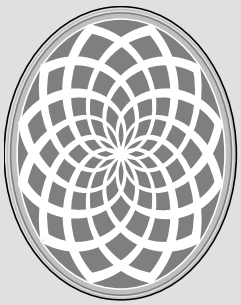

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HIGHLY EFFECTIVE STABLE EVALUATION OF BANDLIMITED FUNCTIONS ON THE SPHERE

KAMEN IVANOV AND PENCHO PETRUSHEV

Abstract. An algorithm for fast and accurate evaluation of band-limited functions at many scattered points on the unit 2-d sphere is developed. The algorithm is based on trigonometric representation of spherical harmonics in spherical coordinates and highly localized tensor-product trigonometric kernels (needlets). It is simple, fast, local, memory efficient, numerically stable and with guaranteed accuracy. Comparison of this algorithm with other existing algorithms in the literature is also presented.

1. Introduction

The fast and accurate evaluation of band limited functions on the sphere is important for many areas such as Geodesy and Geomagnetism. In this article the *band limited functions* on the 2-d sphere $\mathbb{S}^2$ will be termed *spherical polynomials* and we are interested in developing an algorithm for efficient evaluation of high degree (> 2000) spherical polynomials at many (millions) scattered locations on $\mathbb{S}^2$. Our requirements on the algorithm are to be fast, simple, local, numerically stable, and memory efficient with the emphasis on its speed and stability.

A natural standard approach to evaluation of a spherical polynomial given by its coefficients in the standard basis is to

(i) compute the values of the polynomial at regular grid points on $\mathbb{S}^2$, and

(ii) use these to evaluate the polynomial at arbitrary points on $\mathbb{S}^2$.

We shall utilize this approach in the present paper.

Regular grid points on $\mathbb{S}^2$ will be points which are equally spaced with respect to their spherical coordinates (θ, λ). This kind of grid points have the obvious drawback that they concentrate around the poles, but this is fully compensated by the possibility of applying fast Fourier methods. Here we adhere to the fundamental principle, put forward in [8], that high degree spherical polynomials are better represented by their values at regular grid points than by their coefficients.

In the present paper we focus entirely on problem (ii). Our key observation is that a simple extension of every spherical polynomial expressed in spherical coordinates is a trigonometric polynomial in two variables and, therefore, can be represented by highly localized tensor product trigonometric kernels (needlets). This allows us to develop a simple and effective evaluation algorithm.

Thus our algorithm for evaluation of spherical polynomials hinges on our ability to rapidly and accurately compute the values of high degree univariate trigonometric polynomials given their values at equally distributed points. For this we deploy highly localized reproducing kernels of the form $\mathcal{K}_N(x) = 1 + 2\sum_n \varphi\big(\frac{n}{N}\big)\cos nx$, where the cutoff function φ is smooth, $\varphi = 1$ on $[0,1]$ and $\operatorname{supp}\varphi \subset [0, 1+\tau]$, for some $\tau > 0$. We term this kind of kernels *trigonometric needlets*. Clearly,

$\mathcal{K}_N * f = f$ for every trigonometric polynomial f of degree $\le N$. Denoting by $\mathcal{X}$ a set of M (sufficiently large) equally spaced grid points on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{N}$ the approximating needlet operator takes the form

$$\Phi_{N,\delta} f(x) = \sum_{\xi\in\mathcal{X}:\rho(\xi,x)\le\delta} M^{-1}\mathcal{K}_N(x-\xi)f(\xi),$$

where $\delta > 0$ is a small parameter and $\rho(x, y)$ is the distance on $\mathbb{T}$. As will be shown in Section 3 the superb localization of the needlet kernel leads to a short sum above and hence to fast algorithm, while keeping the error of approximation small. A thorough analysis of the relationship between the selection of the cutoff function φ, the parameters δ, τ, the (relative) error ε and the degree N is conducted. It is shown that for practical purposes the optimal $\delta \approx \frac{2\ln 1/\varepsilon}{\tau N}$ in the case of our best cutoff function φ. The operator norms of $\|\Phi_{N,\delta}\|_{\infty\to\infty}$ are explored for different values of τ, ε and N. It turns out that $\|\Phi_{N,\delta}\|_{\infty\to\infty} \approx 2$ and these norms are practically independent of the degree N. A surprising feature of the trigonometric needlet operator $\Phi_{N,\delta}$ is that $\Phi_{N,\delta} f$ interpolates f for the minimum value of M and symmetric cutoff functions φ.

The trigonometric needlet algorithm is compared with two other algorithms for fast evaluation of trigonometric polynomials: (i) Spline interpolation based on Lagrange interpolation, and (ii) Nonequispaced fast Fourier transform (NFFT) with Kaiser–Bessel window function. A quantitative analysis of the error and complexity of the algorithm is presented. The upsides and downsides of these algorithms are clearly delineated.

As indicated above our algorithm for fast evaluation of spherical polynomials at many scattered points relies on the simple fact that every spherical polynomial expressed in spherical coordinates readily extends as a trigonometric polynomial in two variables. The tensor product needlet operator takes the form

$$\Phi^2_{N,\delta} f(\theta,\lambda) = \frac{1}{4KL} \sum_{k:\rho(\theta,\theta_k)\le\delta} \sum_{\ell:\rho(\lambda,\lambda_k)\le\delta} \mathcal{K}_N(\theta-\theta_k)\mathcal{K}_N(\lambda-\lambda_\ell) f(\theta_k,\lambda_\ell), \tag{1.1}$$

where $\{(\theta_k, \lambda_\ell)\}$ are $4KL$ equally distributed grid points in spherical coordinates. Our algorithms uses the operator $\Phi^2_{N,\delta}$ for fast and accurate evaluation or approximation of high degree spherical polynomials. This algorithm inherits all valuable features of the univariate algorithm for evaluation of trigonometric polynomials at scattered points mentioned above, namely, it is fast, numerically stable, local, memory efficient, and simple.

To put this algorithm in prospective we would like to compare it briefly with our spherical needlet algorithm from [8]. In [8] we used reproducing kernels of the form $\mathcal{K}_N(x\cdot\xi)$ with $\mathcal{K}_N(u) = \sum_n \varphi\big(\frac{n}{N}\big)(2n+1)P_n(u)$, where P_n is the nth degree Legendre polynomial and φ is a cutoff function as above. This kernel is highly localised and the operator $\Phi_N f(x) := (4\pi)^{-1}\int_{\mathbb{S}^2} \mathcal{K}_N(x\cdot\xi) f(\xi) d\sigma(\xi)$ reproduces spherical polynomials of degree $\le N$. In [8] we discretize this operator by using a cubature formula on the sphere with nodes $\mathcal{X}$ and weights $w_\xi, \xi\in\mathcal{X}$, and then truncate it to derive an approximating operator of the form:

$$\Phi^\star_{N,\delta} f(x) := \sum_{\xi\in\mathcal{X}:\rho(\xi,x)\le\delta} w_\xi \mathcal{K}_N(x\cdot\xi) f(\xi). \tag{1.2}$$

Here $\mathcal{X}$ is a set of regular grid points on $\mathbb{S}^2$, ρ is the geodesic distance on $\mathbb{S}^2$ and $\delta > 0$ is a small parameter. The idea is the same, the superb localization

of $\mathcal{K}_N(x\cdot\xi)$ allows to use a short sum in (1.2) and at the same time to keep the error under control. The algorithm that uses the operator $\Phi^{\star}_{N,\delta}$ in the place of $\Phi^{2}_{N,\delta}$, however, is more complicated and is computationally more costly. More detailed comparison between these two methods is given in Subsection 5.4. In Subsection 5.4 we also compare our tensor product trigonometric needlet method with the tensor product Lagrange interpolation method and the nonequispaced fast spherical Fourier transform algorithm of Kunis and Potts [15].

A Matlab realization of the tensor product trigonometric needlet algorithm is created and its performance is compared to our spherical needlet software on examples of spherical polynomials of degree 2160.

This article is a followup of [8], where the spherical needlet algorithm mentioned above is developed. The main ideas of this paper are rooted in [8], which is also a useful source for references on the subject.

The organization of the paper is as follows. The spherical harmonics standard basis is reviewed in Section 2 and the problem for fast evaluation of spherical polynomials is stated clearly. The development of our needlet algorithm for fast evaluation of univariate trigonometric polynomials and its comparison with relevant algorithms occupy Section 3. The general idea of the tensor product trigonometric needlet algorithm is presented in Section 4. The tensor product trigonometric needlet algorithm on $\mathbb{S}^2$ is developed in Section 5, where it is also compared with other algorithms. The results of experiments are also provided in Section 5.

Notation. We shall denote by $c, c_1, c_2, \dots$ positive constants which may vary at every appearance and by $\bar{c}, \tilde{c}, c', c''$ and the alike positive constants which preserve their values throughout the paper. The relation $f\sim g$ between functions f and g means $c_1 f\le g\le c_2 f$, while $f\approx g$ is used when $f/g\to 1$ under an appropriate limit of the argument.

2. Background and statement of the problem

We next review the basics of spherical harmonics and state precisely the problems of interest to us.

2.1. **Spherical harmonics: Background.**

Denote by $\mathcal{H}_n$ ($n\ge 0$) the space of all *spherical harmonics of degree* n on $\mathbb{S}^2$. We shall represent spherical harmonics in spherical coordinates. Recall the relationship between the cartesian coordinates (x_1,x_2,x_3) and the spherical coordinates (θ,λ), $0\le\theta\le\pi$, $0\le\lambda<2\pi$, of a point x on the unit 2-d sphere $\mathbb{S}^2$: $x=(x_1,x_2,x_3)=(\sin\theta\cos\lambda,\sin\theta\sin\lambda,\cos\theta)$.

The standard *orthonormal* basis $\{\tilde{\mathcal{C}}_{nm}\}_{m=0}^{n}\cup\{\tilde{\mathcal{S}}_{nm}\}_{m=1}^{n}$ for $\mathcal{H}_n$ is defined in terms of the associated Legendre functions P_{nm}. Namely, for $x=(\theta,\lambda)$

$$\begin{aligned}\tilde{\mathcal{C}}_{nm}(x)&=q_{nm}P_{nm}(\cos\theta)\cos m\lambda,\quad m=0,1,\dots,n,\\ \tilde{\mathcal{S}}_{nm}(x)&=q_{nm}P_{nm}(\cos\theta)\sin m\lambda,\quad m=1,2,\dots,n,\end{aligned}\tag{2.1}$$

where $P_{nm}(u)=(1-u^2)^{m/2}\big(\frac{d}{du}\big)^m P_n(u)$ with P_n being the nth degree Legendre polynomial and the coefficients q_{nm} are selected so that $\tilde{\mathcal{C}}_{nm}$, $\tilde{\mathcal{S}}_{nm}$ are normalized in $L^2(\mathbb{S}^2)$. We refer the reader to [19] and [16] for more details about spherical harmonics.

In the standard basis (2.1) a spherical polynomial f of degree $\leq N$ is usually given by its coefficients $\{a_{nm}, b_{nm}\}$, i.e.

$$f(x) = \sum_{n=0}^{N} \sum_{m=0}^{n} \left(a_{nm} \tilde{\mathcal{C}}_{nm}(x) + b_{nm} \tilde{\mathcal{S}}_{nm}(x) \right). \tag{2.2}$$

We shall denote by $\mathcal{P}_N$ the set of all spherical polynomials of degree $\leq N$.

Extension of spherical polynomials in spherical coordinates. From (2.1) we have for $\theta \in [0, \pi]$ and $\lambda \in [0, 2\pi)$

$$\begin{aligned} \tilde{\mathcal{C}}_{nm}(\theta, \lambda) &= q_{nm} \sin^m \theta \, P_n^{(m)}(\cos\theta) \cos m\lambda, \\ \tilde{\mathcal{S}}_{nm}(\theta, \lambda) &= q_{nm} \sin^m \theta \, P_n^{(m)}(\cos\theta) \sin m\lambda. \end{aligned} \tag{2.3}$$

We use these identities to extend $\tilde{\mathcal{C}}_{nm}$ and $\tilde{\mathcal{S}}_{nm}$ for all $\theta, \lambda \in \mathbb{R}$, and in turn we use the latter and (2.2) to extend the spherical polynomial $f(\theta, \lambda)$ for $\theta, \lambda \in \mathbb{R}$.

The following claim will play a key role in this article:

Proposition 2.1. *Let f be a spherical polynomial of degree $\leq N$ and assume that in spherical coordinates f is extended as above. Then $f(\theta, \lambda)$ can be expressed in the form*

$$f(\theta, \lambda) = \sum_{k=-N}^{N} \sum_{\ell=-N}^{N} c_{k\ell} e^{i(k\theta + \ell\lambda)}, \tag{2.4}$$

where c_{nm} are (complex) coefficients. More generally, the restriction of f over every circle on $\mathbb{S}^2$ is a trigonometric polynomial of degree $\leq N$.

Furthermore,

$$f(-\theta, \lambda + \pi) = f(\theta, \lambda) \quad \textit{for} \quad \theta, \lambda \in \mathbb{R}. \tag{2.5}$$

Proof. The representation (2.4) is immediate from (2.3). It is well known that each space $\mathcal{H}_n$ of spherical harmonics is invariant with respect to the rotation group on $\mathbb{S}^2$. Therefore, by applying an appropriate rotation (2.2) and (2.1) imply that the restriction of f over every circle on $\mathbb{S}^2$ is a trigonometric polynomial of degree $\leq N$. It is easily seen that $\tilde{\mathcal{C}}_{nm}$ and $\tilde{\mathcal{S}}_{nm}$ verify property (2.5) and hence (2.5) holds in general. □

2.2. **The problem of spherical polynomial evaluation.** We are interested in the following

Problem 1. Given a spherical polynomial f with its coefficients $\{a_{nm}, b_{nm}\}$, evaluate $f(x)$ at arbitrary (scattered) points $x \in \mathcal{Z}$, on the sphere $\mathbb{S}^2$ with prescribed precision $\varepsilon_0 > 0$, measured in the uniform norm.

We split this problem into two problems:

Problem 2. Given a spherical polynomial f with its coefficients $\{a_{nm}, b_{nm}\}$, evaluate $f(\xi)$ at all points ξ from a regular grid $\mathcal{X}$ on $\mathbb{S}^2$.

Problem 3. Given the values $f(\xi)$ of a spherical polynomial f at regular grid points $\xi \in \mathcal{X}$, evaluate $f(x)$ at arbitrary (scattered) points $x \in \mathcal{Z}$, on the sphere $\mathbb{S}^2$ with precision ε_0.

Regular grid points on $\mathbb{S}^2$ will be points $\{(\theta_k, \lambda_\ell)\}$ in spherical coordinates, where $\{\lambda_\ell\}$ and $\{\theta_k\}$ are equally distributed. Given $K, L \geq 1$ we define two sets of regular

grid poits $\mathcal{X}^{(i)} = \{\xi_{k,\ell}^{(i)} = (\theta_k^{(i)}, \lambda_\ell^{(i)})\}$, $i = 1, 2$, by

$$\theta_k^{(1)} = \frac{\pi}{K}k,\ k = 0, 1, \dots, K;\ \lambda_\ell^{(1)} = \frac{\pi}{L}\ell,\ \ell = 0, 1, \dots, 2L-1; \tag{2.6}$$

and

$$\theta_k^{(2)} = \frac{\pi}{K}\left(k + \frac{1}{2}\right),\ k = 0, 1, \dots, K-1;\ \lambda_\ell^{(2)} = \frac{\pi}{L}\ell,\ \ell = 0, 1, \dots, 2L-1. \tag{2.7}$$

Here in $\mathcal{X}^{(1)}$ we consider only one node for $k = 0$ (the North Pole) and one node for $k = K$ (the South Pole).

The relations between K, L and N above will be given in Section 5.

In this paper we focus on Problem 3. We shall use the representation of spherical polynomials from Proposition 2.1 to develop an effective algorithm for this problem.

3. Fast and accurate evaluation of trigonometric polynomials

The first step in developing our method for evaluation of spherical polynomials is to develop such an algorithm in the univariate case. Our method relies on highly localized kernels (needlets) which reproduce trigonometric polynomials.

3.1. **Trigonometric needlets.**

As is well known the Nth partial sum of the Fourier series of a 2π-periodic function f takes the form

$$S_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-y)f(y)dy, \quad \text{where} \quad D_N(x) = 1 + 2\sum_{n=1}^{N}\cos nx$$

is the Dirichlet kernel. Clearly $f = S_N f$ for every $f \in \Pi_N$, where Π_N denotes the set of all trigonometric polynomials of degree $\le N$.

The Dirichlet kernel is poorly localized and hence not suitable for evaluation of trigonometric polynomials. Instead we shall utilize reproducing operators with highly localized kernels defined by

$$\mathcal{K}_N(x) = 1 + 2\sum_{n=1}^{\infty}\varphi\Big(\frac{n}{N}\Big)\cos nx, \tag{3.1}$$

where φ is a cutoff function with the properties:

$$\varphi \in C[0,\infty);\ \varphi(t) = 1,\ t \in [0,1];\ 0 \le \varphi(t) \le 1,\ t \in [1, 1+\tau]; \quad \text{and } \varphi(t) = 0,\ t \ge 1+\tau; \tag{3.2}$$

for some $\tau > 0$.

Consider the linear operator

$$\widetilde{\Phi}_N f(x) = \mathcal{K}_N * f(x) := \frac{1}{2\pi}\int_0^{2\pi}\mathcal{K}_N(x-y)f(y)\,dy. \tag{3.3}$$

Clearly,

$$\widetilde{\Phi}_N f = f, \quad \forall f \in \Pi_N, \quad \text{and} \tag{3.4}$$

$$\widetilde{\Phi}_N f \in \Pi_{N_\tau - 1}, \quad \forall f \in L^1(\mathbb{T}),\ \text{with } N_\tau = \lceil N + \tau N \rceil. \tag{3.5}$$

The kernel $\mathcal{K}_N$ can be viewed as a mollified version of the Dirichlet kernel.

Our method relies heavily on the fact that for cutoff functions $\varphi \in C^\infty$ the kernel $\mathcal{K}_N$ from (3.1) has nearly exponential localization. This follows from the following somewhat more general assertion.

Theorem 3.1. *Let $\varphi \in C^\infty(\mathbb{R})$ be compactly supported and set*

$$\mathcal{K}_N(x) := \sum_{n\in\mathbb{Z}} \varphi\Big(\frac{n}{N}\Big) e^{inx}.$$

Then for any $\sigma > 0$ there exists a constant $c_\sigma > 0$ such that

$$|\mathcal{K}_N(x)| \le c_\sigma N(1+N|x|)^{-\sigma}, \quad |x| \le \pi. \tag{3.6}$$

This claim is known and easy to prove but nowhere to find in the standard Fourier series literature, see [9] for a proof.

The nearly exponential localization of $\mathcal{K}_N$ can be improved to sub-exponential by selecting the cutoff function $\varphi \in C^\infty[0,\infty)$ to be with "small" derivatives. As is shown in [9, Theorem 3.1] for any $\varepsilon > 0$ there exists a cutoff function φ satisfying (3.2) such that

$$\|\varphi\|_\infty \le c, \quad \frac{1}{k!}\|\varphi^{(k)}\|_\infty \le c\left(c'[\ln(e+k-1)]^{1+\varepsilon}\right)^k, \quad k = 1,2,\dots \tag{3.7}$$

for some constants $c, c' > 0$ depending only on ε and τ. (A more general statement is established in [10, Theorem 2.3].)

Using this kind of cutoff functions we get the following sub-exponential localization result for the kernels $\mathcal{K}_N$ (see [9, Theorem 5.1]):

Theorem 3.2. *If φ satisfies* (3.2) *and* (3.7)*, then the kernels $\mathcal{K}_N$ from* (3.1) *obeys*

$$|\mathcal{K}_N(x)| \le c_1 N \exp\left\{-\frac{c_2 N|x|}{[\ln(e+N|x|)]^{1+\varepsilon}}\right\}, \quad |x| \le \pi, \tag{3.8}$$

where $c_1, c_2 > 0$ are constants depending only on ε and τ. Here $\varepsilon > 0$ cannot be removed.

Observe that for cutoff functions $\varphi \in C^\infty$ (in fact, much less is needed) the sequence of operators $\left\{\widetilde{\Phi}_N\right\}_{N=0}^\infty$ is uniformly bounded on L^p, i.e.

$$\|\widetilde{\Phi}_N f\|_{L^p(\mathbb{T})} \le c\|f\|_{L^p(\mathbb{T})}, \quad \forall f \in L^p(\mathbb{T}), \quad 1 \le p \le \infty, \tag{3.9}$$

with a constant $c > 0$ depending only on φ. This is immediate from the inequality $\frac{1}{2\pi}\int_{\mathbb{T}} |\mathcal{K}_N(x)|\,dx \le c < \infty$, which follows from (3.6) with some $\sigma > 1$.

3.2. **Discrete trigonometric "needlet" operators.** We next discretize the operator $\widetilde{\Phi}_N$ from (3.3) by using the simple quadrature formula

$$\frac{1}{2\pi}\int_0^{2\pi} F(y)\,dy \sim \sum_{\xi\in\mathcal{X}} M^{-1} F(\xi), \tag{3.10}$$

where

$$\mathcal{X} = \mathcal{X}_M := \{\xi_k = 2\pi M^{-1}k : k = 0,1,\dots,M-1\}. \tag{3.11}$$

It is readily seen that this quadrature formula is exact for trigonometric polynomials F of degree $\le M-1$.

Applying quadrature (3.10) to the integral in (3.3) we obtain a discrete counterpart of the operator $\widetilde{\Phi}_N$, namely,

$$\Phi_N f(x) := \sum_{\xi\in\mathcal{X}} M^{-1}\mathcal{K}_N(x-\xi) f(\xi). \tag{3.12}$$

In the next theorem we collect some simple properties of this operator.

Theorem 3.3. *Let φ satisfy (3.2) for some $\tau > 0$ and assume $\mathcal{K}_N$, defined in (3.1), obeys (3.6) with some $\sigma > 1$. If $M \geq N$, then Φ_N satisfies:*

$$\Phi_N : \ell^\infty(\mathcal{X}) \to C(\mathbb{T}) \text{ is a bounded linear operator}; \tag{3.13}$$

$$\|\Phi_N\|_{\ell^\infty(\mathcal{X})\to C(\mathbb{T})} \leq C, \text{ where } C > 0 \text{ is a constant independent of } N; \tag{3.14}$$

$$\Phi_N f \in \Pi_{N_\tau - 1} \quad \forall f \in \ell^\infty(\mathcal{X}) \text{ with } N_\tau = \lceil N + \tau N \rceil . \tag{3.15}$$

Moreover, if $M \geq N + N_\tau$, then

$$\Phi_N f = f \quad \forall f \in \Pi_N; \tag{3.16}$$

$$\|f - \Phi_N f\|_{C(\mathbb{T})} \leq (\|\Phi_N\| + 1) E_N(f)_\infty \quad \forall f \in C(\mathbb{T}). \tag{3.17}$$

Here $E_N(f)_\infty := \inf_{g\in\Pi_N} \|f - g\|_\infty$.

Proof. The boundedness of the operator Φ_N in (3.13) follows from (3.12). For (3.14) we use (3.6) to obtain

$$\|\Phi_N\|_{\ell^\infty(\mathcal{X})\to C(\mathbb{T})} \leq c_\sigma(N/M) \sum_{m=0}^{M/2} 2(1 + 2\pi m N/M)^{-\sigma} \leq C.$$

Further, (3.15) follows from (3.12) and $\mathcal{K}_N \in \Pi_{N_\tau - 1}$. Identity (3.16) follows from (3.4) because $\mathcal{K}_N(x - \cdot) f(\cdot)$ is a trigonometric polynomial of degree $N + N_\tau - 1$ for every x. Finally, (3.17) follows from (3.16) and (3.13). □

Observe that Theorem 3.3 yields

$$E_{N_\tau - 1}(f)_\infty \leq \|f - \Phi_N f\|_{C(\mathbb{T})} \leq c E_N(f)_\infty \quad \text{for} \quad f \in C(\mathbb{T}),$$

which shows the excellent approximation property of the sequence of operators Φ_N.

However, the number of terms in (3.12) is quite large for any meaningful practical application. Having in mind the excellent localization of the kernel $\mathcal{K}_N$ (see Theorems 3.1 and 3.2) we truncate the sum in (3.12) and define

$$\Phi_{N,\delta} f(x) := \sum_{\substack{\xi\in\mathcal{X} \\ \rho(x,\xi)\leq\delta}} M^{-1} \mathcal{K}_N(x - \xi) f(\xi), \tag{3.18}$$

where $\delta > 0$ is a small parameter and $\rho(x, y) = \min_{n\in\mathbb{Z}} |x - y - 2\pi n|$ is the distance on $\mathbb{T}$.

In the following theorem we shall prove that a sufficient condition for evaluating $f(x), f \in \Pi_N$, by $\Phi_{N,\delta}(f, x)$ with error $\varepsilon\|f\|_\infty$ is

$$|\mathcal{K}_N(x)| \leq \varepsilon \quad \text{for} \quad \delta \leq |x| \leq \pi. \tag{3.19}$$

However, the bounds obtained in Theorem 3.6 bellow for δ satisfying (3.19) are not quite satisfactory. Observe that the majorants of $|\mathcal{K}_N(x)|$ given in (3.6) and (3.8), after reaching the value ε for $x = \delta$ from (3.19), preserve their fast decay for $\delta < x < \pi$. This means that it may be possible to select a smaller value of δ in the operator $\Phi_{N,\delta}$ and still have the same relative error ε. This can be achieved, for example, by replacing the uniform condition in (3.19) by an integral one:

$$\delta = \delta_1 + 2\pi M^{-1}, \quad \frac{1}{\pi} \int_{\delta_1}^{\pi} \mathcal{M}(\mathcal{K}_N, t)\, dt = \varepsilon, \tag{3.20}$$

where $\mathcal{M}$ is the maximal function

$$\mathcal{M}(g, x) := \sup_{y\in[x,2\pi-x]} |g(y)|, \quad 0 \leq x \leq \pi. \tag{3.21}$$

Theorem 3.4. *Let $N, M \in \mathbb{N}$, $\tau > 0$, $M \geq N + N_\tau$, and $0 < \varepsilon \leq 1$. Assume that φ satisfies* (3.2)*, $\mathcal{K}_N$ obeys* (3.6) *for some $\sigma > 1$ and δ is determined by* (3.19) *or* (3.20)*. Then the operator $\Phi_{N,\delta}$, defined in* (3.18)*, satisfies*

$$\Phi_{N,\delta} : \ell^\infty(\mathcal{X}) \to L^\infty(\mathbb{T}) \text{ is a bounded linear operator}; \tag{3.22}$$

$$\|\Phi_N f - \Phi_{N,\delta} f\|_{L^\infty(\mathbb{T})} \leq \varepsilon \|f\|_{\ell^\infty(\mathcal{X})} \quad \forall f \in \ell^\infty(\mathcal{X}); \tag{3.23}$$

$$\|f - \Phi_{N,\delta} f\|_{L^\infty(\mathbb{T})} \leq \varepsilon \|f\|_{\ell^\infty(\mathcal{X})} \quad \forall f \in \Pi_N; \tag{3.24}$$

$$\|f - \Phi_{N,\delta} f\|_{L^\infty(\mathbb{T})} \leq (C+1) E_N(f)_\infty + \varepsilon \|f\|_{\ell^\infty(\mathcal{X})} \quad \forall f \in C(\mathbb{T}), \tag{3.25}$$

where C is the constant from (3.14)*.*

Proof. The boundedness of the operator $\Phi_{N,\delta}$ in (3.22) follows from (3.18). For the proof of (3.23) we first assume that δ is determined by (3.19). Then

$$\sum_{\substack{\xi \in \mathcal{X} \\ \rho(x,\xi) > \delta}} M^{-1} |\mathcal{K}_N(x-\xi)| \leq \sum_{\substack{\xi \in \mathcal{X} \\ \rho(x,\xi) > \delta}} M^{-1} \, \varepsilon \leq \varepsilon. \tag{3.26}$$

Second, assume that δ is determined by (3.20). Using that $\mathcal{M}(g, \cdot)$ is a non-increasing function in $[0, \pi]$ we infer from (3.21)

$$\frac{1}{M} |\mathcal{K}_N(y)| \leq \frac{1}{2\pi} \int_{|y| - 2\pi M^{-1}}^{|y|} \mathcal{M}(\mathcal{K}_N, t) \, dt \quad \text{for} \quad 2\pi M^{-1} \leq |y| \leq \pi.$$

Now we use this inequality separately for $x - \xi = y \in (\delta, \pi]$ and for $x - \xi = y \in [-\pi, -\delta)$ and apply (3.20) to obtain for every $x \in \mathbb{T}$

$$\sum_{\substack{\xi \in \mathcal{X} \\ \rho(x,\xi) > \delta}} M^{-1} |\mathcal{K}_N(x-\xi)| \leq 2 \frac{1}{2\pi} \int_{\delta - 2\pi M^{-1}}^{\pi} \mathcal{M}(\mathcal{K}_N, t) \, dt = \varepsilon. \tag{3.27}$$

In light of (3.12) and (3.18), estimates (3.26)–(3.27) yield

$$|\Phi_N f(x) - \Phi_{N,\delta} f(x)| \leq \sum_{\substack{\xi \in \mathcal{X} \\ \rho(x,\xi) > \delta}} M^{-1} \, |\mathcal{K}_N(x-\xi)| \, \|f\|_{\ell^\infty(\mathcal{X})} \leq \varepsilon \|f\|_{\ell^\infty(\mathcal{X})}.$$

This completes the proof of (3.23). We now appeal to Theorem 3.3 to complete the proof of Theorem 3.4. □

3.3. **Selection of the cutoff function φ.** Our choice of a cutoff function φ satisfying (3.2) will be guided by the following rule: *Find the smallest possible δ for which* (3.23) *holds.* We first satisfy (3.19), which implies (3.23).

3.3.1. *The minimum δ for which* (3.19) *holds.* Given $\tau > 0$, a cutoff function φ satisfying (3.2), $0 < \varepsilon \leq 1$, and $N \geq 1$ we denote by $\delta_\infty(\varphi; \varepsilon, \tau, N)$ the minimum δ for which (3.19) holds. Functions φ that produce small δ are deemed good cutoff functions. Set

$$\delta_\infty(\varepsilon, \tau, N) = \inf_\varphi \delta_\infty(\varphi; \varepsilon, \tau, N),$$

where the infimum is taken on all φ satisfying (3.2). We are interested in establishing lower and upper bounds on $\delta_\infty(\varepsilon, \tau, N)$.

The upper bound relies on the following

Theorem 3.5. *Let $N \in \mathbb{N}$, $\sigma \in \mathbb{R}$, $1 \le \sigma < N$, and $\tau > 0$. There exist a cutoff function φ (defined by (7.4) in [8]) satisfying (3.2) and absolute constants $c_0, c_1 > 0$ such that*

$$|\mathcal{K}_N(x)| \le c_0(1+\tau)N \min\left\{1, \left(\frac{c_1\sigma}{\tau N|x|}\right)^{\sigma}\right\}, \quad |x| \le \pi. \tag{3.28}$$

This theorem follows from [8, Theorem 7.1] with $\alpha = \beta = -1/2$ and $k = \lceil\sigma\rceil$.

Theorem 3.5 provides an explicit form for the constant $c_\sigma = c_0(1+\tau)(c_1\sigma/\tau)^\sigma$ in Theorem 3.1 under the additional conditions (3.2). Note that the form of dependence of c_σ on the length of support $2+2\tau$ and the rate of decay σ cannot be essentially improved as follows from [11].

Matching lower and upper bounds for $\delta_\infty(\varepsilon, \tau, N)$ are given in

Theorem 3.6. *Let $N \in \mathbb{N}$, $0 < \varepsilon \le 1$ and $\tau \ge 1$. There exist absolute constants $c^-, c^+ > 0$ such that for $N > 2\ln(c_0(1+\tau)/\varepsilon)$ with c_0 from Theorem 3.5 we have*

$$c^- \frac{\ln(1/\varepsilon) + \ln N + \ln(1+\tau)}{\tau N} \le \delta_\infty(\varepsilon, \tau, N) \le c^+ \frac{\ln(1/\varepsilon) + \ln N + \ln(1+\tau)}{\tau N}. \tag{3.29}$$

Proof. For the upper bound in (3.29) we apply Theorem 3.5 with $\delta = ec_1\sigma/(\tau N)$ and $\sigma = \ln(c_0(1+\tau)N/\varepsilon)$ (hence $\sigma = \ln(c_0(1+\tau)/\varepsilon) + \ln N < N/2 + N/2 = N$) to obtain (3.19).

According to the proof of Theorem 3.2 in [8] a lower bound $\delta_0 \le \delta_\infty(\varepsilon, \tau, N)$ is determined as

$$\delta_0 = \arccos(8S(S+1)^{-2} - 1), \quad S = \left(R + \sqrt{R^2-1}\right)^{1/\nu}, \quad R = \varepsilon^{-1}\mathcal{K}_N(0),$$

where $\nu = \lceil N + \tau N\rceil - 1$ is the degree of $\mathcal{K}_N$. From the above we infer the following asymptotic representation of δ_0 for large N and small ε

$$\delta_0 \approx \frac{\ln(2\mathcal{K}_N(0)/\varepsilon)}{\nu} > \frac{\ln(4(1+\tilde{c}\tau)N/\varepsilon)}{2\tau N}, \tag{3.30}$$

where the estimate $\mathcal{K}_N(0) > 2(1+\tilde{c}\tau)N$ with some constant $\tilde{c}$ depending only on φ is used in the last inequality. This completes the proof. □

Remark 3.1. In fact, the upper bound in (3.29) holds for $0 < \tau < \infty$. The condition $\tau \ge 1$ is used only in obtaining the lower bound via the inequality in (3.30). For $0 < \tau < 1$ the denominator of the lower bound in (3.29) is $(1+\tau)N$ and does not match the denominator of the upper bound. Note that the localization of $\mathcal{K}_N$ gets worse when $\tau \to 0$ and necessarily $\delta_\infty(\varepsilon, \tau, N) \to \pi$ in this case.

Remark 3.2. The upper estimate in (3.29) is achieved by W^σ_∞ functions φ which *vary* with σ and ε. This estimate cannot be achieved by any single "universal" (i.e. independent on N and ε) cutoff function φ, no matter how smooth it might be.

3.3.2. *The minimum δ for which (3.20) holds.* As before, we set

$$\delta_1(\varepsilon, \tau, N) = \inf_{\varphi} \delta_1(\varphi; \varepsilon, \tau, N),$$

where the infimum is taken over all φ satisfying (3.2) and $\delta_1(\varphi; \varepsilon, \tau, N)$ is δ_1 from (3.20). From (3.19) and (3.20) we get immediately

$$\delta_1(\varphi; \varepsilon, \tau, N) \le \delta_\infty(\varphi; \varepsilon, \tau, N).$$

Hence, the upper bound from Theorem 3.6 holds for $\delta_1(\varepsilon, \tau, N)$ as well. But using again Theorem 3.5 one can improve this estimate as follows:

Theorem 3.7. *There exist absolute constants $c^{\#}, c^{*} > 0$ such that for $N \in \mathbb{N}$, $0 < \varepsilon \le e^{-1}$, $N > \ln(1/\varepsilon) + c^{*}$ and $\tau \ge 1$ we have*

$$\delta_1(\varepsilon, \tau, N) \le c^{\#} \frac{\ln(1/\varepsilon)}{\tau N}. \tag{3.31}$$

Proof. Estimate (3.28) holds with $\mathcal{K}_N(x)$ replaced by $\mathcal{M}(\mathcal{K}_N, x)$ due to the monotonicity of the right-hand side in (3.28). Using this we obtain

$$\begin{aligned} \frac{1}{\pi}\int_{\delta_1}^{\pi} \mathcal{M}(\mathcal{K}_N, t)\, dt &\le \frac{c_0(1+\tau)N}{\pi} \int_{\delta_1}^{\infty} \Big(\frac{c_1\sigma}{\tau N t}\Big)^{\sigma} dt \\ &= \frac{c_0 c_1 \sigma(1+\tau)}{\pi\tau} \int_{\frac{\tau N}{c_1\sigma}\delta_1}^{\infty} v^{-\sigma}\, dv = \frac{c_0 c_1 \sigma(1+\tau)}{\pi(\sigma-1)\tau} \left(\frac{c_1\sigma}{\tau N \delta_1}\right)^{\sigma-1}. \end{aligned} \tag{3.32}$$

Set $\kappa := |\ln(c_0 c_1)| + 2$, $\sigma := \ln(1/\varepsilon) + \kappa$ and $\delta_1 := e c_1 \sigma/(\tau N)$. Noticing that $\sigma \ge 3$ and $\tau \ge 1$ imply $\frac{\sigma(1+\tau)}{\pi(\sigma-1)\tau} < 1$, we infer from (3.32)

$$\frac{1}{\pi}\int_{\delta_1}^{\pi} \mathcal{M}(\mathcal{K}_N, t)\, dt < c_0 c_1 e^{-\sigma+1} \le c_0 c_1 e^{-\ln(1/\varepsilon)-\kappa+1} < \varepsilon. \tag{3.33}$$

Thus, (3.33) proves the theorem with $c^{\#} = e c_1(\kappa+1)$ and $c^{*} = \kappa$ (which leads to $N > \sigma$). □

Note that for a fixed ε the upper bound for $\delta_1(\varepsilon, \tau, N)$ in (3.31) with the increase of N becomes smaller than the lower bound for $\delta_\infty(\varepsilon, \tau, N)$ in (3.29). This fact justifies the replacement of criterion (3.19) by criterion (3.20). Note that the product $N\delta_1(\varepsilon, \tau, N)$ is bounded from above by a quantity depending on ε and τ but not on N. This means that the number of therms in (3.18) is independent of N and we can use $\Phi_{N,\delta}$ for very high degrees N.

3.3.3. *Selection of φ.* During the testing of our algorithm we determined δ_1 according to two criteria: (3.20) and

$$\delta = \delta_1 + 2\pi M^{-1}, \quad \frac{1}{\pi}\int_{\delta_1}^{\pi} |\mathcal{K}_N(t)|\, dt = \varepsilon. \tag{3.34}$$

For our best cutoff functions φ both criteria give very close values of δ_1 for various ε, τ and N. Therefore, for practical applications one can determine δ_1 using (3.34) rather than (3.20). On the other hand, one cannot drop the second term in $\delta := \delta_1 + 2\pi M^{-1}$ because there are functions f for which $\|\Phi_N f - \Phi_{N,\delta} f\|_{L^\infty(\mathbb{T})} > \varepsilon \|f\|_{\ell^\infty(\mathcal{X})}$, whenever δ is taken to be δ_1. Note that the use of δ_1 in (3.18) in the place of δ would decrease the number of terms exactly by 2.

For fixed φ, ε, τ and N it is easy to write a code for approximate computation of $\delta_1(\varphi; \varepsilon, \tau, N)$ from (3.20) or (3.34) and thus to compare the values of δ_1 for different φ's. This approach guided us in selecting a good cutoff function φ for our purposes.

We work with cutoff functions φ satisfying (3.2), which for $t \in [1, 1+\tau]$ are given by

$$\varphi(t) = \kappa^{-1} \int_{(t-1)/\tau}^{1} e^{b\sqrt{v(1-v)}}\, dv, \quad \kappa = \int_0^1 e^{b\sqrt{v(1-v)}}\, dv, \ b > 0. \tag{3.35}$$

In (3.35) b is a parameter, which is given by

$$b = 4.64 \log_{10}(1/\varepsilon) - 0.52 \tag{3.36}$$

provided $4 < \log_{10}(1/\varepsilon) < 11$ and $\tau \ge 1$.

The values of $\delta_1(\varphi;\varepsilon,\tau,1000)$ for various ε and τ are given in Table 1.

$\tau\backslash\varepsilon$	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
1	0.02259	0.02744	0.03219	0.03678	0.04136	0.04585
2	0.01147	0.01374	0.01614	0.01834	0.02071	0.02300
3	0.00762	0.00922	0.01073	0.01224	0.01370	0.01537
4	0.00573	0.00689	0.00803	0.00917	0.01030	0.01141

TABLE 1. Values of $\delta_1(\varphi;\varepsilon,\tau,1000)$ for φ from (3.35)–(3.36)

The computed values of $\delta_1(\varphi;\varepsilon,\tau,N)$ for φ from (3.35) with b from (3.36) can be very well approximated by the expression

$$\delta_1(\varphi;\varepsilon,\tau,N) \approx \tilde{c}\frac{\ln(1/\varepsilon)}{\tau N}, \quad \tilde{c} = 2. \tag{3.37}$$

In fact all values of $\delta_1(\varphi;\varepsilon,\tau,N)\tau N/\log(1/\varepsilon)$ for $N = 1000$ in Table 1 are between 1.9622 and 2.0031. Note that the approximation (3.37) corresponds to the upper limit from (3.31). Moreover, our computations show that the quantity $N\delta_1(\varphi;\varepsilon,\tau,N)$ is practically a constant (less than 2% deviation) for $100 \le N \le 10\,000$ for any fixed ε and τ in the specified range.

3.3.4. *Operator norms.* The norms of $\widetilde{\Phi}_N$, Φ_N and $\Phi_{N,\delta}$ as operators from $C[0,2\pi)$ into $L_\infty[0,2\pi)$ are very small, quite like the norms of their analogues on the sphere (cf. [7, Subsection 5.2]). The norm of the integral needlet operator $\widetilde{\Phi}_N$ from (3.3) is given by

$$\|\widetilde{\Phi}_N\|_{C[0,2\pi)\to C[0,2\pi)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathcal{K}_N(t)|dt$$

and the norms of the discrete needlet operators Φ_N from (3.12) and $\Phi_{N,\delta}$ from (3.18) are given by

$$\|\Phi_N\|_{C[0,2\pi)\to L_\infty[0,2\pi)} = \sup_{x\in[0,2\pi)}\frac{1}{M}\sum_{\xi\in\mathcal{X}}\left|\mathcal{K}_N\left(x-\xi\right)\right|,$$

$$\|\Phi_{N,\delta}\|_{C[0,2\pi)\to L_\infty[0,2\pi)} = \sup_{x\in[0,2\pi)}\frac{1}{M}\sum_{\substack{\xi\in\mathcal{X}\\ \rho(x,\xi)\le\delta}}\left|\mathcal{K}_N\left(x-\xi\right)\right|.$$

Their values for various ε, τ, $M = \lceil(2+\tau)N\rceil$ and φ from (3.35)–(3.36) are given in Table 2. Note that φ depends on both ε and τ and that $0 < \|\Phi_N\| - \|\Phi_{N,\delta}\| < \varepsilon$.

	$\|\widetilde{\Phi}_N\|$				$\|\Phi_N\|$ and $\|\Phi_{N,\delta}\|$			
$\tau\backslash\varepsilon$	10^{-5}	10^{-7}	10^{-9}	10^{-11}	10^{-5}	10^{-7}	10^{-9}	10^{-11}
1	1.6874	1.7515	1.8002	1.8395	2.0583	2.1591	2.2357	2.2975
2	1.5227	1.5869	1.6357	1.6750	1.7987	1.8999	1.9768	2.0387
3	1.4485	1.5127	1.5616	1.6010	1.6816	1.7830	1.8600	1.9221
4	1.4056	1.4699	1.5187	1.5581	1.6136	1.7153	1.7925	1.8546

TABLE 2. Norms of the integral operator $\widetilde{\Phi}_N$ and the discrete operators Φ_N and $\Phi_{N,\delta}$ for $N = 1000$

Our computations also show that these norms are practically independent of the degree N.

3.3.5. *Trigonometric needlet algorithm.* We put forward the following algorithm for solving Problem 3.

Input: N, ε, M $(M > 2N)$, $\mathcal{X} = \{\xi_k = 2\pi k/M, k = 0, \dots, M-1\}$, polynomial values $f(\xi), \xi \in \mathcal{X}$, and scattered points $x \in \mathcal{Z}$.

Pre-computation:

(1) Determine $\tau = M/N - 2$.
(2) For ε, τ and φ given by (3.35)–(3.36) determine $\varphi(n/N)$.
(3) Determine $\delta = \delta_1 + 2\pi/M$ with δ_1 from (3.37).
(4) For φ and δ from Steps 2–3 approximate $\mathcal{K}_N(x)$ for $x \in [0, \delta]$ (see [8, Subsection 3.3]).

Computation: For every $x \in \mathcal{Z}$ compute the approximate value $\tilde{f}(x) = \Phi_{N,\delta} f(x)$ of $f(x)$ using (3.18).

Output: The approximate values $\tilde{f}(x)$, $x \in \mathcal{Z}$.

We next determine the complexity of all steps. The values $\varphi(n/N)$ in Step 2 can be computed in $O(N)$ operations. Step 4 requites $O(N \ln 1/\varepsilon)$ operations. (We follow the kernel evaluation approach described in [8, Subsection 3.3]. Trigonometric polynomial evaluation is done by the Newbery modification of the Clenshaw recurrence [17].) The total complexity of the preparatory Steps 1–4 is $O(N \ln 1/\varepsilon)$.

From inequality (3.31) it follows that the approximate evaluation of f by (3.18) at a single point requires $O(\ln 1/\varepsilon)$ operations. Thus, the total count of operations is $O(N \ln 1/\varepsilon + |\mathcal{Z}| \ln 1/\varepsilon)$, where $|\mathcal{Z}|$ stands for the number of elements in $\mathcal{Z}$.

3.4. **Interpolating needlets.** One surprising property of the trigonometric needlet operators is that $\Phi_N f$ and $\Phi_{N,\delta} f$ interpolate f at the knots of $\mathcal{X}$ for the minimum possible value of M, i.e. $M = (2+\tau)N$, and for symmetric cutoff functions (see (3.38)); no smoothness of the cutoff function is required.

We start with

Lemma 3.1. *Let $N \in \mathbb{N}$ and $\tau > 0$ be such that $\tau N \in \mathbb{N}$. Assume that the cutoff function φ satisfies the conditions: $\varphi(t) = 0$ for $t > 2 + \tau$ and*

$$\varphi(2 + \tau - t) + \varphi(t) = 1, \quad 0 \le t \le 1 + \tau/2. \tag{3.38}$$

Let $M = 2N + \tau N$ and $x_k = 2\pi M^{-1} k$, $k = 0, 1, \dots, M-1$. Then the needlet kernel $\mathcal{K}_N$, defined in (3.1)*, satisfies $\mathcal{K}_N(x_0) = M$ and $\mathcal{K}_N(x_k) = 0$ for $k = 1, 2, \dots, M-1$.*

Proof. From $\sin Mx_k/2 = \sin k\pi = 0$ we get for $n \le M/2$

$$\cos(M-n)x_k - \cos nx_k = 2 \sin Mx_k/2 \sin(M/2 - n)x_k = 0.$$

Also (3.38) implies

$$\varphi((M-n)/N) + \varphi(n/N) = 1, \quad n \le M/2.$$

If $M = 2m$ (M even), then for $k \ge 1$ we have

$$\mathcal{K}_N(x_k) = 1 + 2\sum_{n=1}^{m-1} \cos nx_k + \cos mx_k = \frac{\sin mx_k \cos x_k/2}{\sin x_k/2} = 0$$

and for $k = 0$ we have $\mathcal{K}_N(0) = \mathcal{K}_N(x_0) = 1 + 2(m-1) + 1 = M$.

If $M = 2m + 1$ (M odd), then for $k \ge 1$ we have

$$\mathcal{K}_N(x_k) = 1 + 2\sum_{n=1}^{m} \cos nx_k = \frac{\sin(m + 1/2)x_k}{\sin x_k/2} = 0$$

and for $k = 0$ we have $\mathcal{K}_N(0) = \mathcal{K}_N(x_0) = 1 + 2m = M$. □

Note that in the case of a symmetric cutoff function φ satisfying (3.2) Lemma 3.1 identifies $2N+\tau N-1$ of the zeros of the needlet kernel $\mathcal{K}_N$, which is a trigonometric polynomial of degree $N+\tau N-1$. The remaining $\tau N-1$ (complex) zeros apparently are used for generating the superb localization of the kernel for cutoff functions as those considered in Theorems 3.1 and 3.2.

As an immediate consequence of Lemma 3.1 we get

Theorem 3.8. *Let $N \in \mathbb{N}$, $\tau > 0$ be such that $\tau N \in \mathbb{N}$ and $M = 2N+\tau N$. Assume that φ satisfies* (3.38) *and $\mathcal{X}$ is given by* (3.11)*. Then the needlet operators defined by* (3.12) *and by* (3.18) *for some $\delta > 0$ satisfy $\Phi_N f(\eta) = f(\eta)$ and $\Phi_{N,\delta} f(\eta) = f(\eta)$ for every $\eta \in \mathcal{X}$.*

Proof. Note that $(\eta - \xi)M/(2\pi) \in \mathbb{Z}$ for every $\eta, \xi \in \mathcal{X}$. Therefore, Lemma 3.1 implies that all terms but one (for $\eta = \xi$) in the sums in (3.12) and (3.18) are equal to zero. □

Note that if φ is defined by (3.35) then it satisfies (3.38). Hence, the corresponding needlets $\Phi_N f$ and $\Phi_{N,\delta} f$ and the trigonometric needlet algorithm from Subsection 3.3 interpolate f at the equidistant knots in the case $M = 2N + \tau N$.

3.5. **Comparison with other methods for fast evaluation.** The trigonometric needlets are the main component in the construction of the tensor product needlets to be used on the sphere. In this subsection we compare them with two other algorithms for solving Problem 1 on $\mathbb{T}$, that is, the problem for fast evaluation of trigonometric polynomials, given by their coefficients, at scattered points within a prescribed accuracy. From several existing algorithms for solving Problem 1 we selected two algorithms that entail fast, stable and accurate methods for evaluation of multi-dimensional trigonometric polynomials.

We begin with a short description of the algorithms:

- **A1:** Trigonometric needlets;
- **A2:** Spline interpolation with maximal defect (i.e. piece-wise polynomials) based on Lagrange interpolation;
- **A3:** Nonequispaced fast Fourier transform (NFFT) with Kaiser–Bessel window function.

3.5.1. *Trigonometric needlets.* The algorithm in Subsection 3.3 solves Problem 3 when trigonometric polynomial values are given at equispaced points. If the polynomial is given by its coefficients we modify the algorithm by replacing the *Input* and the first step of the *pre-computation* part with

Input: N, ε, τ, polynomial coefficients $a_{-N}, a_{-N+1}, \dots, a_N$ and scattered points $x \in \mathcal{Z}$.

(1) Determine $M = \lceil (2+\tau)N \rceil$, $\mathcal{X} = \{\xi_k = 2\pi k/M, k = 0, \dots, M-1\}$ and evaluate $f(\xi), \xi \in \mathcal{X}$, using FFT.

This step increases the total count of operations by $O(N \ln N)$.

3.5.2. *Lagrange interpolation.* Let $\mathcal{X} = \mathcal{X}_M$ be defined just as in (3.11) and let $h := 2\pi/M$. For $x \in [0, 2\pi)$ set

$$j = j(x) := \lfloor (xM/(2\pi) - \mu/2 \rfloor .$$

Then $x-\mu h/2\in[\xi_j,\xi_{j+1}]$ (with the convention $\xi_{m+M}=\xi_m$, $m\in\mathbb{Z}$). Observe that for even $\mu=2m$ we have $x\in[\xi_{j+m},\xi_{j+m+1})$ and for odd $\mu=2m-1$ we have $x\in[\xi_{j+m}-h/2,\xi_{j+m}+h/2)$.

Given $M,\mu\in\mathbb{N}$, $M\geq\mu$, we define the operator $\mathcal{L}_{M,\mu}$ by

$$\mathcal{L}_{M,\mu}(f,x)=L_{[\xi_{j(x)+1},\dots,\xi_{j(x)+\mu}]}(f,x):=\sum_{k=1}^{\mu}\ell_{k,\mu}(x-\xi_{j(x)})f(\xi_{j(x)+k}), \tag{3.39}$$

$$\ell_{k,\mu}(t):=\frac{\omega(t)}{(t-\xi_k)\omega'(\xi_k)},\quad \omega(t):=\prod_{i=1}^{\mu}(t-\xi_i).$$

Here $L_{[\xi_{j+1},\dots,\xi_{j+\mu}]}(f)$ stands for the Lagrange interpolation polynomial of the 2π periodic function f with knots $\xi_{j+1},\dots,\xi_{j+\mu}$.

From (3.39) it follows that $\mathcal{L}_{M,\mu}(f,\xi)=f(\xi)$ for $\xi\in\mathcal{X}$ and $\mathcal{L}_{M,\mu}(f)$ is a piecewise polynomial, more precisely, it is an algebraic polynomial of degree $\mu-1$ on every interval of the form $[\xi_k,\xi_{k+1})$ for even μ and of the form $[\xi_k-h/2,\xi_k+h/2)$ for odd μ. Hence $\mathcal{L}_{M,\mu}(f)\in C[0,2\pi)$ for even μ or $\mathcal{L}_{M,\mu}(f)$ may be discontinuous at $\xi+/h2,\xi\in\mathcal{X}$, for odd μ. Thus, $\mathcal{L}_{M,\mu}(f)$ is an interpolating spline with maximal defect.

The following simple claim will be needed.

Theorem 3.9. *Let $M,\mu\in\mathbb{N}$, $M\geq\mu$. Then*

$$\|\mathcal{L}_{M,\mu}(f)-f\|_\infty\leq\kappa(\lfloor\mu/2\rfloor)\left(\frac{\pi}{M}\right)^{\mu}\|f^{(\mu)}\|_\infty,\quad\forall f\in C^{\mu}[0,2\pi); \tag{3.40}$$

$$\|\mathcal{L}_{M,\mu}(f)-f\|_\infty\leq\kappa(\lfloor\mu/2\rfloor)\left(\frac{\pi N}{M}\right)^{\mu}\|f\|_\infty,\quad\forall f\in\Pi_N, \tag{3.41}$$

where

$$\kappa(m)=\frac{(2m)!}{m!^2 2^{2m}}=\frac{1}{\sqrt{\pi m}}(1+o(1)).$$

Moreover the constant in (3.41) *cannot be improved.*

Here (3.40) follows from the error expression for the Lagrange interpolation formula at the middle of the interpolating knots (cf. [6, §9.3]), (3.41) follows from (3.40) and the Bernstein inequality. The examples of $f(x)=\cos N(t+h/2)$ and $x=0$ for even μ and $f(x)=\sin Nt$ and $x=h/2$ for odd μ and $M\to\infty$ show that the constant in (3.41) cannot be improved.

Observe also that the operators $\mathcal{L}_{M,\mu}:L_\infty[0,2\pi)\to L_\infty[0,2\pi)$ have small norms, which are independent of M and increase logarithmically with μ.

The fast evaluation of a trigonometric polynomial f at many scattered points can be done in two steps by: first, pre-computing the values $f(\xi)$, $\xi\in\mathcal{X}$, using FFT and, second, applying (3.39) for every scattered point. The error estimate of this algorithm, given in (3.41), indicates that the prescribed error ε will be guaranteed if the parameters M and μ are selected so that

$$\kappa(\lfloor\mu/2\rfloor)\left(\frac{\pi N}{M}\right)^{\mu}\leq\varepsilon.$$

3.5.3. *Nonequispaced fast Fourier transform.* The work on fast Fourier transforms for nonequispaced data was initiated by Dutt and Rokhlin [3] and had attracted a number of followers, see e.g. [1, 4, 13]. This method approximates a trigonometric polynomial $f(x) = \sum_{k=-N}^{N} a_k e^{ikx}$ given by its coefficients by a periodic function of the form $g_1(x) = \sum_{\xi\in\mathcal{X}} b_\xi \psi(x-\xi)$. Here ψ is a window function with excellent localization in both the space/time and frequency domains, $M = \sigma(2N+1)$ or $M = 2\sigma N$ with $\sigma > 1$ (called oversampling factor) and the coefficients $b_\xi, \xi \in \mathcal{X}$, depend in a simple manner on the Fourier coefficients of ψ and f.

The evaluation of g_1 at a point x is done by truncation of the sum to the closest $2m+1$ knots to x, i.e.

$$g(x) = \sum_{\substack{\xi\in\mathcal{X} \\ \rho(x,\xi)\le\pi(2m+1)/M}} b_\xi \psi(x-\xi). \tag{3.42}$$

The algorithm proceeds as follows: (i) it pre-computes b_ξ, $\xi \in \mathcal{X}$, using FFT and (ii) it applies (3.42) for every scattered point.

The error of the method has two components – the approximation error $\|f - g_1\|$ and the truncation error $\|g_1 - g\|$. If ψ is the Kaiser–Bessel window function (the best known choice for small oversampling σ) then an error estimate given in [18, Satz 1.10] reads

$$\|f-g\|_\infty \le 4\pi(\sqrt{m}+m)\sqrt[4]{1-\frac{1}{\sigma}}\, e^{-2\pi m\sqrt{1-1/\sigma}} \sum_{k=-N}^{N} |a_k|. \tag{3.43}$$

The relation between the NFFT notation and our notation is given by

$$\sigma = 1 + \frac{\tau}{2} = \frac{M}{2N}, \quad m = \left\lfloor \frac{M}{2\pi}\delta + \frac{1}{2} \right\rfloor. \tag{3.44}$$

3.5.4. *Comparison.* Each of the three algorithms consists of two parts: first, precompute some quantities using FFT of length M and, second, apply *local* summation formulas: (3.18) for **A1**, (3.39) for **A2** and (3.42) for **A3**. The amount of computation in the first parts is approximately the same and in the second parts the computations depend on the number of terms in these formulas and the computation time for evaluating the basis functions ($\mathcal{K}_N$, $\ell_{k,\mu}$ and ψ, respectively). Here $\ell_{k,\mu}$ and ψ are given in close forms. There is no close form for $\mathcal{K}_N$ known to us, but we use an appropriate approximation to this kernel and achieve evaluation speed which is independent of the degree N. Hence, the evaluation of these basis functions for $2m+1$ arguments requires cm operatons (with eventually different constants c). Therefore, we shall consider the number of terms $2m+1$ (in the **A3** notation) and how they achieve the prescribed accuracy ε for varying oversampling parameter σ.

The comparison of the three error estimates – (3.24) along with (3.31) for **A1**, (3.41) for **A2** and (3.43) for **A3** – is not straightforward, because the first two employ the uniform norm of the polynomial and the third uses the ℓ_1 norm of the polynomial coefficients. While **A1** and **A3** require only $\sigma > 1$ (i.e. $\tau > 0$) to achieve the prescribed accuracy, **A2** requires oversampling $\sigma > \pi/2$ (i.e. $\tau > \pi - 2$) and cannot work with lower oversampling because the estimate (3.41) is sharp. This is not a problem in dimension 1, however, could cause memory problems in higher dimensions.

The leading terms of the error bounds for the three algorithms are:

$$\exp(-\pi(1-1/\sigma)m); \quad \exp(-2\ln(2\sigma/\pi)m); \quad \exp(-2\pi\sqrt{1-1/\sigma}\, m),$$

where the term for **A1** is obtained from (3.37) with $\tilde{c}=2$. Thus, all three algorithms have exponential error decay in m but the constants in front of m are different.

In Table 3 we give the smallest values of m for algorithms **A1**, **A2** and **A3** which ensure error bound ε for various values of $\sigma = 1+\tau/2$. For algorithm **A1** m is computed as the closest integer to $\delta_1 M/(2\pi)+1.5$ and the values of δ_1 are taken from Table 1. Also φ is defined by (3.35)–(3.36) and $M=2\sigma N$. For algorithm **A2** m is computed as the smallest integer such that the constant in front of the uniform norm in (3.41) (with $\mu=2m+1$ and $M=2\sigma N$) is at most ε. For algorithm **A3** m is computed as the smallest integer such that the constant in front of the ℓ^1 coefficient norm in (3.43) is at most ε.

	A1				**A2**				**A3**			
$\varepsilon\backslash\sigma$	1.5	2.0	2.5	3.0	1.5	2.0	2.5	3.0	1.5	2.0	2.5	3.0
10^{-5}	12	8	7	6	–	20	11	8	5	4	4	4
10^{-6}	14	10	8	8	–	24	13	9	5	5	4	4
10^{-7}	16	11	10	9	–	29	15	11	6	5	5	4
10^{-8}	19	13	11	10	–	33	18	13	7	6	5	5
10^{-9}	21	14	12	11	–	38	20	15	7	6	6	5
10^{-10}	23	16	13	12	–	43	22	16	8	7	6	6

TABLE 3. Values of m for algorithms **A1**, **A2** and **A3**

Table 3 shows that the number of terms that **A1** uses is approximately twice bigger (for $\sigma \geq 2$) than the terms in **A3**. Algorithm **A2** is inferior to **A1** and **A3** for small oversampling factor σ. On the other hand, algorithm **A2** has smaller number of terms than **A1** and **A3** for large oversampling factor σ ($\sigma \geq 6$ if compared with **A1** and $\sigma \geq 35$ if compared with **A3**). Currently we see no practical application with such significant oversampling.

We now turn our attention to the norms in the the error estimates. They are related by

$$\|f\|_{C(\mathbb{T})} \leq \sum_{k=-N}^{N} |a_k| \leq \sqrt{2N+1}\|f\|_{C(\mathbb{T})} \text{ for any } f(x)=\sum_{k=-N}^{N} a_k e^{ikx}.$$

The constant 1 in the first inequality is exact and the constant $\sqrt{2N+1}$ in the second inequality cannot be replaced by $\gamma\sqrt{2N+1}$ for any $\gamma<1$ and independent of N as the Kahane ultraflat polynomials [12] show. This gives some advantage to algorithms **A1** and **A2** over **A3** when absolute error $\varepsilon\|f\|_{C(\mathbb{T})}$ is to be achieved. In such cases m for **A3** in Table 3 is to be increased in order to compensate for the additional $\sqrt{2N+1}$ factor. Of course, such increase would be bigger when going to the multidimensional case.

Finally, let us look at the structure of the approximating expressions. The dependence of the summation formulas for **A1** and **A2** on the polynomial values at the knots will be an *essential stability advantage* over the summation formulas for **A3** when going to the 2-d sphere in Section 5. Moreover, if $f(\xi), \xi \in \mathcal{X}$, are not

values of a trigonometric polynomial of degree N, then **A1** can be used for approximation of f and (3.18) will be an approximation operator with very small error, cf. (3.25).

4. Tensor product trigonometric needlets

The generalization of the trigonometric needlets from Section 3 to d-dimensional tensor product operators is straightforward.

The function domain is $\mathbb{T}^d$ with points $\boldsymbol{x} = (x_1, \dots, x_d)$ and distance $\rho(\boldsymbol{x}, \boldsymbol{y}) := \max\{\rho(x_1, y_1), \dots, \rho(x_d, y_d)\}$. Denote by Π_N^d the set of all d-dimensional trigonometric polynomials of coordinate degrees N. The tensor product trigonometric needlet kernel is defined by

$$\boldsymbol{\mathcal{K}}_N^d(\boldsymbol{x}) := \prod_{k=1}^d \mathcal{K}_N(x_k).$$

Using the knots $\mathcal{X}^d = \mathcal{X}_M \times \cdots \times \mathcal{X}_M$, where $\mathcal{X}_M$ is defined in (3.11), we define the truncated tensor product needlet operator by

$$\Phi_{N,\delta}^d f(\boldsymbol{x}) := \sum_{\substack{\boldsymbol{\xi} \in \mathcal{X}^d \\ \rho(\boldsymbol{\xi}, \boldsymbol{x}) \le \delta}} M^{-d} \boldsymbol{\mathcal{K}}_N^d(\boldsymbol{x} - \boldsymbol{\xi}) f(\boldsymbol{\xi}). \tag{4.1}$$

The next theorem follows directly from Theorems 3.3 and 3.4.

Theorem 4.1. *Let $d, N, M \in \mathbb{N}$, $\tau > 0$, $M \ge N + N_\tau$, and $0 < \varepsilon \le 1$. Assume that φ satisfies* (3.2)*, $\mathcal{K}_N$ obeys* (3.6) *with some $\sigma > 1$ and δ is determined by* (3.19) *or* (3.20) *with $\varepsilon_1 = \varepsilon/(d\|\Phi_N\|^{d-1})$ in the place of ε. Then we have*

$$\Phi_{N,\delta}^d : \ell^\infty(\mathcal{X}^d) \to L^\infty(\mathbb{T}^d) \text{ is a bounded linear operator}; \tag{4.2}$$

$$\|f - \Phi_{N,\delta}^d f\|_{L^\infty(\mathbb{T}^d)} \le \varepsilon \|f\|_{\ell^\infty(\mathcal{X}^d)} \quad \forall f \in \Pi_N^d; \tag{4.3}$$

$$\|f - \Phi_{N,\delta}^d f\|_{L^\infty(\mathbb{T}^d)} \le c E_N(f)_\infty + \varepsilon \|f\|_{\ell^\infty(\mathcal{X}^d)} \quad \forall f \in C(\mathbb{T}^d), \tag{4.4}$$

where $E_N(f)_\infty := \inf_{g \in \Pi_N^d} \|f - g\|_{L^\infty(\mathbb{T}^d)}$.

Estimate (4.3) shows that (4.1) can be the base for a fast and stable needlet algorithm for evaluating $f \in \Pi_N^d$ at many scattered points on $\mathbb{T}^d$. To be more specific, first, the values of $f(\boldsymbol{\xi}), \boldsymbol{\xi} \in \mathcal{X}^d$, are pre-computed using multidimensional FFT and, second, (4.1) is applied to evaluate $f(\boldsymbol{x})$ at every scattered point $\boldsymbol{x}$. The number of operations for a scattered point consists of $d\mu$ evaluations of the *one-dimensional* kernel $\mathcal{K}_N$ plus μ^d multiplications and additions, where μ denotes the number of terms in (3.18).

5. Spherical trigonometric needlets

We now turn to the problem for effective evaluation of spherical polynomials. We focus on Problem 3, stated in Subsection 2.2.

5.1. The main tenet of our method.

Our method relies on the following representation of spherical polynomials $f \in \mathcal{P}_N$ in spherical coordinates:

$$f(\theta, \lambda) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathcal{K}_N(\theta - \theta') \mathcal{K}_N(\lambda - \lambda') f(\theta', \lambda') d\lambda' d\theta', \tag{5.1}$$

where $\mathcal{K}_N$ is the kernel from (3.1). Here $f(\theta,\lambda)$ is extended, first, by

$$f(\theta,\lambda) := f(2\pi-\theta,\lambda+\pi) \quad \text{for} \quad \pi<\theta<2\pi,\ \lambda\in\mathbb{R}, \tag{5.2}$$

and then 2π-periodically in θ. This identity is an immediate consequence of Proposition 2.1 and Theorem 3.3.

We now introduce the operator

$$\widetilde{\Phi}_N^2 f(\theta,\lambda) := \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_N(\theta-\theta')\mathcal{K}_N(\lambda-\lambda')f(\theta',\lambda')d\lambda' d\theta', \tag{5.3}$$

acting on any function $f\in L^1(\mathbb{S}^2)$ defined on $\mathbb{S}^2$ in spherical coordinates and extended as in (5.2). From (5.1) it follows that $\widetilde{\Phi}_N^2 f = f$ for $f\in\mathcal{P}_N$.

Our next step is to discretize the operator $\widetilde{\Phi}_N^2$ by using quadrature formula (3.10). Let $\mathcal{X}=\{(\theta_k,\lambda_\ell)\}$ be one of the regular grids (2.6) or (2.7). We extend the set $\mathcal{X}$ for $\pi\le\theta<2\pi$ by simply letting $k=K,\dots,2K-1$ in (2.6) or (2.7). We define

$$\Phi_N^2 f(\theta,\lambda) := \frac{1}{4KL}\sum_{k=0}^{2K-1}\sum_{\ell=0}^{2L-1} \mathcal{K}_N(\theta-\theta_k)\mathcal{K}_N(\lambda-\lambda_\ell)f(\theta_k,\lambda_\ell), \tag{5.4}$$

where $f(\theta,\lambda)$ is extended as in (5.2). Note that $\lambda_\ell=\pi+\lambda_{\ell-L}$ (it is necessary for this equation that we use even number of knots in the λ direction), $\theta_k^{(1)}=2\pi-\theta_{2K-k}^{(1)}$, $\theta_k^{(2)}=2\pi-\theta_{2K-k-1}^{(2)}$. Hence, in view of (5.2) only the values of f at the original regular grids $\mathcal{X}^{(1)}$ or $\mathcal{X}^{(2)}$ are used in (5.4).

We next record some useful properties of the operators Φ_N^2, which follow by (5.4) and Theorem 3.3.

Theorem 5.1. *Let $\mathcal{K}_N$ be just as in Theorem 3.3. Then the operator Φ_N^2 from (5.4) is a bounded operator as a map $\Phi_N^2:\ell^\infty(\mathcal{X})\to C(\mathbb{S}^2)$ and*

$$\|\Phi_N^2\|_{\ell^\infty(\mathcal{X})\to C(\mathbb{S}^2)}\le C \quad \textit{with } C>0 \textit{ a constant independent of } N. \tag{5.5}$$

Also, $\Phi_N^2 f$ for any $f\in\ell^\infty(\mathcal{X})$ is a trigonometric polynomial in both θ and λ of degree $<N_\tau$ with $N_\tau:=\lceil N+\tau N\rceil$.

Furthermore, if $2K\ge N+N_\tau$ and $2L\ge N+N_\tau$, then

$$\Phi_N^2 f = f \quad \forall f\in\mathcal{P}_N \quad \textit{and} \tag{5.6}$$

$$\|f-\Phi_N^2 f\|_{C(\mathbb{S}^2)}\le(\|\Phi_N\|^2+1)E_N(f)_\infty \quad \forall f\in C(\mathbb{S}^2). \tag{5.7}$$

Here $E_N(f)_\infty:=\inf_{g\in\mathcal{P}_N}\|f-g\|_{C(\mathbb{S}^2)}$ and $\|\Phi_N\|$ is the norm of the one-dimensional operator Φ_N studed in Subsection 3.3.

In order to achieve fast evaluation of spherical polynomials we introduce the following truncated version of the operator Φ_N^2:

$$\Phi_{N,\delta}^2 f(\theta,\lambda) := \frac{1}{4KL}\sum_{\substack{0\le k<2K\\ \rho(\theta,\theta_k)\le\delta}}\sum_{\substack{0\le\ell<2L\\ \rho(\lambda,\lambda_\ell)\le\delta}} \mathcal{K}_N(\theta-\theta_k)\mathcal{K}_N(\lambda-\lambda_\ell)f(\theta_k,\lambda_\ell), \tag{5.8}$$

where δ is a small parameter.

As a consequence of Theorems 3.4 and 3.3 we obtain the following basic properties of the operators $\Phi_{N,\delta}^2$.

Theorem 5.2. *Let $N, K, L \in \mathbb{N}$, $\tau > 0$, $2K \ge N + N_\tau$, $2L \ge N + N_\tau$, and $0 < \varepsilon \le 1$. Assume that φ satisfies (3.2), $\mathcal{K}_N$ obeys (3.6) with some $\sigma > 1$ and δ is determined by (3.19) or (3.20) with $\varepsilon_1 = \varepsilon/(2\|\Phi_N\|)$ in the place of ε. Then $\Phi^2_{N,\delta} : \ell^\infty(\mathcal{X}) \to L^\infty(\mathbb{S}^2)$ is a bounded linear operator,*

$$\|\Phi^2_N f - \Phi^2_{N,\delta} f\|_{L^\infty(\mathbb{S}^2)} \le \varepsilon \|f\|_{\ell^\infty(\mathcal{X})} \quad \forall f \in \ell^\infty(\mathcal{X}); \tag{5.9}$$

$$\|f - \Phi^2_{N,\delta} f\|_{L^\infty(\mathbb{S}^2)} \le \varepsilon \|f\|_{\ell^\infty(\mathcal{X})} \quad \forall f \in \mathcal{P}_N; \tag{5.10}$$

$$\|f - \Phi^2_{N,\delta} f\|_{L^\infty(\mathbb{S}^2)} \le (C+1) E_N(f)_\infty + \varepsilon \|f\|_{\ell^\infty(\mathcal{X})} \quad \forall f \in C(\mathbb{S}^2), \tag{5.11}$$

where C is the constant from (5.5) and $\|\Phi_N\|$ is the norm of the one-dimensional operator Φ_N studed in Subsection 3.3.

Proof. Only (5.9) need to be verified; it follows from

$$\begin{aligned}
\frac{1}{4KL} &\left(\sum_{k=0}^{2K-1} \sum_{\ell=0}^{2L-1} - \sum_{\substack{0\le k<2K \\ \rho(\theta,\theta_k)\le\delta}} \sum_{\substack{0\le \ell<2L \\ \rho(\lambda,\lambda_\ell)\le\delta}} \right) |\mathcal{K}_N(\theta-\theta_k)||\mathcal{K}_N(\lambda-\lambda_\ell)| \\
&\le \frac{1}{2K} \sum_{k=0}^{2K-1} |\mathcal{K}_N(\theta-\theta_k)| \frac{1}{2L} \sum_{\substack{0\le \ell<2L \\ \rho(\lambda,\lambda_\ell)>\delta}} |\mathcal{K}_N(\lambda-\lambda_\ell)| \\
&\quad + \frac{1}{2K} \sum_{\substack{0\le k<2K \\ \rho(\theta,\theta_k)>\delta}} |\mathcal{K}_N(\theta-\theta_k)| \frac{1}{2L} \sum_{\ell=0}^{2L-1} |\mathcal{K}_N(\lambda-\lambda_\ell)| \le 2\|\Phi_N\|\varepsilon_1 = \varepsilon
\end{aligned}$$

on account of (3.26) or (3.27) applied with $M = 2K$ or $M = 2L$. □

5.2. **Spherical trigonometric needlet algorithm.** Inequality (5.10) is the base of the spherical trigonometric needlet algorithm. Theorem 5.2 specifies (3.19) or (3.20) with $\varepsilon_1 = \varepsilon/(2\|\Phi_N\|)$ in the place of ε as sufficient conditions for (5.10). Here the decrease of ε to ε_1 is not big because $\|\Phi_N\|$ is bounded by an absolute constant in view of (3.14). From Table 2 we observe that $\|\Phi_N\| < 2.5$ for the values of ε and τ under consideration. Thus, $\varepsilon_1 = \varepsilon/5$ will ensure (5.10).

Now, we put forward the following algorithm for solving Problem 3.

Input: N, ε, K, L ($K > N, L > N$), $\mathcal{X} = \mathcal{X}^{(1)}$ or $\mathcal{X} = \mathcal{X}^{(2)}$, polynomial values $f(\xi), \xi \in \mathcal{X}$, and scattered points $x \in \mathcal{Z} \subset \mathbb{S}^2$.

Pre-computation:

1. Determine $\tau = 2(\min\{K, L\}/N - 1)$.
2. For ε, τ and φ given by (3.35)–(3.36) with $\varepsilon/5$ in the place of ε compute $\varphi(n/N)$.
3. Set $\delta = \delta_1 + \pi/\min\{K, L\}$, where δ_1 is from (3.37) with $\varepsilon/5$ in the place of ε.
4. For φ and δ from Steps 2–3 approximate the one-dimensional kernel $\mathcal{K}_N(x)$ for $x \in [0, \delta]$ (see [8, Subsection 3.3]).

Computation: For every $x \in \mathcal{Z}$ compute the approximate value $\tilde{f}(x) = \Phi^2_{N,\delta} f(x)$ of $f(x)$ using (5.8).

Output: The approximate values $\tilde{f}(x)$, $x \in \mathcal{Z}$.

We next determine the complexity of all steps. The values $\varphi(n/N)$ in Step 2 can be computed in $O(N)$ operations. Step 4 requites $O(N \ln 1/\varepsilon)$ operations. (We

follow the kernel evaluation approach described in [8, Subsection 3.3]. Trigonometric polynomial evaluation is done by the Newbery modification of the Clenshaw recurrence [17].) The total complexity of the preparatory Steps 1–4 is $O(N \ln 1/\varepsilon)$.

From inequality (3.31) we get that the approximate evaluation of f by (5.8) at a single point requires $O(\ln 1/\varepsilon)$ kernel evaluations and $O(\ln^2 1/\varepsilon)$ multiplications and additions. Thus, the total count of operations is $O(N \ln 1/\varepsilon + |\mathcal{Z}| \ln^2 1/\varepsilon)$, where $|\mathcal{Z}|$ stands for the number of elements in $\mathcal{Z}$.

Remark 5.1. This algorithm and Theorem 5.2 cover the case when the regular grid steps in the latitude and longitude directions are approximately equal ($K \approx L$). In the case when the steps are not equal, say $K - N > 1.2(L - N)$, we can take advantage of the additional information we have about the trigonometric polynomial and modify operator (5.8). We set $\tau_1 = 2(K-N)/N$, $\tau_2 = 2(L-N)/N$ and choose different cutoff functions and kernels: φ_1 and $\mathcal{K}_{1,N}$ corresponding to τ_1 and φ_2 and $\mathcal{K}_{2,N}$ corresponding to τ_2. Then (5.8) is replaced by

(5.12)
$$\Phi^2_{N,\tilde\delta_1,\tilde\delta_2} f(\theta,\lambda) := \frac{1}{4KL} \sum_{\substack{0\le k<2K \\ \rho(\theta,\theta_k)\le\tilde\delta_1}} \sum_{\substack{0\le \ell<2L \\ \rho(\lambda,\lambda_\ell)\le\tilde\delta_2}} \mathcal{K}_{1,N}(\theta-\theta_k)\mathcal{K}_{2,N}(\lambda-\lambda_\ell) f(\theta_k,\lambda_\ell),$$

where $\tilde\delta_1, \tilde\delta_2$ are small parameters. The appropriate modification of Theorem 5.2 holds for the operator (5.12) and the spherical trigonometric needlet algorithm can take advantage of it. According to (3.37) the reduction of the number of terms in (5.12) compared to (5.8) is approximately $\tau_2/\tau_1 = (L-N)/(K-N)$.

Remark 5.2. In our experiments we have extensively used (3.20) with $\varepsilon_1 = \varepsilon/2$ (instead of $\varepsilon_1 = \varepsilon/5$) in the place of ε. In all cases the relative error of the computed polynomial values have not exceeded ε.

5.3. **Interpolating needlets.** The spherical trigonometric needlet operators preserve the interpolation property of the one-dimensional trigonometric needlet operators. Indeed, from (5.4), (5.8) and Lemma 3.1 we immediately get

Theorem 5.3. *Let $N \in \mathbb{N}$ and $\tau > 0$ be such that τN is an even integer. Assume that φ satisfies* (3.38)*. Let $K = L = N + \tau N/2$ and $\mathcal{X}$ be given by* (2.6) *or* (2.7)*. Then the spherical trigonometric needlet operators defined by* (5.4) *and by* (5.8) *for some $\delta > 0$ satisfy $\Phi^2_N f(\eta) = f(\eta)$ and $\Phi^2_{N,\delta} f(\eta) = f(\eta)$ for every $\eta \in \mathcal{X}$.*

Note that if φ is defined by (3.35) then it satisfies (3.38). Hence, the corresponding operators $\Phi^2_N f$ and $\Phi^2_{N,\delta} f$ and the spherical trigonometric needlet algorithm from Subsection 5.2 interpolate f at the regular grid knots in the case $K = L = N + \tau N/2$.

5.4. **Comparisons.**

5.4.1. *Comparison with spherical needlets.* We have designed tensor product trigonometric needlets in an attempt to improve the speed and to simplify the algorithm of the spherical needlets developed in [8]. The implemented simplifications are:

- use of equispaced knots instead of Gaussian quadrature knots in the latitude direction;
- no need of separating the sphere in three regions;

- no need of rotating the spherical polynomial in the polar regions in the pre-computation part.

The improvement of the speed comes from:

- (mainly) the reduction of the number of kernel evaluations per scattered point;
- the number of terms in the truncated needlet operator does not depend on the latitude of the scattered point, as is the case of a spherical cap of fixed radius.

Note that *the number of kernel evaluations* per scattered point is approximately $4(\delta M/(2\pi))$ for the tensor product needlets and $\pi(\bar{\delta}M/(2\pi))^2$ for the spherical needlets (when the evaluation point is on the equator or at the poles). Here $\bar{\delta}$ is the spherical needlet truncation parameter and in view of (3.37) and [8, Subsection 3.2] we may take $\bar{\delta} \approx 1.15\delta$. Hence the number of kernel evaluations reduces $\delta M/(2\pi) \approx m$ times, where the values of m are the entries for algorithm **A1** in Table 3.

On the other hand, the number of terms in the local approximation formulas is approximately the same for both truncated operators (for one and the same values of ε and τ), which gives similar *number of multiplications and additions*.

The above reasoning is confirmed by the entries in Table 4. Here we compare the speed of the *computation* part of tensor product trigonometric needlet and of spherical needlet algorithms written in MATLAB 2012b with double-precision variables. The experiments were conducted on an Intel Core i7, 2.4 GHz PC with 16 GB of RAM. The spherical polynomial degree is $N = 2160$, the parameters of the regular grid (2.6) are $K = L = \lceil (1+\tau/2)N \rceil$, and the number of scattered points is 1 000 000.

	Tensor product needlet algorithm				Spherical needlet algorithm			
$\tau\backslash\varepsilon$	10^{-5}	10^{-7}	10^{-9}	10^{-11}	10^{-5}	10^{-7}	10^{-9}	10^{-11}
1	95 520	75 735	63 804	50 551	8 958	5 155	3 535	2 553
2	129 820	101 740	84 767	65 436	16 592	11 007	7 565	5 131
3	140 647	119 847	98 309	82 795	19 457	14 621	10 113	6 108
4	145 117	126 231	106 838	90 645	21 454	16 515	11 573	7 752

TABLE 4. Computed values per second of a spherical polynomial of degree $N = 2160$

The two algorithms are stable.

5.4.2. *Comparison with tensor product piece-wise polynomials (Lagrange interpolation).* The tensor product Lagrange interpolation operator is given by (cf. (3.39))

$$\mathcal{L}^2_{M,\mu} f(\theta,\lambda) := \sum_{i=1}^{\mu}\sum_{k=1}^{\mu} \ell_{i,\mu}(\theta - \xi_{j(\theta)})\ell_{k,\mu}(\lambda - \xi_{j(\lambda)}) f(\xi_{j(\theta)+i,j(\lambda)+k}). \tag{5.13}$$

with an error estimate

$$\|f - \mathcal{L}^2_{M,\mu} f\|_{C(\mathbb{S}^2)} \le 2\|\mathcal{L}_{M,\mu}\|\kappa(\lfloor \mu/2 \rfloor)\left(\frac{\pi N}{M}\right)^{\mu} \|f\|_{C(\mathbb{S}^2)}, \quad \forall f \in \mathcal{P}_N.$$

The difference between tensor product Lagrange interpolation operator (5.13) and the tensor product trigonometric needlet operator (5.8) is in the basis functions $\ell_{i,\mu}(\theta-\xi_{j(\theta)})\ell_{k,\mu}(\lambda-\xi_{j(\lambda)})$ and $\mathcal{K}_N(\theta-\theta_k)\mathcal{K}_N(\lambda-\lambda_\ell)$ they use. So, they would achieve approximately the same speed for equal number of terms in (5.13) and (5.8). But the comparisons from Subsection 3.5 show that the tensor product Lagrange interpolation operator will need more terms than tensor product trigonometric needlet operator to achieve the same error for small or moderate oversampling.

The tensor product piece-wise polynomial algorithm is stable (if the Lagrange interpolation is properly realized).

5.4.3. *Comparison with nonequispaced fast spherical Fourier transform.* We turn our attention to Problem 1, where the spherical polynomial is given by its coefficients.

The first step in the needlet algorithm is the evaluation of $f(\xi)$ for all knots ξ from a regular grid, say (2.6) or (2.7). This can be done within the prescribed precision ε_2 with $O(N^2\ln N)$ operations by Tygert's algorithm [20, 21], who asserts [20] that the algorithm is numerically stable. Then, as second step we apply the spherical trigonometric needlet algorithm for polynomial evaluation at scattered points. The error in the polynomial values introduced in the first step will slightly increase, say two times, due to the small norm of the needlet operator. Therefore, the total error of this algorithm is bounded by $\|\Phi_{N,\delta}\|\varepsilon_2+\varepsilon_3$, where ε_3 stands for the relative error of the spherical trigonometric needlet algorithm.

The first step in the nonequispaced fast spherical Fourier transform developed by Kunis and Potts [15, 14] computes an approximation of the polynomial coefficients $c_{k\ell}$ in (2.4) if the polynomial is given by a_{mn} and b_{mn} in (2.2). This step is called discrete Legendre function transform and in some parts follows the (transposed version of) Driscoll and Healy algorithm [2, 5]. The discrete Legendre function transform, however, seems in principle unstable and various modifications were designed in order to overcome the problem. This instability makes the application of this approach problematic for high degree spherical polynomials. The second step is the two-dimensional nonequispaced fast Fourier transform.

We could not find in the literature error estimates for nonequispaced fast spherical Fourier transform algorithm. But, if we extrapolate the discussion on the one-dimensional nonequispaced fast Fourier transform from Subsection 3.5 then in the second step the nonequispaced fast spherical Fourier transform algorithm will require approximately 2 times less basis functions evaluations and 4 times less multiplications and summations than the tensor product needlet algorithm (cf. Table 3). This advantage should be reduced because the norm multiplier $\sum_{k,\ell=-N}^N|c_{k\ell}|$ in the error estimate for the two-dimensional nonequispaced fast Fourier transform is related to the norm multiplier $\|f\|_{C(\mathbb{S}^2)}$ for the tensor product trigonometric needlets by

$$\|f\|_{C(\mathbb{S}^2)}\le\sum_{k,\ell=-N}^N|c_{k\ell}|\le(2N+1)\|f\|_{C(\mathbb{S}^2)}.$$

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