mathrm{Ave}_{\varepsilon=\pm1}\left\|\sum_{k\in\Lambda}\varepsilon_k\psi_k\right\|^p\right)^{1/p} \leq C\left(\sum_{k\in\Lambda}\|\psi_k\|^p\right)^{1/p} \ll |\Lambda|^{1/p}.$$

Our assumption that Ψ is UCC implies

$$\left\|\sum_{k\in\Lambda}\psi_k\right\| \asymp \left(\mathrm{Ave}_{\varepsilon=\pm1}\left\|\sum_{k\in\Lambda}\varepsilon_k\psi_k\right\|^p\right)^{1/p} \ll |\Lambda|^{1/p}.$$

We now need the following lemma.

Lemma 3.5 *Assume that Ψ satisfies the Definition 3. Then for any finite subset $A \subset \mathbb{N}$ one has*

$$\left\|\sum_{k\in A}c_k\psi_k\right\| \leq C_2\max_{k\in A}|c_k|\left\|\sum_{k\in A}\psi_k\right\|.$$

The proof of this lemma repeats the argument from the proof of Theorem 2.13.

So we obtain

$$\left\|\sum_{k\in\Lambda}c_k\psi_k\right\| \ll \max_{k\in\Lambda}|c_k|\cdot|\Lambda|^{1/p}.$$

Therefore, for any $f \in F^r$ we get

$$\|f - G_m(f)\| \leq \sum_{s=0}^{\infty}\left\|\sum_{k=2^s m+1}^{2^{s+1}m}c_k(f)\psi_k\right\| \ll \sum_{s=0}^{\infty}(2^s m)^{-r}(2^s m)^{1/p} \asymp m^{1/p-r}.$$

□

Let us make some comparison of Theorem 3.1 with Theorem 3.4. It is known that $L_p([0,1]^d)$, $1 \leq p < \infty$, has type $\min(2,p)$. It is also known that a greedy basis Ψ is an unconditional basis and, therefore, Ψ^d is an unconditional basis for $L_p([0,1]^d)$, $1 < p < \infty$. Thus, in the case $1 < p \leq 2$, Theorem 3.1 follows from Theorem 3.4. In the case $2 < p < \infty$ Theorem 3.1 gives a better bound than Theorem 3.4.

4 Greedy approximation with regard to quasi-greedy bases

We now proceed to a further discussion of quasi-greedy bases. In particular, Lemma 2.12 implies that a quasi-greedy basis is a UCC basis. We begin our discussion with the case of a Hilbert space. It is easy to see that for a normalized basis Ψ

$$\mathrm{Ave}_{\varepsilon_k=\pm 1}\left\langle \sum_{k\in\Lambda}\epsilon_k\psi_k, \sum_{l\in\Lambda}\epsilon_l\psi_l\right\rangle = |\Lambda|.$$

Therefore, for a normalized UCC basis Ψ of a Hilbert space one has

$$\left\|\sum_{k\in\Lambda}\psi_k\right\| \asymp |\Lambda|^{1/2}.$$

This means that a quasi-greedy basis of a Hilbert space is automatically a democratic basis, which in general is defined as follows. **Definition** We recall that a basis $\{\psi_n\}_{n=1}^{\infty}$ in a Banach space X is called democratic if for any two finite sets of indices P and Q with the same cardinality, we have

$$\left\|\sum_{n\in P}\psi_n\right\| \le D\left\|\sum_{n\in Q}\psi_n\right\|$$

with a constant D independent of P and Q. The above property of quasi-greedy bases in Hilbert spaces was observed in [17]. In [4] it was proved that for any quasi-greedy and democratic basis (almost greedy basis) Ψ of a Banach space X the following inequality holds for any $f\in X$ and $\lambda>1$

$$\|f-G_{\lambda m}(f,\Psi)\| \le C(\lambda)\sigma_m(f,\Psi).$$

Concluding the above discussion, we can formulate the following theorem.

Theorem 4.1 *Let Ψ be a normalized quasi-greedy basis of a Hilbert space H. Then, for any $f\in H$ and $\lambda>1$*

$$\|f-G_{\lambda m}(f,\Psi)\| \le C(\lambda)\sigma_m(f,\Psi).$$

We pointed out in the Introduction that it follows from the known results that we cannot let λ to take value 1 in Theorem 4.1. It is mentioned in [17] that in this case ($\lambda=1$) one has the following inequality

$$\|f-G_m(f,\Psi)\| \le C(\log m)\sigma_m(f,\Psi).$$

We do not know if the above inequality is sharp in the sense that an extra factor $\log m$ cannot be replaced by a slower growing factor. It follows from the above discussion that it cannot be replaced by a constant.

We now proceed to a discussion of quasi-greedy bases in L_p spaces. The results that we present extend the corresponding results for a Hilbert space from [17]. Following [17] we will use the following notations here. For a sequence $\{a_k\}\in c_0$ we denote

$$|a_{k_1}|\ge|a_{k_2}|\ge\dots,\qquad a_n^* := |a_{k_n}|.$$

Theorem 4.2 *Let $\{\psi_k\}_{k=1}^\infty$ be a quasi-greedy basis of the L_p space, $1<p<\infty$. Then for each $f=\sum_{k=1}^\infty a_k\psi_k$ we have*

$$C_1(p)\sup_n n^{1/p}a_n^* \le \|f\|_p \le C_2(p)\sum_{n=1}^\infty n^{-1/2}a_n^*, \quad 2\le p<\infty;$$

$$C_3(p)\sup_n n^{1/2}a_n^* \le \|f\|_p \le C_4(p)\sum_{n=1}^\infty n^{1/p-1}a_n^*, \quad 1<p\le 2.$$

Proof: Denote $\mathcal{N}_s := \{n : a_n^* \ge 2^{-s}\}$ and $N_s := |\mathcal{N}_s|$. The proofs in both cases $1<p\le 2$ and $2\le p<\infty$ are similar. We will give a proof only in the case $2\le p<\infty$. First, we prove the upper bound for $\|f\|_p$. In this case we have

$$\|f\|_p \le |a_{k_1}| + \left\|\sum_s \sum_{n\in\mathcal{N}_s\setminus\mathcal{N}_{s-1}} a_{k_n}\psi_{k_n}\right\|_p.$$

Using Lemma 2.12 we get

$$\|f\|_p \le |a_{k_1}| + 4K\sum_s 2^{-s}\left\|\sum_{n\in\mathcal{N}_s\setminus\mathcal{N}_{s-1}} \psi_{k_n}\right\|_p. \tag{4.7}$$

The L_p space has type 2 for $2\le p<\infty$. Therefore,

$$\left\|\sum_{n\in\mathcal{N}_s\setminus\mathcal{N}_{s-1}} \psi_{k_n}\right\|_p \le C(p)N_s^{1/2}, \tag{4.8}$$

and

$$\|f\| \le |a_1^*| + C(p)\sum_s 2^{-s}N_s^{1/2} \le |a_1^*| + C(p)\sum_s 2^{-s}\sum_{n=1}^{N_s} n^{-1/2}$$

$$\le |a_1^*| + C(p)\sum_{n=1}^\infty n^{-1/2}a_n^* \le C_2(p)\sum_{n=1} n^{-1/2}a_n^*.$$

Second, we prove the lower bound for the $\|f\|_p$. From the definition of quasi-greedy basis we have for each n

$$\|f\|_p \ge K^{-1}\left\|\sum_{l=1}^n a_{k_l}\psi_{k_l}\right\|_p. \tag{4.9}$$

By Lemma 2.12 we get

$$\left\|\sum_{l=1}^n a_{k_l}\psi_{k_l}\right\|_p \ge (4K^2)^{-1}|a_{k_n}|\left\|\sum_{l=1}^n \psi_{k_l}\right\|_p. \tag{4.10}$$

The L_p space with $2 \le p < \infty$ is of cotype p. Therefore,

$$\left\| \sum_{l=1}^{n} \psi_{k_l} \right\|_p \ge C(p) n^{1/p}. \tag{4.11}$$

Combining (4.9) – (4.11) we obtain the required lower bound. $\square$

As a direct corollary of Theorem 4.2 we get the following inequality for any P and Q of cardinality m

$$\left\| \sum_{k \in P} \psi_k \right\|_p / \left\| \sum_{k \in Q} \psi_k \right\|_p \le C(p) m^{|1/2-1/p|}, \qquad 1 < p < \infty. \tag{4.12}$$

It is well known (see [13], [17]) how inequalities like (4.12) can be used in estimating $\|f - G_m(f)\|$ in terms of $\sigma_m(f)$. In particular, (4.12) and Theorem 4 from [17] imply the following result.

Theorem 4.3 *Let $1 < p < \infty$ and let Ψ be a quasi-greedy basis of the L_p space. Then for each $f \in L_p$ we have*

$$\|f - G_m(f, \Psi)\|_p \le C(p) m^{|1/2-1/p|} \sigma_m(f, \Psi).$$

We note (see [11]) that similar inequality holds for the trigonometric system that is not a quasi-greedy basis for L_p, $p \neq 2$.

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